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On delta Schur-convex mappings

By ANDRZEJ OLBRYŚ (Katowice)

Abstract. The aim of the present paper is to combine the notions of Schur-convex and delta-convex mappings in the sense of Veselý and Zajiček. Our main result gives necessary and sufficient conditions on maps $F_j, j = 1, \dots, n$, under which the sum $\sum_{j=1}^{n} F_j(x_j)$ is delta Schur-convex.

1. Introduction and terminology

Throughout the whole paper (unless explicity stated) $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ denote real linear Banach spaces and, $D \subset X$ will be a non-empty open and convex set. Let us fix some notation and terminology. Recall that a function $f : D \to \mathbb{R}$ is said to be convex on D if it satisfies the following inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y),$$

for every $x, y \in D$ and every $t \in [0,1]$.

Definition 1. A map $F : D \to Y$ is said to be affine, if it satisfies Jensen equation, i.e., for every $x, y \in D$

$$F\Big(\frac{x+y}{2}\Big) = \frac{F(x)+F(y)}{2}.$$

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Definition 2. For $x, y \in \mathbb{R}^n$

$$x \prec y \quad \text{if} \quad \begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, & k = 1, \ldots, n-1 \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, & \end{cases}$$

where, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x_{[1]} \geq \cdots \geq x_{[n]}$ denote the components of x in decreasing order. When $x \prec y$, x is said to be majorized by y.

This notation and terminology was introduced by Hardy, Littlewood, and Pólya [4]. Let us recall that an $n \times n$ matrix $P = [p_{ij}]$ is doubly stochastic if

$$p_{ij} \geq 0, \quad \text{for } i, j = 1, \ldots, n,$$

and

$$\sum_{i=1}^{n} p_{ij} = 1, \; j = 1, \ldots, n, \quad \sum_{j=1}^{n} p_{ij} = 1, \; i = 1, \ldots, n.$$

Particularly interesting examples of doubly stochastic matrices are provided by the permutation matrices. Recall that, matrix Π is said to be a permutation matrix if each row and column has a single unite entry, and all other entries are zero.

The well-known Hardy, Littlewood and Pólya theorem says that $x \prec y$, if and only if, $x = yP$ for some doubly stochastic matrix P. (In general, the matrix P is not unique.)

Motivated by this concept, we introduce the following natural generalization of the definition of majorization $\prec$ on vectors having not necessary real components.

Definition 3. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be n-tuples of vectors $x_i, y_i \in X$, $i = 1, \ldots, n$. We say that x is majorized by y, written $x \prec y$, if

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_n)P,$$

for some doubly stochastic $n \times n$ matrix P.

In 1923 [13] Schur has introduced the following class of functions, which in Schur's honor are said to be convex in the sense of Schur (or Schur-convex).

Definition 4. A real valued function Φ defined on a set D^n is said to be Schur-convex on D^n if

$$x \prec y \quad \text{on} \quad D^n \;\Rightarrow\; \Phi(x) \leq \Phi(y).$$

Similarly Φ is said to be Schur-concave on D^n if

$$x \prec y \quad \text{on} \quad D^n \ \Rightarrow \ \Phi(x) \geq \Phi(y).$$

Of course, Φ is Schur-concave if and only if $-\Phi$ is Schur-convex.

A survey of results concerning Schur-convex functions may be found in the positions [1], [4], [11], [12], [13]. In particular C.T. Ng in [8] has given a characterization of functions generating Schur-convex sums. In fact in [8] Ng proved the equivalence of the following four conditions:

Theorem 1 (Ng, [8])**.** *Let $D \subset \mathbb{R}^m$ be a non-empty open and convex set, and let $f : D \to \mathbb{R}$ be a function. The following conditions are pairwise equivalent:*

(i) $\sum_{i=1}^n f(x_i)$ *is Schur-convex on D^n for some $n \geq 2$,*

(ii) $\sum_{i=1}^n f(x_i)$ *is Schur-convex on D^n for every $n \geq 2$,*

(iii) *f is convex in the sense of Wright, i.e., it satisfies the following inequality*

$$f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y), \quad x, y \in D, \ t \in [0,1],$$

(iv) *f admits the representation*

$$f(x) = w(x) + a(x), \qquad x \in D,$$

where a is additive, i.e., $a(x+y) = a(x) + a(y)$, $x, y \in \mathbb{R}^m$, and w is convex on D.

