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On the planar, outer planar, cut vertices and end-regular comaximal graph of lattices

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Abstract. In this paper, we determine the cut vertices in the comaximal graphs of lattices and study the lattices with end-regular comaximal graphs. Also, we investigate the comaximal graph of the product of two lattices. In particular, we determine all lattices with planar and outerplanar comaximal graphs.

1. Introduction

The investigation of graphs related to various algebraic structures is a very large and growing area of research. In particular, Cayley graphs have attracted serious attention in the literature and in the last 50 years, it has been grown into a substantial branch in algebraic graph theory. Cayley graphs have found many useful applications in solving and understanding a variety of problems of scientific interest, see [10], [13], [15], [18], [20], [24] for examples of recent results and further references. Several other classes of graphs associated with algebraic structures have been also actively investigated. For example, power graphs and divisibility graphs have been considered in [11], [12], zero-divisor graphs have been studied in [3], [4], [5], [6], [7], [8] and cozero-divisor graphs have been introduced in [2].

Let R be a commutative ring with non-zero identity. In [19], Sharma and Bhatwadekar defined the comaximal graph on R, denoted by $\Gamma(R)$, with all elements of R being the vertices of $\Gamma(R)$, where two distinct vertices a and b are

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adjacent if and only if $Ra + Rb = R$. In [16] and [21], the authors considered a subgraph $\Gamma_2(R)$ of $\Gamma(R)$ consisting of non-unit elements of R, and studied several properties of the comaximal graph. Also the comaximal graph of a non-commutative ring was defined and studied in [22]. Recently, the first two authors defined and studied the comaximal graph of a lattice, which is denoted by $\Gamma_2(L)$, in [1].

In Section 2, we study the cut vertices in the comaximal graph $\Gamma_2(L)$. Also, we investigate some basic properties of $\Gamma_2(L_1 \times L_2)$, where L_1 and L_2 are two bounded lattices. In Section 3, we study the planarity of $\Gamma_2(L_1 \times L_2)$. In Section 4, we investigate the outerplanarity in the comaximal graphs of lattices, and in the last section, we study the end-regularity in $\Gamma_2(L)$.

In a graph G, the *distance* between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we set $\mathrm{d}(a, b) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(a, b) : a \text{ and } b \text{ are distinct vertices of } G\}$. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. For a vertex x of a graph G, The *neighborhood* (resp, *degree*) of x, denoted by $\mathrm{N}(x)$ (resp, $\mathrm{d}(x)$), is the set of vertices which are adjacent to x (resp, the cardinality of $\mathrm{N}(x)$). A *clique* of a graph is a maximal complete subgraph of it and the number of vertices in a largest clique of G is called the *clique number* of G and is denoted by $\omega(G)$. Also, a vertex a is an end vertex, if there is only one edge incident to a. In the graph theory, a *unicycle graph* is a graph that has exactly one cycle. The graph with no vertices and no edges is the *null graph.*

A *lattice* is a set L with two binary operations $\wedge$ and $\vee$ on L satisfying the following conditions: for all $a, b, c \in L$,

1. $a \wedge a = a$, $a \vee a = a$,
2. $a \wedge b = b \wedge a$, $a \vee b = b \vee a$,
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $a \vee (b \vee c) = (a \vee b) \vee c$, and
4. $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

Note that in every lattice $a \wedge b = a$ always implies that $a \vee b = b$.

By [17, Theorem 2.1], one can define an order $\leq$ on a lattice L as follows:

For any $a, b \in L$, we set $a \leq b$ if and only if $a \wedge b = a$. Then $(L, \leq)$ is an ordered set in which every pair of elements has a greatest lower bound (g.l. b.) and a least upper bound (l.u. b.). Conversely, let P be an ordered set such that, for every pair $a, b \in P$, g.l.b.(a, b), l.u.b.$(a, b) \in P$. For each a and b in P, we define $a \wedge b :=$ g.l.b.(a, b) and $a \vee b :=$ l.u.b.(a, b). Then $(P, \wedge, \vee)$ is a lattice.

A lattice L is said to be *bounded* if there are elements 0 and 1 in L such that $0 \wedge a = 0$ and $a \vee 1 = 1$, for all $a \in L$. Clearly every finite lattice is bounded. A lattice with two elements is called a trivial lattice. In a partially ordered set $(P, \leq)$, we say that a covers b or b is covered by a, in notation $b \prec a$, if and only if $b < a$ and there is no element p in P such that $b < p < a$. Assume that S is a subset of P. Then an element x in P is a lower bound of S if $x \leqslant s$ for all $s \in S$. The set of all lower bounds of S is denoted by S^ℓ i.e. $S^\ell := \{x \in P \mid x \leqslant s, \text{ for all } s \in S\}$. An element a in L is called an *atom* if $0 \prec a$. Similarly, a is called a *co-atom* if $a \prec 1$. Throughout the paper, all lattices are finite. Also, $C(L) = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$ is the set of all co-atoms of L.

2. Cut vertices in the comaximal graph of a lattice

An element a in L is said to be *unit* if there exits an element b in L such that $a \wedge b = 1$. It is easy to see that 1 is the only unit element in every lattice. The comaximal graph of a lattice L, denoted by $\Gamma(L)$, is an undirected graph with all elements of L being the vertices, and two distinct vertices a and b are adjacent if and only if $a \vee b = 1$ (see [1]). In $\Gamma(L)$ the vertex 1, which is the only unit element in L, is adjacent to all other vertices. Thus we consider $L \setminus \{1\}$ as vertex-set and denote this set by $W(L)$. In [1, Proposition 3.1], it was mentioned that an induced subgraph of the comaximal graph with vertex-set $\bigcap\{\mathfrak{m}\}^\ell$, where $\mathfrak{m} \in C(L)$, is totally disconnected and it is disjoint from an induced subgraph with vertices in $W(L) \setminus \bigcap_{\mathfrak{m} \in C(L)}\{\mathfrak{m}\}^\ell$. Hence, all vertices in $\bigcap_{\mathfrak{m} \in C(L)}\{\mathfrak{m}\}^\ell$ are isolated vertices. Therefore we ignore these isolated vertices and consider an induced subgraph with vertex-set $W(L) \setminus \bigcap_{\mathfrak{m} \in C(L)}\{\mathfrak{m}\}^\ell$, which is denoted by $\Gamma_2(L)$.

