Publ. Math. Debrecen
86/3-4 (2015), 325–346
DOI: 10.5486/PMD.2015.6078

Geometry of *H*-paracontact metric manifolds

By GIOVANNI CALVARUSO (Lecce) and DOMENICO PERRONE (Lecce)

Abstract. We introduce and study H-paracontact metric manifolds, that is, paracontact metric manifolds whose Reeb vector field ξ is harmonic. We prove that they are characterized by the condition that ξ is a Ricci eigenvector. We then investigate how harmonicity of the Reeb vector field ξ of a paracontact metric manifold is related to some other relevant geometric properties, like infinitesimal harmonic transformations and paracontact Ricci solitons.

1. Introduction

In parallel with contact and complex structures in the Riemannian case, paracontact metric structures were introduced in [19] in semi-Riemannian settings, as a natural odd-dimensional counterpart to para-Hermitian structures. Up to recently, the study of paracontact metric manifolds mainly concerned the special case of para-Sasakian manifolds.

A systematic study of paracontact metric manifolds started with the paper [30], were the Levi–Civita connection, the curvature and a canonical connection (analogous to the Tanaka–Webster connection of the contact metric case) of a paracontact metric manifold have been described. The technical apparatus introduced in [30] is essential for further investigations of paracontact metric geometry. Since then, paracontact metric manifolds have been studied under several different points of view. The case when the Reeb vector field satisfies a nullity condition was studied in [14]. Conformal paracontact curvature, and its applications, were

Mathematics Subject Classification: 53D10, 53C50, 53C43, 53C25.
Key words and phrases: paracontact metric structures, Reeb vector field, Ricci tensor, harmonic vector fields, infinitesimal harmonic transformations, Ricci solitons.
Supported by funds of the Universitá del Salento and of the M.I.U.R. (PRIN 2010-11).

investigated in [20]. In [7] the first author studied three-dimensional homogeneous paracontact metric manifolds, while a large class of paracontact metric structures on the unit tangent sphere bundle was described and studied in [9].

Because of the recent studies of harmonicity conditions in semi-Riemannian geometry, it is a natural problem to investigate when the Reeb vector field of a paracontact metric manifold is a harmonic vector field. Given a (smooth, oriented, connected) semi-Riemannian manifold (M, g) and a unit vector field V on M, the *energy* of V is the energy of the corresponding smooth map $V : (M, g) \to (T_1M, g^s)$, where (T_1M, g^s) is the unit tangent bundle of (M, g), equipped with the Sasaki metric. V is said to be a *harmonic vector field* if $V : (M, g) \to (T_1M, g^s)$ is a critical point for the energy functional restricted to maps defined by unit vector fields. We may refer to the recent monograph [17] and references therein for an overview on harmonic vector fields.

The second author [24] proved that the Reeb vector field ξ of a contact Riemannian manifold is harmonic if and only if ξ is a Ricci eigenvector. This led to define *H-contact Riemannian manifolds* as contact metric manifolds, whose Reeb vector field is harmonic. Since then, H-contact Riemannian manifolds have been intensively studied and their relations to other contact geometry properties are now well understood (see, for example, [3, Section 10.3.1], [15], [17, Chapter 4], [25] and references therein)

In this paper we introduce the corresponding notion of *H-paracontact (metric) manifolds*, that is, paracontact metric manifolds whose Reeb vector field is harmonic. We prove that a paracontact metric manifold is H-paracontact if and only if the Reeb vector field is a Ricci eigenvector. This result is not a direct adaptation of its contact Riemannian analogue, because of the deep differences arising between Riemannian and semi-Riemannian settings. In fact, the results proved in [24] use in an essential way the fact that in the Riemannian case, a self-adjoint operator admits an orthonormal basis of eigenvectors, while this property does not hold in semi-Riemannian settings. We then investigate the relationship between H-paracontact manifolds and some relevant geometric properties, like the Reeb vector field being an infinitesimal harmonic transformation or the paracontact metric structure being a paracontact Ricci soliton. Under these points of view, the Riemannian case presents some strong rigidity results (see [26] and references therein), which do not directly extend to general semi-Riemannian settings. This makes interesting to study contact semi-Riemannian structures, whose Reeb vector field is an infinitesimal harmonic transformation or determines a Ricci soliton.