Remark 1. The characterization of Wright-convex functions defined on an algebraically open and convex subset of arbitrary real linear spaces independently was given by Z. Kominek in [6] (see also [5], [7], [9]).

Delta-convex mappings between normed linear spaces provide a generalization of functions which are representable as a differences of two convex functions. An interesting study of the class of delta-convex mappings has been given by Veselý and L. Zajíček in [15]. The definition of delta-convexity reads as follows:

Definition 5. A map $F : D \to Y$ is called delta-convex, if there exists a continuous and convex functional $f : D \to \mathbb{R}$ such that $f + y^* \circ F$ is continuous and convex for any member y^* of the space Y^* dual to Y with $\|y^*\| = 1$. If this is the case, then we say that F is a delta-convex mapping with a control function f.

In [15] the authors have given many properties of delta-convexity, in particular they have proved that if a map F is a delta-convex with control function f, then the following inequality of Jensen-type holds

$$\Big\| \sum_{i=1}^{n} t_i F(x_i) - F\Big(\sum_{i=1}^{n} t_i x_i\Big)\Big\| \le \sum_{i=1}^{n} t_i f(x_i) - f\Big(\sum_{i=1}^{n} t_i x_i\Big), \tag{1}$$

for all $x_1, \dots, x_n \in D$, $t_1, \dots, t_n \in [0,1]$ such that $t_1 + \cdots + t_n = 1$.

Moreover, it turns out that a continuous function $F : D \to Y$ is a delta-convex mapping controlled by a continuous function $f : D \to \mathbb{R}$ if and only if the functional inequality

$$\Big\| F\Big(\frac{x+y}{2}\Big) - \frac{F(x)+F(y)}{2}\Big\| \le \frac{f(x)+f(y)}{2} - f\Big(\frac{x+y}{2}\Big), \tag{2}$$

is satisfied for all $x, y \in D$. (Corollary 1.18 in [15])

Remark 2. Note that inequality (1) may obviously be investigated without any regularity assumption upon F and f. In the present paper by delta-convex map we will mean a map $F : D \to Y$, for which there exists a function $f : D \to \mathbb{R}$ such that $f + y^* \circ F$ is convex (not necessary continuous), for any member y^* of the space Y^*, dual to Y with $\|y^*\| = 1$. This definition is equivalent to the fact that a pair (F, f) satisfies the inequality (1).

Below we give a joint generalization of Schur-convexity and delta-convexity.

Definition 6. A map $F : D^n \to Y$ is said to be delta Schur-convex with control function $f : D^n \to \mathbb{R}$, if

$$\|F(y) - F(x)\| \le f(y) - f(x), \tag{3}$$

whenever $x \prec y$ on D^n.

2. Results

We begin the study of (3) with the following

Observation 1. Every delta Schur-convex mapping $F : D^n \to Y$ is symmetric i.e.

$$F(Px) = F(x),$$

for every $n \times n$ permutation matrices P.

Proof. By assumption

$$\|F(Sx) - F(x)\| \leq f(x) - f(Sx),$$

holds for all $x \in D$ and every doubly stochastic matrix S. Because an arbitrary permutation matrix P and its inverse are doubly stochastic, then if F is a delta Schur-convex we have

$$\|F(Px) - F(x)\| \leq f(x) - f(Px),$$

and,

$$\|F(Px) - F(x)\| = \|F(P^{-1}Px) - F(Px)\| \leq f(Px) - f(x),$$

so $F(Px) = F(x)$. □

The following result establishes necessary and sufficient conditions for a given map to be delta Schur-convex.

Theorem 2. *The following conditions are pairwise equivalent:*

(i) *F is a delta Schur-convex mapping controlled by f,*

(ii) *for every $y^* \in Y^*$, $\|y^*\| = 1$, the function $y^* \circ F + f$ is Schur-convex,*

(iii) *for every $y^* \in Y^*$, $\|y^*\| = 1$, the function $y^* \circ F - f$ is Schur-concave.*

Proof. (i) implies (ii). Assume that

$$\|F(x) - F(y)\| \leq f(y) - f(x),$$

whenever $x \prec y$ on D. Let $y^* \in Y^*$, $\|y^*\| = 1$ be arbitrary. From the above inequality it follows that for $x \prec y$,

$$y^*(F(x) - F(y)) \leq f(y) - f(x),$$

or, equivalently,

$$x \prec y \Rightarrow y^*(F(x)) + f(x) \leq y^*(F(y)) + f(y).$$

(ii) implies (iii). Replace y^* by $-y^*$ in (ii).