Clearly, if $|C(L)| = 1$, then $\Gamma_2(L)$ is a null graph. Therefore, we suppose that $n = |C(L)| \geq 2$.

Recall that a vertex a of a graph G is called a *cut vertex* if the removal of a and all edges incident to a creates a graph with more connected components than G.

Theorem 2.1. *If $\mathfrak{m}$ is a cut vertex in $\Gamma_2(L)$, then $\mathfrak{m}$ is a co-atom of L.*

PROOF. Assume that $\mathfrak{m}$ is not a co-atom. Since $\mathfrak{m}$ is a cut vertex, $\Gamma_2(L) \setminus \{\mathfrak{m}\}$ has at least two components X and Y. We claim that $C(L) \subseteq X$ or $C(L) \subseteq Y$. Otherwise there are co-atoms $\mathfrak{m}_i$ and $\mathfrak{m}_j$, where $1 \leq i \neq j \leq n$, such that $\mathfrak{m}_i \in X$

and $\mathfrak{m}_j \in Y$. Now we have that $\mathfrak{m}_i$ is adjacent to $\mathfrak{m}_j$, which is impossible. Without loss of generality, we may assume that $C(L) \subseteq X$. Then, for all $y \in Y$, we have $y \in \{\mathfrak{m}_i\}^\ell$, for $i = 1, 2, \ldots, n$. Thus $y \in \cap_{i=1}^n \{\mathfrak{m}_i\}^\ell$, which is impossible. Therefore $\mathfrak{m} \in C(L)$. □

The following example shows that the converse of Theorem 2.1 is not true in general.

Example 2.2. Suppose that L is a lattice in Figure 1. Then, it is easy to see that $\mathfrak{m}_1$ is a co-atom, but it is not a cut vertex in $\Gamma_2(L)$.

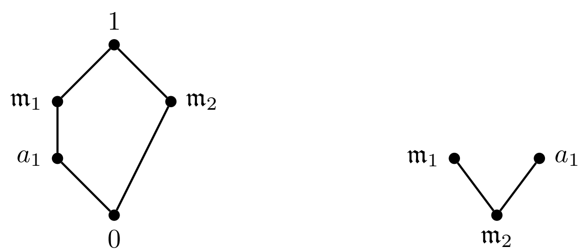

Figure 1. L and $\Gamma_2(L)$

Notation 2.3. Let $i_1, i_2, \ldots, i_k$ be integers with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. The notation $S^L_{i_1 i_2 \ldots i_k}$ stands for the following set:

$$\{x \in L \mid x \in \cap_{t=1}^k \{\mathfrak{m}_{i_t}\}^\ell \setminus \cup_{j \neq i_1, i_2, \ldots, i_k} \{\mathfrak{m}_j\}^\ell\}.$$

Note that no two distinct elements in $S^L_{i_1 i_2 \ldots i_k}$ are adjacent in $\Gamma_2(L)$. Also if the index sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_{k'}\}$ of $S^L_{i_1 i_2 \ldots i_k}$ and $S^L_{j_1 j_2 \ldots j_{k'}}$, respectively, are distinct, then one can easily check that $S^L_{i_1 i_2 \ldots i_k} \cap S^L_{j_1 j_2 \ldots j_{k'}} = \emptyset$, and also each vertex in $S^L_{i_1 i_2 \ldots i_k}$ is adjacent to all vertices in $S^L_{j_1 j_2 \ldots j_{k'}}$. Moreover $L \setminus \{1\} = \cup^n_{k=1, 1 \leq i_1 < i_2 < \cdots < i_k \leq n} S^L_{i_1 i_2 \ldots i_k}$. If there is no ambiguity, we denote the set $S^L_{i_1 i_2 \ldots i_k}$ by $S_{i_1 i_2 \ldots i_k}$. Also by $1 \ldots \hat{i} \ldots n$ we mean that $1 \ldots i-1 \; i+1 \ldots n$.

In the next theorem, we provide conditions under which the converse of Theorem 2.1 holds.

Theorem 2.4. *Suppose that $|V(\Gamma_2(L))| \geq 3$. Then there exists a positive integer i with $1 \leqslant i \leqslant n$ such that $\mathfrak{m}_i$ is a cut vertex in $\Gamma_2(L)$, if $S_i = \{\mathfrak{m}_i\}$ and $S_{1 \ldots \hat{i} \ldots n} \neq \emptyset$, for some $1 \leq i \leq n$.*

PROOF. It is enough to show that there exist vertices b and c in L such that $\mathfrak{m}_i$ is in every path from b to c in $\Gamma_2(L)$. Since $S_{1\dots\hat{i}\dots n} \neq \emptyset$, there is an element b in $S_{1\dots\hat{i}\dots n}$. Also, for some $j \neq i$, consider $c \in S_j$. Now, in view of Notation 2.3, since $S_i \cap S_{1\dots\hat{i}\dots n} = \emptyset$, we have that $\mathfrak{m}_i$ is a unique neighbor of b. Also, $\mathfrak{m}_i$ is adjacent to c. Therefore, $\mathfrak{m}_i$ is a cut vertex in $\Gamma_2(L)$. □

Definition 2.5. For any x in L, set

$$Z_x := \{y \in L \mid x \vee y = 1\}.$$

We say that Z_x is properly maximal if $Z_x \subseteq Z_b$, for some $b \in L$, then we have $Z_x = Z_b$.