The paper is organized in the following way. In Section 2 we report some basic information about paracontact metric manifolds and harmonicity properties of vector fields. The characterization of H-paracontact metric manifolds in terms of the Ricci operator is proved in Section 3, where we also prove that the notion of H-paracontact manifold is invariant under D-homothetic deformations, as described in (3.9). In Section 4 we prove that several classes of paracontact metric manifolds (para-Sasakian and K-paracontact manifolds, paracontact (κ, μ)-spaces, three-dimensional homogeneous paracontact metric manifolds) are H-paracontact, so showing that the class of H-paracontact metric manifolds is rather large. The relationship between H-paracontact metric manifolds and paracontact metric manifolds, whose Reeb vector field is 1-harmonic (equivalently, an infinitesimal harmonic transformation) or whose vector field determines a Ricci soliton, are then investigated in Section 5. Differently from the contact Riemannian case, the class of paracontact metric structures, whose Reeb vector field is an infinitesimal harmonic transformation, is strictly larger than the one of K-paracontact structures.

2. Preliminaries

2.1. Paracontact metric manifolds. The aim of this Subsection is to report some basic facts about paracontact metric manifolds. All manifolds are assumed to be connected and smooth. We may refer to [19], [30] and references therein for more information about paracontact metric geometry.

A $(2n+1)$-dimensional manifold M is said to be a *contact manifold* if it admits a global 1-form η, such that $\eta \wedge (d\eta)^n \neq 0$. Given such a form η, there exists a unique vector field ξ, called the *characteristic vector field* or the *Reeb vector field*, such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. A semi-Riemannian metric g is said to be an *associated metric* if there exists a tensor φ of type $(1,1)$ such that

$$\eta = g(\xi, \cdot), \quad d\eta(\cdot, \cdot) = g(\cdot, \varphi \cdot), \quad \varphi^2 = I - \eta \otimes \xi.$$

Then, (φ, ξ, η, g) (more briefly, (η, g)) is called a *paracontact metric structure*, and $(M, \varphi, \xi, \eta, g)$ a *paracontact metric manifold*.

As shown in [30], any almost paracontact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ admits a *φ-basis*, that is, a local orthonormal basis of the form $\{\xi, e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_n, \}$, where $\xi, e_1, \ldots, e_n$ are space-like vector fields and $\varphi e_1, \ldots, \varphi e_n$ are time-like vector fields.

We now report some results on the Levi–Civita connection and curvature of a paracontact metric manifold [30], which shall be used in the next Section. Let

∇ and R respectively denote the Levi–Civita connection and the corresponding Riemann curvature tensor, taken with the sign convention

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all smooth vector fields X, Y. Moreover, we shall denote by ϱ the Ricci tensor of type $(0,2)$, by Q the corresponding endomorphism field and by r the scalar curvature. The tensor $h = \frac{1}{2}\mathcal{L}_\xi\varphi$, where $\mathcal{L}$ denotes the Lie derivative, is symmetric and satisfies [30]:

$$\nabla\xi = -\varphi + \varphi h, \quad \nabla_\xi\varphi = 0, \quad h\varphi = -\varphi h, \quad h\xi = 0, \quad \operatorname{tr} h = \operatorname{tr} h\varphi = 0 \tag{2.1}$$

and

$$(\nabla_{\varphi X}\varphi)\varphi Y - (\nabla_X\varphi)Y = 2g(X,Y)\xi - \eta(Y)\big(X - hX + \eta(X)\xi\big). \tag{2.2}$$

In the above equation (2.1) and throughout the paper, if not explicitly stated otherwise, by $\operatorname{tr} = \operatorname{tr}_g$ we mean the trace with respect to the metric tensor g. The Ricci curvature of any paracontact metric manifold (M^{2n+1}, η, g) satisfies

$$\varrho(\xi,\xi) = -2n + \operatorname{tr} h^2. \tag{2.3}$$

A paracontact metric manifold (M, η, g) is said to be

- *η-Einstein* if its Ricci operator Q is of the form

$$Q = aI + b\eta \otimes \xi, \tag{2.4}$$

  where a, b are smooth functions.

- a *(κ,μ)-space* if its curvature tensor satisfies

$$R(X,Y)\xi = \kappa(\eta(X)Y - \eta(Y)X) + \mu(\eta(X)hY - \eta(Y)hX), \tag{2.5}$$

  for all tangent vector fields X, Y, where κ, μ are smooth functions on M.