(iii) implies (i). For every $y^* \in Y^*$, $\|y^*\| = 1$ and $x \prec y$ we have

$$y^*(F(y)) - f(y) \leq y^*(F(x)) - f(x),$$

and, consequently,

$$\|F(y) - F(x)\| = \sup\{y^*(F(y) - F(x)) : \|y^*\| = 1\} \leq f(y) - f(x),$$

which completes the proof. □

Let us observe, that delta Schur-convex mappings provide a generalization of functions which are representable as a differences of two Schur-convex functions.

Proposition 1. *In the case where* $(Y, \|\cdot\|) = (\mathbb{R}, |\cdot|)$ *a map* $F : D^n \to \mathbb{R}$ *is a delta Schur-convex, if and only if,* F *is a difference of two Schur-convex functions.*

Proof. Assume that $f : D^n \to \mathbb{R}$ is a control function for F. For all $x, y \in D^n$ such that $x \prec y$ we have

$$|F(y) - F(x)| \leq f(y) - f(x).$$

Put

$$\phi_1 := \frac{1}{2}(F + f) \quad \text{and} \quad \phi_2 := \frac{1}{2}(f - F).$$

It is easy to see that both ϕ_1 and ϕ_2 are Schur-convex functions, moreover, $F = \phi_1 - \phi_2$. Conversely, let $F = \phi_1 - \phi_2$, where ϕ_1, ϕ_2 are Schur-convex. Setting $f := \phi_1 + \phi_2$ we infer that both $f - F$ and $f + F$ are Schur-convex, whence, for every $x, y \in D^n$ we obtain

$$x \prec y \Rightarrow |F(y) - F(x)| \leq f(y) - f(x),$$

which finishes the proof. $\square$

The following result is a consequence of Jensen inequality for delta-convex mapping (1).

Theorem 3. *If* $F : D \to Y$ *is a delta-convex map with a control function* $f : D \to \mathbb{R}$ *then a map* $H : D^n \to Y$ *given by the formula*

$$H(x_1, \ldots, x_n) := \sum_{j=1}^{n} F(x_j),$$

is a delta Schur-convex with a control function $h(x_1, \ldots, x_n) := \sum_{j=1}^{n} f(x_j)$.

Proof. Assume that $x \prec y$. There exists a doubly stochastic matrix P such that $x = yP$. Since

$$x_j = \sum_{i=1}^{n} y_i p_{i,j}, \quad \text{where} \quad \sum_{i=1}^{n} p_{i,j} = 1,$$

it follows from the inequality (1) that

$$\left\| F(x_j) - \sum_{i=1}^{n} p_{i,j} F(y_i) \right\| \leq \sum_{i=1}^{n} p_{i,j} f(y_i) - f(x_j),$$

so because $\sum_{j=1}^n p_{i,j} = 1$ and using the triangle inequality several times we obtain

$$\begin{aligned}\Big\|\sum_{i=1}^n F(y_i) - \sum_{j=1}^n F(x_j)\Big\| &= \Big\|\sum_{i=1}^n\sum_{j=1}^n p_{i,j}F(y_i) - \sum_{j=1}^n F(x_j)\Big\| \\ &= \Big\|\sum_{j=1}^n\sum_{i=1}^n p_{i,j}F(y_i) - \sum_{j=1}^n F(x_j)\Big\| = \Big\|\sum_{j=1}^n\Big(\sum_{i=1}^n p_{i,j}F(y_i) - F(x_j)\Big)\Big\| \\ &\le \sum_{j=1}^n\Big\|\sum_{i=1}^n p_{i,j}F(y_i) - F(x_j)\Big\| \le \sum_{j=1}^n\Big(\sum_{i=1}^n p_{i,j}f(y_i) - f(x_j)\Big) \\ &= \sum_{i=1}^n\sum_{j=1}^n p_{i,j}f(y_i) - \sum_{j=1}^n f(x_j) = \sum_{i=1}^n f(y_i) - \sum_{j=1}^n f(x_j),\end{aligned}$$

which was to be proved. □

In the proof of our next result we will use the following theorem, which is a particular case of Theorem 4 proved in [10].