Theorem 2.6. *If $\mathfrak{m}$ is a cut vertex in $\Gamma_2(L)$, then $Z_{\mathfrak{m}}$ is properly maximal.*

PROOF. Assume on the contrary that $Z_{\mathfrak{m}} \subsetneqq Z_b$, for some vertices b in $\Gamma_2(L)$ with $b \neq \mathfrak{m}$. Then clearly all vertices adjacent to $\mathfrak{m}$ are also adjacent to b. This is a contradiction with the fact that $\mathfrak{m}$ is a cut vertex. □

Let $(L_1, \wedge_1, \vee_1)$ and $(L_2, \wedge_2, \vee_2)$ be two bounded lattices. Then, clearly $(L_1 \times L_2, \wedge, \vee)$ is also a bounded lattice with the following relations. For two distinct elements $(a, b), (c, d) \in L_1 \times L_2$ we say that $(a, b) \wedge (c, d) = (a \wedge_1 c, b \wedge_2 d)$ and $(a, b) \vee (c, d) = (a \vee_1 c, b \vee_2 d)$. Clearly $(L_1 \times L_2, \wedge, \vee)$ has the minimum element $(0, 0)$ and the maximum element $(1, 1)$. Suppose that L_1 and L_2 are two finite lattices such that $C(L_1) = \{\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n\}$ and $C(L_2) = \{\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_m\}$. In the following, we study some properties of the comaximal graph $\Gamma_2(L_1 \times L_2)$.

Lemma 2.7. *In the lattice $L_1 \times L_2$, we have $C(L_1 \times L_2) = (C(L_1) \times \{1\}) \cup (\{1\} \times C(L_2))$, and so $|C(L_1 \times L_2)| = |C(L_1)| + |C(L_2)|$.*

PROOF. Suppose that $(\mathfrak{m}, \mathfrak{t})$ is an arbitrary element in $C(L_1 \times L_2)$. If $\mathfrak{m}, \mathfrak{t} \neq 1$, then we have $(\mathfrak{m}, \mathfrak{t}) < (\mathfrak{m}, 1) < (1, 1)$ which is impossible. Thus we have $\mathfrak{m} = 1$ or $\mathfrak{t} = 1$. Without loss of generally, we may assume that $\mathfrak{t} = 1$. If $\mathfrak{m} \notin C(L_1)$, then there exists a co-atom $\mathfrak{m}_i \in C(L_1)$, for some $1 \leq i \leq n$, such that $\mathfrak{m} < \mathfrak{m}_i$. Hence we have that $(\mathfrak{m}, 1) < (\mathfrak{m}_i, 1) < (1, 1)$ which is impossible. Therefore $\mathfrak{m} \in C(L_1)$, and so the result holds. □

We can extend the concept of $L_1 \times L_2$ for a product of finite number of lattices.

Corollary 2.8. *Let $L = L_1 \times L_2 \times \cdots \times L_n$, where for $i = 1, 2, \dots, n$, $(L_i, \leq_i)$ is a lattice with 1. Then $C(L)$ consists of elements $(\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n)$ such that there exists $1 \leq j \leq n$ with $\mathfrak{m}_j \in C(L_j)$, and, for all i with $1 \leq i \neq j \leq n$, $\mathfrak{m}_i = 1$.*

Proposition 2.9. *Assume that* $L_1, \dots, L_n$ *are non-trivial lattices. Let* $L = L_1 \times L_2 \times \cdots \times L_n$ *be a lattice such that* $L \neq L_1 \times L_2$, *with* $|L_1| = |L_2| = 3$. *If* $\mathfrak{m} = (1, \dots, 1, u_i, 1, \dots, 1) \in V(\Gamma_2(L))$, *is a cut vertex with component* u_i *such that* $u_i \notin V(\Gamma_2(L_i))$, *then* $|L_i| = 3$.

Proof. Assume on the contrary that L_i has at least four elements, and so there exists v_i in $L_i \setminus \{0, 1, u_i\}$. It is easy to see that $Z_{\mathfrak{m}} \subseteq Z_{(1,\dots,1,v_i,1,\dots,1)}$. Since $\mathfrak{m}$ is a cut vertex, by Theorem 2.6, we have that $Z_{\mathfrak{m}} = Z_{(1,\dots,1,v_i,1,\dots,1)}$. If $\mathfrak{m} \neq (1, \dots, 1, v_i, 1, \dots, 1)$, then $\mathfrak{m}$ is not a cut vertex. So $\mathfrak{m} = (1, \dots, 1, v_i, 1, \dots, 1)$. Hence $u_i = v_i$, which is a contradiction. □

Lemma 2.10. *In the graph* $\Gamma_2(L)$, *we have* $\omega(\Gamma_2(L)) \geq |C(L)|$.

Proof. The induced subgraph of $\Gamma_2(L)$ with vertex-set $C(L)$ is a complete graph. So we have $\omega(\Gamma_2(L)) \geq |C(L)|$. □

Now the following corollary follows from Lemmas 2.7 and 2.10.

Corollary 2.11. *In the graph* $\Gamma_2(L_1 \times L_2)$, *we have*

$$\omega(\Gamma_2(L_1 \times L_2)) \geq |C(L_1)| + |C(L_2)|.$$

For a graph G, the *girth* of G is the length of the shortest cycle in G and is denoted by $\mathrm{gr}(G)$. If G has no cycles, we put $\mathrm{gr}(G) := \infty$. In 1916, the Hungarian mathematician, Dénes König (1884–1944) deduced that a graph is bipartite if and only if it contains no cycles of odd length.

One can easily see that $\Gamma_2(L)$ is a star graph if and only if L has two co-atoms $\mathfrak{m}_1$ and $\mathfrak{m}_2$, such that $|S_1| = 1$ or $|S_2| = 1$. Also, $\Gamma_2(L)$ is a bipartite graph but not a star graph if and only if L has two co-atoms $\mathfrak{m}_1$ and $\mathfrak{m}_2$ such that $|S_1|, |S_2| > 1$.