- *K-paracontact* if ξ is a Killing vector field, or equivalently, $h = 0$.
- *para-Sasakian* if the paracontact structure (ξ, η, φ, g) is *normal*, that is, satisfies $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$. This condition is equivalent to

$$(\nabla_X\varphi)Y = -g(X,Y)\xi + \eta(Y)X.$$

Any para-Sasakian manifold is K-paracontact, and the converse also holds when $n = 1$, that is, for three-dimensional spaces. An alternative definition of para-Sasakian manifolds, in terms of *cones* over paraKähler manifolds, was given in [1]. We also recall that any para-Sasakian manifold satisfies

$$R(X, Y)\xi = -(\eta(X)Y - \eta(Y)X), \tag{2.6}$$

so that it is a (κ, μ)-space with $\kappa = -1$. Note that, differently from the contact metric case, condition (2.6) is necessary but not sufficient for a paracontact metric manifold to be para-Sasakian. This fact was already pointed out in other papers (see for example [14]). However, the present authors could not find explicit examples in literature. Example 4.7 describes explicitly a three-dimensional paracontact metric manifold which satisfies (2.6) but is not para-Sasakian.

2.2. Harmonic vector fields. We now provide some basic information on harmonic vector fields over a semi-Riemannian manifold. For more details, we refer to [18], [17, Chapter 8] and [6].

Let (M, g) be an m-dimensional semi-Riemannian manifold, ∇ its Levi–Civita connection and V a smooth vector field on M. The *energy* of V is, by definition, the energy of the corresponding smooth map $V : (M, g) \to (TM, g^s)$, where g^s is the *Sasaki metric* (also referred to as the *Kaluza–Klein metric* in Mathematical Physics) on the tangent bundle TM of M. If M is compact, then

$$(V) = \frac{1}{2}\int_M (\operatorname{tr} V^* g^s) dv = \frac{m}{2}\operatorname{vol}(M, g) + \frac{1}{2}\int_M g(\nabla V, \nabla V) dv,$$

while in the non-compact case, one works over relatively compact domains. Note that the energy of a vector field V, up to a constant, also corresponds to the *total bending* of V [29]. By the Euler-Lagrange equation, a vector field V defines a harmonic map from (M, g) to (TM, g^s) if and only if its *tension field* $\tau(V) = \operatorname{tr}(\nabla \mathrm{d}V)$ vanishes, that is, when

$$\operatorname{tr}[R(\nabla.V, V)\cdot] = 0 \quad \text{and} \quad \bar{\Delta}V = 0.$$

Here, $\bar{\Delta}V := -\operatorname{tr}\nabla^2 V$ is the socalled *rough Laplacian* of V. With respect to any local pseudo-orthonormal frame field $\{E_1, \ldots, E_m\}$ on (M, g), with $\varepsilon_i = g(E_i, E_i) = \pm 1$ for all indices $i = 1, \ldots, m$, we have

$$\bar{\Delta}V = \sum_i \varepsilon_i \left(\nabla_{\nabla_{E_i}E_i} V - \nabla_{E_i}\nabla_{E_i} V\right).$$

If g is Riemannian and M is compact, then the parallel vector fields are the only vector fields defining harmonic maps.

Next, for any real constant $r \neq 0$, let $\mathfrak{X}^r(M) = \{V \in \mathfrak{X}(M) : g(V,V) = r\}$ denote the set of tangent vector fields of constant length r. A vector field $V \in \mathfrak{X}^r(M)$ is said to be *a harmonic vector field* if it is a critical point for the energy functional $E|_{\mathfrak{X}^r(M)}$, restricted to vector fields of the same length. The Euler-Lagrange equation of this variational condition yields that V is a harmonic vector field if and only if

$$\bar{\Delta} V \quad \text{is collinear to} \quad V. \tag{2.7}$$

This characterization was first obtained, in the Riemannian case, by G. Wiegmink [29] and C. M. Wood [28] (see also [17], Chapter 2). In semi-Riemannian settings, the same argument applies for vector fields of constant length, if not light-like [6].

Let T_1M denote the *unit tangent sphere bundle* over M, and g^s the metric induced on T_1M by the Sasaki metric of TM. Then, the map $V : (M,g) \to (T_1M, g^s)$ is harmonic if V is a harmonic vector field and the additional condition

$$\mathrm{tr}[R(\nabla . V, V)\cdot] = 0 \tag{2.8}$$

holds. In analogy with the contact metric case [24], we now introduce the following definition.