Theorem 4. *Let $F : D \to Y$ and $f : D \to \mathbb{R}$ satisfy the inequality* (2). *Then for an arbitrary point $y \in D$ there exist affine maps $A_y : D \to Y$ and $a_y : D \to \mathbb{R}$ such that*

$$A_y(y) = F(y), \quad a_y(y) = f(y),$$

and, for all $x \in D$

$$\|F(x) - A_y(x)\| \le f(x) - a_y(x).$$

Now, we are in position to prove the characterization of delta Schur-convex sums. The following theorem corresponds to the theorem of Ng [8]

Theorem 5. *Let $F : D \to Y$ and $f : D \to \mathbb{R}$ be given mappings. Then the following statements are pairwise equivalent:*

(i) *$\sum_{i=1}^n F(x_i)$ is delta Schur-convex on D^n with control function $\sum_{i=1}^n f(x_i)$, for some $n \ge 2$,*

(ii) *$\sum_{i=1}^n F(x_i)$ is delta Schur-convex on D^n with control function $\sum_{i=1}^n f(x_i)$, for every $n \ge 2$,*

(iii) *F is delta-convex in the sense of Wright i.e. it satisfies the following inequality*

$$\begin{aligned}&\|F(x) + F(y) - F(tx + (1-t)y) - F((1-t)x + ty)\| \\ &\qquad \le f(x) + f(y) - f(tx + (1-t)y) - f((1-t)x + ty),\end{aligned}$$

for all $x, y \in D$, $t \in [0, 1]$.

(iv) *F admits the representation*

$$F(x) = W(x) + A(x), \quad x \in D,$$

where $W : D \to Y$ is a delta-convex on D and $A : X \to Y$ is an additive.

Proof. Assume that, for some fixed $n \geq 2$, the sum $\sum_{j=1}^{n} F(x_j)$ is delta Schur-convex on D^n. Fix $x_3, x_4, \ldots, x_n \in D$ arbitrarily and consider two vectors $x := (x_1, x_2, x_3, \ldots, x_n)$ and $y := (y_1, y_2, x_3, \ldots, x_n)$. Of course $x \prec y$ if and only if $(x_1, x_2) \prec (y_1, y_2)$, so there exists a $t \in [0, 1]$ such that

$$(x_1, x_2) = (y_1, y_2) \begin{bmatrix} t & 1-t \\ 1-t & t \end{bmatrix} = (ty_1 + (1-t)y_2, (1-t)y_1 + ty_2).$$

Then for $(x_1, x_2) \prec (y_1, y_2)$ the inequality

$$\left\| \sum_{j=1}^{2} F(x_j) - \sum_{j=1}^{2} F(y_j) \right\| \leq \sum_{j=1}^{2} f(y_j) - \sum_{j=1}^{2} f(x_j)$$

implies (iii).

Suppose that F is a delta Wright-convex with a control function f. In particular if we put $\lambda = \frac{1}{2}$ the inequality (2) holds true. By Theorem 4 there exist affine maps $\overline{A} : D \to Y$ and $\overline{a} : D \to \mathbb{R}$ such that, for all $x \in D$,

$$\|F(x) - \overline{A}(x)\| \leq f(x) - \overline{a}(x).$$

Without loss of generality we may assume that $\overline{A}$ and $\overline{a}$ are additive maps. (Otherwise we will consider $\overline{A} - \overline{A}(0)$ and $\overline{a} - \overline{a}(0)$ instead of $\overline{A}$ and $\overline{a}$ respectively.) Put

$$G(x) := F(x) - \overline{A}(x), \quad \text{and} \quad g(x) := f(x) - \overline{a}(x), \quad x \in D.$$