In [1, Theorem 3.6], it was proved that $\mathrm{gr}(\Gamma_2(L)) \in \{3, 4, \infty\}$. In the following theorem, we characterize the cases that $\mathrm{gr}(\Gamma_2(L))$ is $3, 4$ or ∞.

Theorem 2.12. *Let* L *be a lattice. Then the following statements hold:*

(i) $\mathrm{gr}(\Gamma_2(L)) = \infty$ *if and only if* $\Gamma_2(L)$ *is a star graph.*

(ii) $\mathrm{gr}(\Gamma_2(L)) = 4$ *if and only if* $\Gamma_2(L)$ *is a bipartite graph which is not a star graph.*

(iii) $\mathrm{gr}(\Gamma_2(L)) = 3$ *if and only if* $\Gamma_2(L)$ *contains an odd cycle.*

Proof. (i) First assume that $\mathrm{gr}(\Gamma_2(L)) = \infty$. Suppose that $\Gamma_2(L)$ is not a star graph. Since by [1, Theorem 3.2], $\Gamma_2(L)$ is connected, we have that $|W(L) \setminus \cap_{\mathfrak{m} \in C(L)} \{\mathfrak{m}\}^{\ell}| \geq 4$. Also, since the graph $\Gamma_2(L)$ is not a star graph, so there is a path of the form $a - x - b - c$ in the graph $\Gamma_2(L)$ such that $a, b, x, c \in W(L) \setminus \cap_{\mathfrak{m} \in C(L)} \{\mathfrak{m}\}^\ell$. If a is adjacent to c, then $a - x - b - c - a$ is a cycle in $\Gamma_2(L)$, a contradiction. If a is not adjacent to c, then there exits $z \in L \setminus \{0, 1\}$ such that $z \geq a$ and $z \geq c$. So z is adjacent to both vertices x and b. Hence we have the cycle $z - x - b - z$ in $\Gamma_2(L)$, which is a contradiction. Therefore $\Gamma_2(L)$ is a star graph.

Conversely, if $\Gamma_2(L)$ is a star graph, then it is easy to see that $\mathrm{gr}(\Gamma_2(L)) = \infty$.

(ii) First assume that $\mathrm{gr}(\Gamma_2(L)) = 4$. Clearly, $\Gamma_2(L)$ is not a star graph. We show that $\Gamma_2(L)$ has no odd cycles, which implies that it is bipartite. On the contrary, let us assume that $\Gamma_2(L)$ has an odd cycle and that $x_1 - x_2 - x_3 - \cdots - x_n - x_1$ be an odd cycle of minimal length n in $\Gamma_2(L)$. Clearly, $n \geq 5$, since $\mathrm{gr}(\Gamma_2(L)) \neq 3$. Now, the minimality of n ensures that $x_2 \vee x_4 \neq 1$. Suppose that $z = x_2 \vee x_4$ and it is not equal to 1. Then we have $x_1 \vee z \geq x_1 \vee x_2 = 1$ and $x_5 \vee z \geq x_5 \vee x_4 = 1$. It follows that $x_1 - z - x_5 - \cdots - x_n - x_1$ is a cycle of length $n - 2$ in $\Gamma_2(L)$. This contradicts the minimality of n. Hence $\Gamma_2(L)$ has no odd cycles.

Conversely, suppose that $\Gamma_2(L)$ is a bipartite graph which is not a star graph. Then, we have $\mathrm{gr}(\Gamma_2(L)) \neq 3$. Also, by (i), $\mathrm{gr}(\Gamma_2(L)) \neq \infty$. Hence, we have $\mathrm{gr}(\Gamma_2(L)) = 4$.

(iii) It follows from (i) and (ii). □

The following corollary follows immedietly from Theorem 2.12.

Corollary 2.13. *In the graph* $\Gamma_2(L)$, $\mathrm{gr}(\Gamma_2(L_1 \times L_2)) \in \{3, 4, \infty\}$.

3. Planarity of $\Gamma_2(L_1 \times L_2)$

Recall that a graph is said to be *planar* if it can be drawn in the plane, so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Theorem 3.1. *If* $\Gamma_2(L_1 \times L_2)$ *is planar, then* $|C(L_1)| + |C(L_2)| \leq 4$.

Proof. Suppose on the contrary that $|C(L_1)| + |C(L_2)| \geq 5$. Since the induced subgraph of $\Gamma_2(L_1 \times L_2)$ with the vertex-set $C(L_1 \times L_2)$ is a complete graph, we can find a subgraph isomorphic to K_5 in $\Gamma_2(L_1 \times L_2)$, and so, by Kuratowski's Theorem, it is not planar. Hence we have $|C(L_1)|+|C(L_2)| \leq 4$. □

By Theorem 3.1, we must study the cases that $|C(L_1)| + |C(L_2)|$ is equal to 2, 3 and 4. In the following proposition, we state the necessary and sufficient condition for planarity of $\Gamma_2(L_1 \times L_2)$, when $|C(L_1)| + |C(L_2)| = 2$.

Proposition 3.2. *Suppose that* $|C(L_1)| + |C(L_2)| = 2$. *Then* $\Gamma_2(L_1 \times L_2)$ *is planar if and only if* $|L_1| \leq 3$ *or* $|L_2| \leq 3$.