Definition 2.1. A paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be *H-paracontact* if its Reeb vector field ξ is a harmonic vector field.

We will show in Theorem 3.5 that this notion is also invariant under D-homothetic deformations of the paracontact metric structure as described in (3.9).

3. Harmonicity of the Reeb vector field of a paracontact manifold

In this Section we shall prove the following characterization of H-paracontact metric manifolds.

Theorem 3.1. *A paracontact metric manifold is H-paracontact if and only if the Reeb vector field ξ is an eigenvector of the Ricci operator.*

The above Theorem 3.1 will be obtained as a consequence of the following result.

Theorem 3.2. *Let (M,η,ξ,g,φ) be a $(2n+1)$-dimensional paracontact metric manifold. Then,*

$$\bar{\Delta}\xi = -4n\xi - Q\xi = \|\nabla\xi\|^2\xi - \mathrm{pr}_{|\ker\eta}Q\xi, \tag{3.1}$$

where $\|\nabla\xi\|^2 = -(2n + \operatorname{tr} h^2)$ *and* $\mathrm{pr}_{|\ker\eta}$ *denotes the projection on* $\ker\eta$.

Proof. Let M be a (2n+1)-dimensional paracontact metric manifold and consider a local pseudo-orthonormal φ-basis $\{E_1,\dots,E_{2n+1}\}=\{e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n,\xi\}$, with $g(e_i,e_i) = -g(\varphi e_i,\varphi e_i) = 1$. We shall use the notation $g(E_i,E_i) = \varepsilon_i = \pm 1$, for $i = 1,\dots,2n+1$. We obtain

$$\begin{aligned}\bar{\Delta}\xi &= -\sum_{i=1}^{2n+1}\varepsilon_i\left(\nabla_{E_i}\nabla_{E_i}\xi - \nabla_{\nabla_{E_i}E_i}\xi\right)\\ &= -\sum_{i=1}^{2n+1}\varepsilon_i\left(\nabla_{E_i}(-\varphi+\varphi h)\right)E_i = \operatorname{tr}\nabla\varphi - \operatorname{div}(\varphi h).\end{aligned}$$

Using formula (2.2), we have

$$\operatorname{tr}\nabla\varphi = (\nabla_{e_i}\varphi)e_i - (\nabla_{\varphi e_i}\varphi)\varphi e_i + (\nabla_\xi\varphi)\xi = -2n\xi \tag{3.2}$$

and so,

$$\bar{\Delta}\xi = -2n\xi - \operatorname{div}(\varphi h). \tag{3.3}$$

By equation (2.1), $\nabla\xi = -\varphi + \varphi h$. Differentiating, we get

$$R(X,Y)\xi = (\nabla_X\varphi)Y - (\nabla_Y\varphi)X - (\nabla_X\varphi h)Y + (\nabla_Y\varphi h)X, \tag{3.4}$$

from which we deduce that the Ricci curvature $\varrho(X,\xi)$ is given by

$$\begin{aligned}\varrho(X,\xi) = \operatorname{tr}R(X,\cdot)\xi &= \sum_{i=1}^{2n}\varepsilon_i g((\nabla_X\varphi)E_i,E_i) - \sum_{i=1}^{2n}\varepsilon_i g((\nabla_{E_i}\varphi)X,E_i)\\ &\quad - \sum_{i=1}^{2n}\varepsilon_i g((\nabla_X\varphi h)E_i,E_i) + \sum_{i=1}^{2n}\varepsilon_i g((\nabla_{E_i}\varphi h)X,E_i).\end{aligned} \tag{3.5}$$

By direct calculation, we find

$$\sum_{i=1}^{2n}\varepsilon_i g((\nabla_X\varphi)E_i,E_i) = 0, \quad \sum_{i=1}^{2n}\varepsilon_i g((\nabla_X\varphi h)E_i,E_i) = \nabla_X\operatorname{tr}(\varphi h) = 0,$$

$$\sum_{i=1}^{2n} \varepsilon_i g((\nabla_{E_i} \varphi\, h)X, E_i) = g(\mathrm{div}(\varphi\, h), X) \tag{3.6}$$

and, taking into account $\mathrm{tr}\, \nabla\varphi = -2n\xi$,

$$\sum_{i=1}^{2n} \varepsilon_i g((\nabla_{E_i} \varphi)X, E_i) = 2n\, \eta(X). \tag{3.7}$$