Inequality

$$\|G(x)\| \leq g(x), \quad x \in D \tag{4}$$

implies that for every $y^* \in Y^*$, $\|y^*\| = 1$ we have

$$y^*(G(x)) \leq g(x), \quad x \in D.$$

To complete the proof of our implication it is enough to show that, for every $y^* \in Y^*$, $\|y^*\| = 1$, the function

$$D \ni x \longrightarrow y^*(G(x)) + g(x),$$

is convex. Obviously the defining function is convex in the sense of Jensen. Fix $x, y \in D$ arbitrary. Since D is open there exists a $\delta > 0$ such that $tx+(1-t)y \in D$, for $t \in (-\delta, 1+\delta)$. Let us define a function $h : (-\delta, 1+\delta) \to \mathbb{R}$ by the formula

$$h(t) := y^*(G(tx + (1-t)y)) + g(tx + (1-t)y).$$

Of course h is convex in the sense of Jensen, moreover, by (4)

$$\begin{aligned} h(t) &\le 2g(tx + (1-t)y) = 2[f(tx + (1-t)y) - \overline{a}(tx + (1-t)y)] \\ &= 2[f(tx + (1-t)y) + \overline{a}((1-t)x + ty) - \overline{a}(x) - \overline{a}(y)] \\ &\le 2[f(tx + (1-t)y) + f((1-t)x + ty) - \overline{a}(x) - \overline{a}(y)] \\ &\le 2[f(x) + f(y) - \overline{a}(x) - \overline{a}(y)]. \end{aligned}$$

Hence h is bounded from above then by a famous Bernstein–Doetsch [2] theorem continuous and convex. In particular

$$\begin{aligned} y^*(G(tx + (1-t)y)) + g(tx + (1-t)y) &= h(t) \\ &= h(t1 + (1-t)0) \le th(1) + (1-t)h(0) \\ &= t[y^*(G(x)) + g(x)] + (1-t)[y^*(G(y)) + g(y)]. \end{aligned}$$

This completes the proof of implication (iii) $\Rightarrow$ (iv).

Suppose F has the representation $F = W + A$, where W is a delta-convex map with control function w and A is additive. On account of Theorem 3 and by additivity of A for an arbitrary $n \ge 2$ we obtain (because $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$)

$$\begin{aligned} \left\| \sum_{j=1}^n F(y_j) - \sum_{j=1}^n F(x_j) \right\| &= \left\| \sum_{j=1}^n W(y_j) - \sum_{j=1}^n W(x_j) + \sum_{j=1}^n A(y_j) - \sum_{j=1}^n A(x_j) \right\| \\ &= \left\| \sum_{j=1}^n W(y_j) - \sum_{j=1}^n W(x_j) \right\| \le \sum_{j=1}^n w(y_j) - \sum_{j=1}^n w(x_j). \end{aligned}$$

This proves implication (iv) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (i) is trivial. $\square$

In the proof of our main result we use the following

Lemma 1. *Let a map $H : D^n \to Y$ be of the form*

$$H(x_1, \ldots, x_n) = \sum_{j=1}^n F_j(x_j), \tag{5}$$

where $F_j : D \to Y$, for $j = 1, \ldots, n$. Then H is symmetric, if and only if, there exist a map $F : D \to Y$ and a constants $C_1, \ldots, C_n \in Y$ such that

$$F_j(x_j) = F(x_j) + C_j, \quad x_j \in D, \quad j = 1, \ldots, n.$$

Proof. The proof of sufficiency is obvious. Assume that H is symmetric. It means that

$$H(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = H(x_1,\ldots,x_n),$$

for all $x_1,\ldots,x_n \in D$ and all $\sigma \in \Pi(n)$, where $\Pi(n)$ denote the set of all permutations of the integers $\{1,\ldots,n\}$. Fix $i,j \in \{1,\ldots,n\}$. Let $x_i = x$, $x_j = y$ and $x_k = z$ for all $k \in \{1,\ldots,n\} \setminus \{i,j\}$. Consider a permutation $\sigma \in \Pi(n)$ such that $\sigma(i) = j$, $\sigma(j) = i$, and $\sigma(k) = k$ for $k \in \{1,\ldots,n\} \setminus \{i,j\}$. By symmetry of H we have

$$F_i(x_i) + F_j(x_j) = F_i(x_j) + F_j(x_i),$$

or, equivalently,

$$F_i(x) - F_j(x) = F_i(y) - F_j(y),$$

for all $x, y \in D$. Put

$$C_{ij} := F_i(x) - F_j(x).$$

Let $F := F_1$, $C_j := C_{j1}$, for $j = 1,\ldots,n$. We obtain a representation