Proof. Since $|C(L_1)| + |C(L_2)| = 2$, one can easily see that $\Gamma_2(L_1 \times L_2) \cong K_{|L_1|-1,|L_2|-1}$. Therefore, the graph $\Gamma_2(L_1 \times L_2)$ is planar if and only if $|L_1| \leq 3$ or $|L_2| \leq 3$. □

Now, suppose that L_1 and L_2 are lattices such that $|C(L_1)| + |C(L_2)| = 3$. Let $|C(L_1)| = 1$ and $|C(L_2)| = 2$. If $|L_1|, |L_2| \geq 5$, then one can find a copy of $K_{3,3}$ in the graph $\Gamma_2(L_1 \times L_2)$. Thus, by Kuratowski's Theorem, $\Gamma_2(L_1 \times L_2)$ is not planar. Therefore, if $\Gamma_2(L_1 \times L_2)$ is planar, then $|L_1| \leq 4$ or $|L_2| \leq 4$. Now, we have the following cases:

Case 1. Suppose that $|L_1| = 3$. If $|S_i^{L_2}| \geq 2$, for all $1 \leq i \leq 2$, then we can find a subdivision of K_5 as in Figure 2, where $y_i \in S_i^{L_2} \setminus \{\mathfrak{t}_i\}$, for all $1 \leq i \leq 2$, and so $\Gamma_2(L_1 \times L_2)$ is not planar.

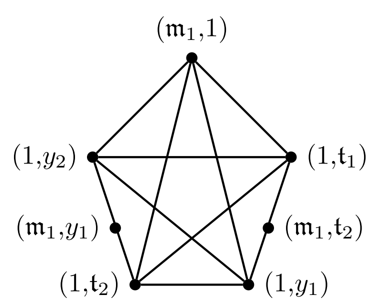

Figure 2

If $|S_i^{L_2}| = 1$ and $|S_j^{L_2}| \geq 2$, for some $1 \leq i \neq j \leq 2$, then we can find a subdivision of K_5 as in Figure 3, where $y_j \in S_j^{L_2} \setminus \{\mathfrak{t}_j\}$, for some $1 \leq i \neq j \leq 2$. Hence $\Gamma_2(L_1 \times L_2)$ is not planar.

Now, if $|S_i^{L_2}| = 1$, for all $1 \leq i \leq 2$, then $\Gamma_2(L_1 \times L_2)$ is pictured in Figure 4, and so $\Gamma_2(L_1 \times L_2)$ is planar.

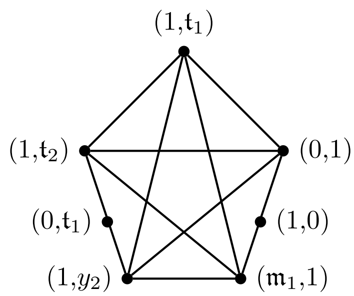

Figure 3

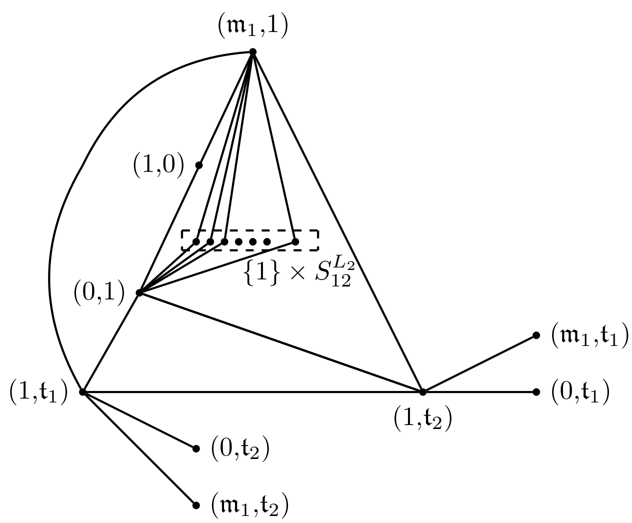

Figure 4

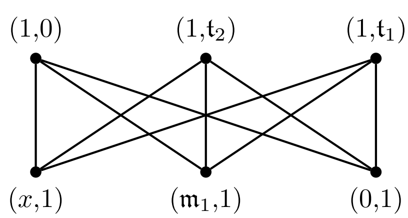

Figure 5

Case 2. Suppose that $|L_1| = 4$. In this situation one can find a copy of $K_{3,3}$ as in Figure 5, where $x \in L \setminus \{0, 1, \mathfrak{m}_1\}$, and so $\Gamma_2(L_1 \times L_2)$ is not planar.

Case 3. Suppose that $|L_2| = 4$ and $|L_1| = 3$. In this situation $\Gamma_2(L_1 \times L_2)$ is pictured in Figure 6, and hence $\Gamma_2(L_1 \times L_2)$ is planar.

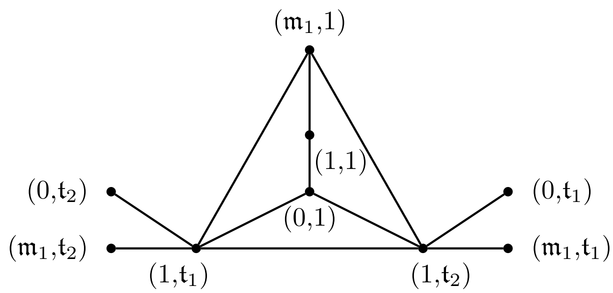

Figure 6

Thus we have the following theorem.

Theorem 3.3. *Suppose that* $|C(L_1)| + |C(L_2)| = 3$ *such that* $|C(L_1)| = 1$ *and* $|C(L_2)| = 2$*. Then* $\Gamma_2(L_1 \times L_2)$ *is planar if and only if one of the following conditions holds.*

(i) $|L_1| = 3$ *and* $|S_i^{L_2}| = 1$, *for all* $1 \leq i \leq 2$.

(ii) $|L_2| = 4$ *and* $|L_1| = 3$.

Finally assume that $|C(L_1)|+|C(L_2)| = 4$. Now, we have the following cases:

Case 1. Suppose that $|C(L_1)| = 1$ and $|C(L_2)| = 3$. Then we can find a subdivision of K_5 as in Figure 7. So $\Gamma_2(L_1 \times L_2)$ is not planar.