We then replace into (3.5) and we obtain

$$\varrho(X, \xi) = -2n\eta(X) + g(\mathrm{div}(\varphi\, h), X). \tag{3.8}$$

On the other hand, $\varrho(\xi, \xi) = -2n + \mathrm{tr}\, h^2 = -4n - \|\nabla\xi\|^2$. So, equations (3.3) and (3.8) yield

$$\bar{\Delta}\xi = -4n\xi - Q\xi = -(2n + \mathrm{tr}\, h^2)\xi - (Q\xi)_{|\ker \eta} = \|\nabla\xi\|^2 \xi - (Q\xi)_{|\ker \eta}$$

and this ends the proof. □

Remark 3.3. The contact metric analogues of the above Theorems 3.1 and 3.2 were proved in [24]. The proof of these contact Riemannian results used in an essential way the fact that the tensor h, being self-adjoint, admits an orthonormal basis of eigenvectors. The lack of such information in the paracontact metric case required a completely different approach to the proof of the above Theorem 3.2.

As an immediate consequence of Theorem 3.1 and the definition of η-Einstein manifolds, we have the following result, which parallels its contact Riemannian anlogue proved in [24].

Corollary 3.4. *η-Einstein paracontact metric manifolds are H-paracontact.*

We now prove that any D-homothetic deformation of a H-paracontact metric structure is again H-paracontact. This fact shows that the harmonicity of the Reeb vector field is rather natural for paracontact metric manifolds, and permits to build new examples of H-paracontact metric structures from the known ones.

Given a paracontact metric structure (η, g, ξ, φ), its *D-homothetic deformation*, determined by any real constant $t \neq 0$, is the new paracontact metric structure $(\eta_t, g_t, \xi_t, \varphi_t)$, defined by

$$\eta_t = t\eta, \qquad \xi_t = t^{-1}\xi, \quad \varphi = \varphi, \quad g_t = tg + t(t-1)\eta \otimes \eta \tag{3.9}$$

(see [30],[14]). In [14], the relationships between the Levi–Civita connections ∇ and ∇_t and curvature tensors R and R_t of g and g_t respectively were investigated.

In particular, by Proposition 3.6 in [14], rewritten for our sign convention of the curvature, we have

$$\begin{aligned} tR_t(X,Y)\xi_t &= R(X,Y)\xi + (t-1)^2(\eta(Y)X - \eta(X)Y) \\ &\quad + (t-1)\big((\nabla_X\varphi)Y - (\nabla_Y\varphi)X + \eta(Y)(X-hX) - \eta(X)(Y-hY)\big). \end{aligned}$$

When both X, Y belong to $\ker\eta$, the above equation reduces to

$$tR_t(X,Y)\xi_t = R(X,Y)\xi + (t-1)\big((\nabla_X\varphi)Y - (\nabla_Y\varphi)X\big). \tag{3.10}$$

Let now $\{E_1,\dots,E_{2n+1}\} = \{e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n,\xi\}$ be a local φ-basis for (η,g,ξ,φ). Note that $\ker\eta_t = \ker\eta$ and $g_t = tg$ on $\ker\eta$. Therefore, $\big\{\frac{1}{\sqrt{t}}e_1,\dots,\frac{1}{\sqrt{t}}e_n,\frac{1}{\sqrt{t}}\varphi e_1,\dots,\frac{1}{\sqrt{t}}\varphi e_n,\xi_t\big\}$ is a local basis of vector fields, pseudo-orthonormal with respect to g_t. We can now calculate the Ricci tensor $\varrho_t(X,\xi_t)$, for any vector field $X \in\ker\eta$, by contraction of (3.10). Taking into account equations (3) and (3.7), we get

$$\begin{aligned} \varrho_t(X,\xi_t) &= \frac{1}{t}\sum_{i=1}^{2n}\varepsilon_i g(R(X,E_i)\xi + \frac{t-1}{t}\sum_{i=1}^{2n} g((\nabla_X\varphi)E_i,E_i) \\ &= -\frac{t-1}{t}\sum_{i=1}^{2n} g((\nabla_{E_i}\varphi)X,E_i)\frac{1}{t}\varrho(X,\xi) - \frac{t-1}{t}\cdot 2n\eta(X) = \frac{1}{t}\varrho(X,\xi). \end{aligned}$$

Thus, like in the contact Riemannian case [24], the property "ξ is an eigenvector of the Ricci operator" is invariant under D-homothetic deformations of a paracontact metric structure. So, Theorem 3.1 yields the following result.