$$F_j(x) = F(x) + C_j, \quad j = 1,\ldots,n.$$

□

Our main result reads as follows

Theorem 6. *Assume that we are given maps $F_j : D \to Y$ and $f_j : D \to \mathbb{R}$, for $j = 1,\ldots,n$. Then $\sum_{j=1}^n F_j(x_j)$ is a delta Schur-convex with a control function $\sum_{j=1}^n f_j(x_j)$, if and only if, there exist constants $C_1,\ldots,C_n \in Y$, additive mapping $A : X \to Y$ and a delta-convex map $W : D \to Y$ such that*

$$F_j(x) = A(x) + W(x) + C_j, \quad j = 1,\ldots,n. \tag{6}$$

Proof. Suppose that a map $\sum_{j=1}^n F_j(x_j)$ is a delta Schur-convex. On account of Observation 1 it is symmetric, consequently, by Lemma 1 there exist a map $F : D \to Y$ and constants $C_1,\ldots,C_n \in Y$ such that

$$F_j(x) = F(x) + C_j, \quad j = 1,\ldots,n.$$

It is not hard to check that a sum $\sum_{j=1}^n F(x_j)$ is a delta Schur-convex, then by Theorem 5 a map F has the form

$$F(x) = A(x) + W(x), \quad x \in D,$$

where $A : X \to Y$ is an additive and $W : D \to Y$ a delta-convex.

Conversely, each map of the form (5) admitting a representation (6) is a delta Schur-convex. □

Remark 3. Observe, that substituting $F := 0$ in our theorems we obtain the results concerning classical Schur-convexity.

References

[1] B. C. Arnold, A. W. Marshall and I. Olkin, Inequalities: Theory of Majorization and Its Applications, (Second edition), *Springer Series in Statistics*, *New York – Dordrecht – Heidelberg – London*, 2011.

[2] F. Bernstein and G. Doetsch, Zur Theorie der konvexen Functionen, *Math. Ann.* **76** (1915), 514–526.

[3] R. Ger, Stability aspects of delta-convexity, In: "Stability of Hyers-Ulam type", (Th. M. Rassias and J. Tabor, ed.), *Hardonic Press, Palm Harbor*, 1994, 99–109.

[4] G. Hardy, J. E. Littlewood and G. Pólya, Inequalities, 1st ed.,2nd ed., *Cambridge Univ. Press*, *Cambridge, London and New York*, 1934, 1952.

[5] Z. Kominek, Convex functions in linear spaces, *Prace Naukowe Uniwersytetu Śląskiego w Katowicach* **1087** (1–68).

[6] Z. Kominek, On additive and convex functionals, *Radovi Math.* **3** (1987), 267–279.

[7] Gy. Maksa and Zs. Páles, Decomposition of higher-order Wright-convex functions, *J. Math. Anal. Appl.* **359** (2009), 439–443.

[8] C. T. Ng, General Inequalities, Vol. 80, Oberwolfach Internat. Ser. Numer. Math., (W. Walter, ed.), *Birkhäuser, Boston*, 1986, 433–438.

[9] K. Nikodem, On some class of midconvex functions, *Ann. Polon. Math.* **50** (1989), 145–151.

[10] A. Olbryś, A support theorem for delta (s,t)-convex mappings, *Aequat. Math.* (2014), DOI 10.1007/s00010-014-0290-6.

[11] A. Ostrowski, Sur quelques application des functions convexes et concave au sens de I. Schur, *J. Math. Pures Appl.* **31** (1952), 253–292.

[12] A. W. Roberts and D. E. Varberg, Convex Functions, *Academic Press*, *New York and London*, 1973.

[13] I. Schur, Über eine Klasse von Mittelbildungen mit Anwendungen auf die Determinanten theorie, *Sitzungsber. Berlin. Math. Ges.* **22** (1923), 9–20.

[14] E. M. Wright, An inequality for convex functions, *Amer. Math. Monthly* **61** (1954), 620–622.

[15] L. Veselý and L. Zajíček, Delta-convex mappings between Banach spaces and applications, *Dissertationes Math. Polish Scientific Publishers Warszawa* **289** (1989).

ANDRZEJ OLBRYŚ
INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
BANKOWA 14
40-007 KATOWICE
POLAND

E-mail: andrzej.olbrys@wp.pl

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