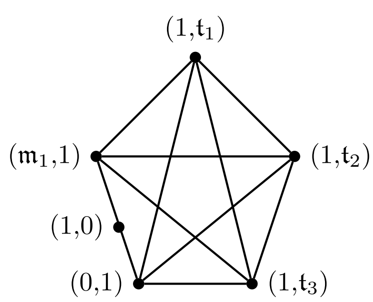

Figure 7

Case 2. Assume that $|C(L_1)| = 2 = |C(L_2)|$. In this situation we can find a subdivision of $K_{3,3}$ as in Figure 8, and so $\Gamma_2(L_1 \times L_2)$ is not planar.

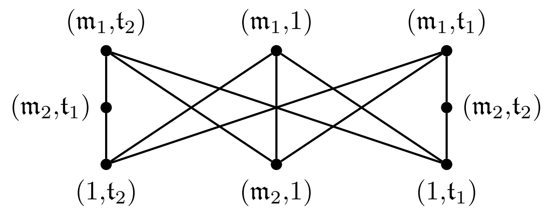

Figure 8

Now we have the following theorem from the above discussion.

Theorem 3.4. *Suppose that* $|C(L_1)| + |C(L_2)| = 4$. *Then* $\Gamma_2(L_1 \times L_2)$ *is not planar.*

4. Outerplanarity of $\Gamma_2(L)$ and $\Gamma_2(L_1 \times L_2)$

An undirected graph is *outerplanar* if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

In the following, we characterize all lattices L such that $\Gamma_2(L)$ is outerplanar.

Lemma 4.1. *If* $\Gamma_2(L)$ *is outerplanar, then* $|C(L)| \leq 3$.

PROOF. Assume to the contrary that $|C(L)| \geq 4$. Since the induced subgraph of $\Gamma_2(L)$ with vertex-set $C(L)$ is a complete subgraph, one can find a copy of K_4 in $\Gamma_2(L)$, and so $\Gamma_2(L)$ is not outerplanar. Hence we have $|C(L)| \leq 3$. □

By Lemma 4.1, we must study the cases that $|C(L)|$ is equal to 2 and 3. In the following proposition, we state the necessary and sufficient condition for outerplanarity of $\Gamma_2(L)$, when $|C(L)| = 2$.

Proposition 4.2. *Suppose that* $|C(L)| = 2$. *Then* $\Gamma_2(L)$ *is outerplanar if and only if* $|S_i| = 1$, *for some* $1 \leq i \leq 2$, *or* $|S_i| \leq 2$, *for all* $1 \leq i \leq 2$.

PROOF. Since $|C(L)| = 2$, we have that $\Gamma_2(L)$ is a complete bipartite graph. Now one can easily see that $\Gamma_2(L)$ is outerplanar if and only if $|S_i| = 1$, for some $1 \leq i \leq 2$, or $|S_i| \leq 2$, for all $1 \leq i \leq 2$. □

In the sequel of this section, we investigate the outerplanarity of $\Gamma_2(L)$, when $|C(L)| = 3$. If $|\cup_{i=1}^3 S_i| \geq 5$, then one can find a copy of $K_{2,3}$ in the structure of $\Gamma_2(L)$, and so $\Gamma_2(L)$ is not outerplanar. Therefore, if $\Gamma_2(L)$ is outerplanar, then $|\cup_{i=1}^3 S_i| \leq 4$. Now, we have the following cases:

Case 1. Suppose that $|\cup_{i=1}^3 S_i| = 3$. In this situation $\Gamma_2(L)$ is a unicyclic graph which is in pictured in Figure 9, and so it is outerplanar.

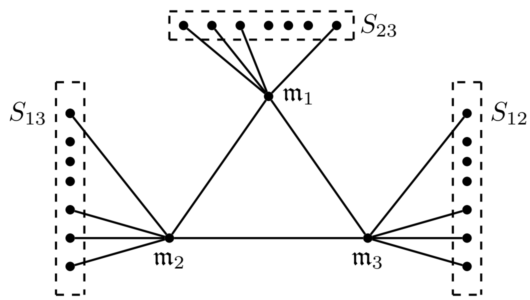

Figure 9

Case 2. Suppose that $|\cup_{i=1}^3 S_i| = 4$. Assume that $|S_i| = 2$. If $|S_{jk}| \geq 1$, for some $1 \leq i \neq j \neq k \leq 3$, then we can find a copy of $K_{2,3}$ as in Figure 10, where $\mathfrak{m}'_i \in S_i \setminus \mathfrak{m}_i$ and $c_{jk} \in S_{jk}$, for some $1 \leq i \neq j \neq k \leq 3$, and so $\Gamma_2(L)$ is not outerplanar.

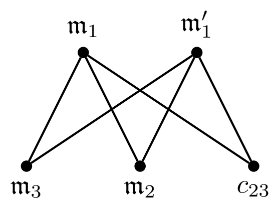

Figure 10

Now, if $S_{jk} = \emptyset$, for all $1 \leq i \neq j \neq k \leq 3$, then $\Gamma_2(L)$ is isomorphic to the graph which is pictured in Figure 11, and so $\Gamma_2(L)$ is outerplanar.

Theorem 4.3. *Suppose that $|C(L)| = 3$. Then $\Gamma_2(L)$ is outerplanar if and only if one of the following conditions holds.*

(i) $|\cup_{i=1}^3 S_i| = 3$.

(ii) $|\cup_{i=1}^3 S_i| = 4$ *and if* $|S_i| = 2$, *for some* $1 \leq i \leq 3$, *then* $S_{jk} = \emptyset$, *for all* $1 \leq i \neq j \neq k \leq 3$.

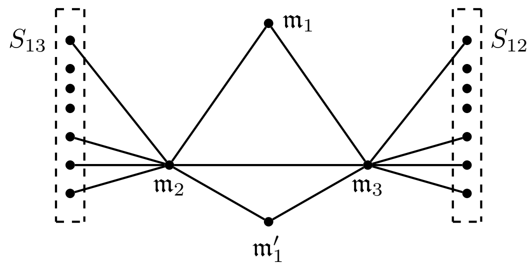

Figure 11

In the following, we characterize all lattices L_1 and L_2 such that $\Gamma_2(L_1 \times L_2)$ is outerplanar. Clearly, if $\Gamma_2(L_1 \times L_2)$ is outerplanar, then, by Lemmas 2.7 and 4.1, $|C(L_1)| + |C(L_2)| \leq 3$. In the next two theorems, we investigate the cases that $|C(L_1)| + |C(L_2)| = 2$ and $|C(L_1)| + |C(L_2)| = 3$.