Theorem 3.5. *The class of H-paracontact manifolds is invariant under D-homothetic deformations.*

4. Examples

We shall now investigate the relationships among the class of H-paracontact spaces and some relevant classes of paracontact metric manifolds.

4.1. K-paracontact and para-Sasakian manifolds. We start considering the K-paracontact case, for which we shall prove the following.

Theorem 4.1. *The Ricci operator of a K-paracontact metric manifold satisfies*

$$Q\xi = -2n\xi.$$

Hence, K-paracontact (in particular, para-Sasakian) manifolds are H-paracontact. Moreover, the Reeb vector field ξ of any K-paracontact metric manifold (M,η,g) defines a harmonic map $\xi : (M,g) \to (T_1M, g^s)$.

Proof. It follows from equation (3.4) that the curvature tensor of a K-paracontact metric manifold satisfies

$$R(Y,Z,\xi,X) = g((\nabla_Y\varphi)Z,X) - g((\nabla_Z\varphi)Y,X). \tag{4.1}$$

Then, using the first Bianchi identity and (4.1), we get

$$\begin{aligned} g(R(Y,Z)\xi,X) &= -g(R(Z,\xi)Y,X) - g(R(\xi,Y)Z,X) \\ &= -g((\nabla_X\varphi)Y,Z) + g((\nabla_Y\varphi)X,Z) - g((\nabla_Z\varphi)X,Y) \\ &\quad + g((\nabla_X\varphi)Z,Y) = 2g((\nabla_X\varphi)Z,Y) - g(R(Y,Z)\xi,X), \end{aligned}$$

that is,

$$R(\xi,X,Y,Z) = g((\nabla_X\varphi)Z,Y). \tag{4.2}$$

Let now $\{e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n,\xi\}$ be a local pseudo-orthonormal φ-basis, with $g(e_i,e_i) = -g(\varphi e_i,\varphi e_i) = 1$. Using the formulae (4.2) and (3.2), we obtain

$$\begin{aligned} \varrho(\xi,Y) &= \sum_{i=1}^n g\left(R(\xi,e_i,Y,e_i\right) - \sum_{i=1}^n g\left(R(\xi,\varphi e_i,Y,\varphi e_i\right) \\ &= \sum_{i=1}^n g((\nabla_{e_i}\varphi)e_i,Y) - g((\nabla_{\varphi e_i}\varphi)\varphi e_i,Y) = g(\operatorname{tr}\nabla\varphi,Y) = -2n\eta(Y), \end{aligned}$$

that is, $Q\xi = -2n\xi$. Hence, (M,η,g) is H-contact. In order to conclude that $\xi : (M,g) \to (T_1M,g^s)$ is a harmonic map, it then suffices to prove that $\operatorname{tr}[R(\nabla\xi,\xi)\cdot] = 0$.

Since M is K-contact, the first equation in (2.1) reduces to $\nabla\xi = -\varphi$. With respect to the above local pseudo-orthonormal φ-basis $\{e_i,\varphi e_i,\xi\}$, using $\nabla\xi = -\varphi$ and the first Bianchi identity, we obtain

$$\operatorname{tr}[R(\nabla\xi,\xi)\cdot] = -\operatorname{tr}[R(\varphi\cdot,\xi)\cdot] = -\sum_{i=1}^n R(e_i,\varphi e_i)\xi.$$

On the other hand, by equations (3.4) and (2.2) with $h=0$, we get

$$R(e_i,\varphi e_i)\xi = \left(\nabla_{e_i}\varphi\right)\varphi e_i - \left(\nabla_{\varphi e_i}\varphi\right)e_i = \left(\nabla_{\varphi e_i}\varphi\right)\varphi^2 e_i - \left(\nabla_{\varphi e_i}\varphi\right)e_i = 0.$$

So, we conclude that $\operatorname{tr}[R(\nabla\xi,\xi)\cdot] = 0$ and this ends the proof. □