Theorem 4.4. *Suppose that* $|C(L_1)| + |C(L_2)| = 2$*. Then* $\Gamma_2(L_1 \times L_2)$ *is outerplanar if and only if* $|L_i| \leq 3$ *or,* $|L_j| \leq 4$ *with* $|L_i| \leq 2$*, for some* $1 \leq i \neq j \leq 2$.

Proof. Since $|C(L_1)| + |C(L_2)| = 2$, one can easily see that $\Gamma_2(L_1 \times L_2) \cong K_{|L_1|-1,|L_2|-1}$. Therefore, $\Gamma_2(L_1 \times L_2)$ is an outerplanar graph if and only if $|L_i| \leq 3$ or, $|L_j| \leq 4$ and $|L_i| \leq 2$, for some $1 \leq i \neq j \leq 2$. □

Now, suppose that L_1 and L_2 are lattices such that $|C(L_1)|$=1 and $|C(L_2)|$=2. If $|L_i| \geq 4$ and $|L_j| \geq 5$, for all $1 \leq i \neq j \leq 2$, then we can find a copy of $K_{2,3}$ in the graph $\Gamma_2(L_1 \times L_2)$. Thus $\Gamma_2(L_1 \times L_2)$ is not outerplanar. Therefore, if $\Gamma_2(L_1 \times L_2)$ is outerplanar, then $|L_1| = 3$, or $|L_2| = 4$ with $|L_1| \leq 4$. Now, in the following two cases, we study the outerplanarity of $\Gamma_2(L_1 \times L_2)$ whenever $|L_1| = 3$, or $|L_1| \leq 4$ with $|L_2| = 4$.

Case 1. Suppose that $|L_1| = 3$. If $|S_i^{L_2}| \geq 2$, for some $1 \leq i \leq 2$, then we can find a subdivision of K_4 as in Figure 12, where $y_i \in S_i^{L_2} \setminus \{\mathfrak{t}_i\}$, and so $\Gamma_2(L_1 \times L_2)$ is not outerplanar.

Now, if $|S_i^{L_2}| = 1$, for all $1 \leq i \leq 2$, then one can find a copy of $K_{2,3}$ as in Figure 13, and so $\Gamma_2(L_1 \times L_2)$ is not outerplanar.

Case 2. Suppose that $|L_2| = 4$ and $|L_1| \leq 4$. If $|L_1| = 4$, then one can find a copy of $K_{2,3}$ as in Figure 13, and so $\Gamma_2(L_1 \times L_2)$ is not outerplanar.

If $|L_1| = 3$, then $\Gamma_2(L_1 \times L_2)$ is pictured in Figure 14, and so it is outerplanar.

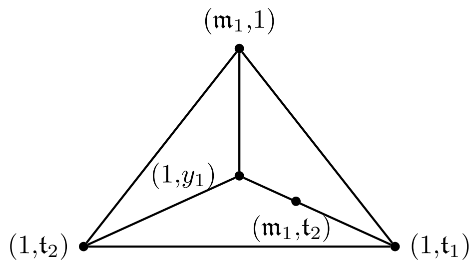

Figure 12

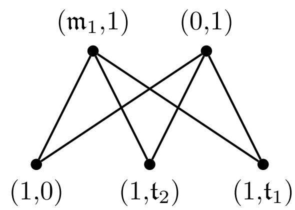

Figure 13

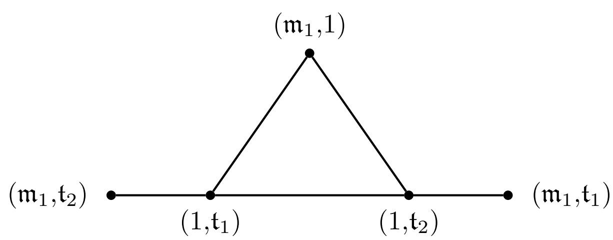

Figure 14

Theorem 4.5. *Suppose that* $|C(L_1)| + |C(L_2)| = 3$ *such that* $|C(L_1)| = 1$ *and* $|C(L_2)| = 2$*. Then* $\Gamma_2(L_1 \times L_2)$ *is outerplanar if and only if* $|L_2| = 4$ *and* $|L_1| = 3$.

Let G be a graph with n vertices and q edges. We recall that a *chord* is any edge of G joining two nonadjacent vertices in a cycle of G. Let C be a cycle of$G$. We say C is a *primitive cycle* if it has no chords. Also, a graph G has the *primitive cycle property* (PCP) if any two primitive cycles intersect in at most one edge. The number frank(G) is called the *free rank* of G and it is the number of primitive cycles of G. Also, the number rank(G)=$q - n + r$ is called the *cycle rank* of G, where r is the number of connected components of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. By [9, Proposition 2.2], we have rank$(G) \leq$ frank (G). A graph G is called a *ring graph* if it satisfies one of the following equivalent conditions (see [9]).

(i) rank(G)= frank(G),

(ii) G satisfies the PCP and G does not contain a subdivision of K_4 as a subgraph.

Clearly, every outerplanar graph is a ring graph and every ring graph is a planar graph.

Now, in view of the proofs of Proposition 4.2 and Theorem 4.3, we have the following result.