Consider $\mathbb{R}^{2n+2}$, equipped with the standard paracomplex structure $\mathbb{I}$ and flat metric g of neutral signature. Then, any non-degenerate hypersurface in $(\mathbb{R}^{2n+2}, \mathbb{I}, g)$ inherits an integrable paracontact hermitian structure [20]. In particular, a standard example of para-Sasakian manifold is given by the hyperboloid

$$HS^{2n+1} := \{(x_0, y_0, \ldots, x_n, y_n) \mid x_0^2 + \cdots + x_n^2 - y_0^2 - \ldots y_n^2 = 1\},$$

with the natural para-CR structure induced by its embedding in $(\mathbb{R}^{2n+2}, \mathbb{I}, g)$. In this case,

$$\eta = \sum_{j=0}^{n} (y_j dx_j - x_j dy_j), \quad \xi = \sum_{j=0}^{n} \left(x_j \frac{\partial}{\partial y_j} + y_j \frac{\partial}{\partial x_j} \right),$$
$$\varphi = \mathbb{I}|_{HS^{2n+1}}, \quad g|_{HS^{2n+1} \times HS^{2n+1}}$$

is a para-Sasakian structure. By the above Theorem 4.1, *the Reeb vector field ξ of the canonical para-Sasakian structure of HS^{2n+1} defines a harmonic map into its unit tangent sphere bundle.*

Remark 4.2. K-contact Riemannian manifolds are *characterized* by the Ricci curvature condition $Q\xi = 2n\xi$. In fact, for such manifolds, equation $Q\xi = 2n\xi$ follows at once from condition $h = 0$. Conversely, being h self-adjoint, it admits an orthonormal basis of Ricci eigenvectors, $he_i = \lambda_i e_i$ for all indices i. Therefore, if $Q\xi = 2n\xi$, then $0 = \mathrm{tr} h^2 = \sum_i \lambda_i^2$ and so, $h = 0$, that is, the manifold is K-contact.

The above argument does not extend to the paracontact metric case. As we proved in the above Theorem 4.1, K-paracontact manifolds satisfy the corresponding condition $Q\xi = -2n\xi$, but the converse does not hold. In fact, if $Q\xi = -2n\xi$, then by (2.3) we find $\mathrm{tr} h^2 = 0$. However, since the metric tensor is not positive definite for a paracontact metric manifold, the tensor h, though self-adjoint, needs not to be diagonalizable [22]. Consequently, $\mathrm{tr} h^2 = 0$ does not imply that a paracontact metric manifold is K-paracontact. Explicit examples of paracontact metric manifolds with $\mathrm{tr} h^2 = 0$ (indeed, with $h^2 = 0$) but $h \neq 0$ will be given in the next subsection 4.3

4.2. (κ, μ)**-paracontact metric manifolds.** We now consider paracontact metric manifolds, whose Reeb vector field satisfies the nullity condition (2.5). By contraction of (2.5), it is easily seen that the Ricci operator of a (κ, μ)-paracontact metric manifold satisfies $Q\xi = 2n\kappa\xi$ (see also [14], p. 670). Therefore, Theorem 3.1 implies at once that (κ, μ)-paracontact metric manifolds are H-paracontact.

Next, the following formula holds for (κ,μ)-paracontact metric manifolds with $\kappa \neq -1$ (see [14], pp. 682 and 690, rewritten here for our sign convention on the curvature tensor):

$$\begin{aligned} R(X,Y)hZ - hR(X,Y)Z &= \big(\kappa(\eta(Y)g(hX,Z) - \eta(X)g(hY,Z)) \\ &+ \mu(\kappa+1)(\eta(Y)g(X,Z) - \eta(X)g(Y,Z))\big)\xi + \kappa\big(g(Y,\varphi Z)\varphi hX - g(X,\varphi Z)\varphi hY \\ &+ g(Z,\varphi hY)\varphi X - g(Z,\varphi hX)\varphi Y\big) + \eta(Z)\big(\eta(Y)hX - \eta(X)hY\big) \\ &+ \mu\big((\kappa+1)\eta(Z)(\eta(Y)X - \eta(X)Y)\big) + 2\mu g(X,\varphi Y)\varphi hZ, \end{aligned} \tag{4.3}$$

for all tangent vector fields X, Y, Z. Let now (M,φ,ξ,η,g) denote any (κ,μ)-paracontact metric manifold with $\kappa \neq -1$, and consider a pseudo-orthonormal φ-basis $\{\xi, e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n\}$. Using the first equation in (2.1) and the first Bianchi identity, we find