Theorem 4.6. *The comaximal graph $\Gamma_2(L)$ is a ring graph if and only if it is an outerplanar graph.*

5. End-regularity of zero-divisor graphs of lattices

Let G and H be graphs. A *homomorphism* f from G to H is a map from $V(G)$ to $V(H)$ such that for any $a, b \in V(G)$, a is adjacent to b implies that $f(a)$ is adjacent to $f(b)$. Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an *isomorphism* from G to H, and in this case we say G is isomorphic to H, denoted by $G \cong H$. A homomorphism (resp, an isomorphism) from G to itself is called an *endomorphism* (resp, *automorphism*) of G. An endomorphism f is said to be *half-strong* if $f(a)$ is adjacent to $f(b)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that c is adjacent to d. By End(G), we denote the set of all the endomorphisms of G. It is well-known that End(G) is a monoid with respect to the composition of mappings. Let S be a semigroup. An element a in S is called *regular* if $a = aba$ for some $b \in S$ and S is called regular if every element in S is regular. Also, a graph G is called *end-regular* if End(G) is regular.

Now, we recall the following Lemma from [14].

Lemma 5.1 ([14, Lemma 2.1])**.** *Let G be a graph. If there are pairwise distinct vertices a, b, c in G satisfying $\mathrm{N}(c) \subsetneqq \mathrm{N}(a) \subseteq \mathrm{N}(b)$, then G is not end-regular.*

Lemma 5.2. *Suppose that $|C(L)| \geq 3$. If $S_{i\ldots j}, S_{i\ldots j\ldots k} \neq \emptyset$, such that $|S_{i\ldots j}| > 1$, for some $1 \leq i < j < k < n$, or $S_{i\ldots j}$, $S_{i\ldots j\ldots k}, S_{i\ldots j\ldots k\ldots t} \neq \emptyset$, for some $1 \leq i < j < k < t < n$, then $\Gamma_2(L)$ is not end-regular.*

Proof. First suppose that $S_{i\ldots j}, S_{i\ldots j\ldots k} \neq \emptyset$ and $|S_{i\ldots j}| > 1$, for some $1 \leq i < j < n$. Let $a, b \in S_{i\ldots j}$ and $c \in S_{i\ldots j\ldots k}$. Then $\mathrm{N}(c) \subsetneqq \mathrm{N}(a)$, since $\mathfrak{m}_k \in \mathrm{N}(a) \setminus \mathrm{N}(c)$. Now, we have $\mathrm{N}(c) \subsetneqq \mathrm{N}(a) \subseteq \mathrm{N}(b)$, and so, by Lemma 5.1, $\Gamma_2(L)$ is not end-regular. If $S_{i\ldots j}$, $S_{i\ldots j\ldots k}, S_{i\ldots j\ldots k\ldots t} \neq \emptyset$, for some $1 \leq i < j < k < t < n$, then consider the elements $a \in S_{i\ldots j}, b \in S_{i\ldots j\ldots k}$ and $c \in S_{i\ldots j\ldots k\ldots t}$. Now, we have $\mathrm{N}(c) \subsetneqq \mathrm{N}(b) \subseteq \mathrm{N}(a)$. Hence $\Gamma_2(L)$ is not end-regular. □

Proposition 5.3. *Suppose that $|C(L)| = 2$. Then $\Gamma_2(L)$ is end-regular.*

Proof. Clearly $\Gamma_2(L)$ is a complete bipartite graph. Now, by [23, Theorem 3.4], we have that $\Gamma_2(L)$ is end-regular. □

Lemma 5.4. *Suppose that $x, y \in W(L) \setminus \bigcap_{\mathfrak{m} \in C(L)} \{\mathfrak{m}\}^{\ell}$. Then $\mathrm{N}(x) \subseteq \mathrm{N}(y)$ if and only if $Z_x \subseteq Z_y$ and $x \vee y \neq 1$.*

Proof. ($\Leftarrow$) It is trivial.

($\Rightarrow$) If $\mathrm{N}(x) \subseteq \mathrm{N}(y)$, then $Z_x \subseteq Z_y$. Suppose to the contrary that $x \vee y = 1$. Then x is adjacent to y. This means that $y \in \mathrm{N}(x) \subseteq \mathrm{N}(y)$, and so $y \in \mathrm{N}(y)$, which is a contradiction. □

Proposition 5.5. *Suppose that $L = L_1 \times L_2 \times \cdots \times L_n$. Then we have the following statements.*

(i) *If $n \geq 3$, then $\Gamma_2(L_1 \times L_2 \times \cdots \times L_n)$ is not end-regular.*

(ii) *If $|C(L_1)| = 1 = |C(L_2)|$, then $\Gamma_2(L_1 \times L_2)$ is end-regular.*

Proof. (i) Suppose that $C(L_1) = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$, $C(L_2) = \{\mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_s\}$ and $C(L_3) = \{v_1, v_2, \ldots, v_r\}$, where $n, s, r \geq 1$. Set $x := (\mathfrak{m}_i, 1, \ldots, 1)$, $y := (\mathfrak{m}_i, \mathfrak{t}_j, 1, \ldots, 1)$ and $z := (\mathfrak{m}_i, \mathfrak{t}_j, v_k, 1, \ldots, 1)$, for some $1 \leq i \leq n$, $1 \leq j \leq s$ and $1 \leq k \leq v$. Then $\mathrm{N}(z) \subsetneqq \mathrm{N}(y) \subsetneqq \mathrm{N}(x)$. Now, by Lemmas 5.1 and 5.4, $\Gamma_2(L)$ is not end-regular.

(ii) Note that in this case, $\Gamma_2(L_1 \times L_2)$ is a complete bipartite graph and, by [23, Theorem 3.4], every complete bipartite graph is end-regular. □

Lemma 5.6. *Assume that $\Gamma_2(L_2)$ has distinct vertices x and y such that $x, y \notin C(L_2)$ and $\mathrm{N}(x) \subseteq \mathrm{N}(y)$. Then $\Gamma_2(L_1 \times L_2)$ is not end-regular.*

Proof. Suppose that $b \in C(L_2)$. Then it follows from the fact that
$\mathrm{N}((1, x)) \subsetneqq \mathrm{N}((1, y)) \subsetneqq \mathrm{N}((1, b))$. □

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