$$-\operatorname{tr}[R(\nabla_\cdot\xi,\xi)\cdot] = -\sum_{i=1}^n R(-\varphi e_i + \varphi h e_i,\xi)e_i + \sum_{i=1}^n R(-e_i - he_i,\xi)\varphi e_i$$
$$= \sum_{i=1}^n R(\xi,e_i)\varphi h e_i + \sum_{i=1}^n R(\xi,\varphi e_i)he_i = -\sum_{i=1}^n R(\xi,e_i)h\varphi e_i + \sum_{i=1}^n R(\xi,\varphi e_i)h\varphi^2 e_i.$$

On the other hand, by (4.3) we have

$$R(\xi,E_j)h\varphi E_j = hR(\xi,E_j)\varphi E_j + \kappa g(hE_j,\varphi E_j)\xi,$$

for any $E_j \in \{e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n\}$. Therefore, we conclude that

$$\begin{aligned} &-\operatorname{tr}[R(\nabla_\cdot\xi,\xi)\cdot] \\ &\quad= \sum_{i=1}^n \big(-hR(\xi,e_i)\varphi e_i - \kappa g(he_i,\varphi e_i)\xi + hR(\xi,\varphi e_i)e_i + \kappa g(h\varphi e_i,\varphi^2 e_i)\xi\big) \\ &\quad= \sum_{i=1}^n \big(-hR(\xi,e_i)\varphi e_i - \kappa g(he_i,\varphi e_i)\xi + hR(\xi,\varphi e_i)e_i + \kappa g(he_i,\varphi e_i)\xi\big) \\ &\quad= -\sum_{i=1}^n h\big(R(\xi,e_i)\varphi e_i - R(\xi,\varphi e_i)e_i\big) = -\sum_{i=1}^n h\big(R(e_i,\varphi e_i)\xi\big) = 0, \end{aligned}$$

by the (κ,μ)-nullity condition. Therefore, we proved the following result.

Theorem 4.3. *(κ,μ)-paracontact metric manifolds are H-paracontact. Moreover, whenever $\kappa \neq -1$, the Reeb vector field of a paracontact (κ,μ)-space also defines a harmonic map into its unit tangent sphere bundle.*

(κ,μ)-contact Riemannian manifolds are H-contact [24]. On the other hand, paracontact (κ,μ)-spaces show some peculiar features, which do not have a contact Riemannian counterpart. In fact, Sasakian manifolds can be characterized as (κ,μ)-contact metric manifolds with $\kappa=1$. Instead, para-Sasakian manifolds are (κ,μ)-paracontact metric manifolds with $\kappa=-1$, but not conversely. It is interesting to investigate non-para-Sasakian paracontact (κ,μ)-spaces with $\kappa=-1$ [14], also in order to decide whether their Reeb vector field defines a harmonic map into the unit tangent sphere bundle.

4.3. Three-dimensional homogeneous paracontact metric manifolds. In [7], the first author obtained the complete classification of three-dimensional homogeneous paracontact metric manifolds. The classification result is the following.

Theorem 4.4 ([7])**.** *A simply connected complete homogeneous paracontact metric three-manifold is isometric to a Lie group G with a left-invariant paracontact metric structure (φ,ξ,η,g). More precisely, one of the following cases occurs:*

• *If G is unimodular, then the Lie algebra of G is one of the following:*

(1) $\mathfrak{g}_2: [\xi,e]=-\gamma e+\beta\varphi e$, $[\xi,\varphi e]=\beta e+\gamma\varphi e$, $[e,\varphi e]=2\xi$, *with* $\gamma\neq 0$.

In this case, G is either the identity component of $O(1,2)$, or $\widetilde{SL}(2,\mathbb{R})$.

(2) $\mathfrak{g}_3: [\xi,e]=-\gamma\varphi e$, $[\xi,\varphi e]=-\beta e$, $[e,\varphi e]=2\xi$.

In this case, G is

(2a) *the identity component of $O(1,2)$ or $\widetilde{SL}(2,\mathbb{R})$ if either $\beta,\gamma>0$ or $\beta,\gamma<0$;*

(2b) *$\widetilde{E}(2)$ if either $\beta>0=\gamma$ or $\beta=0>\gamma$;*

(2c) *$E(1,1)$ if either $\beta<0=\gamma$ or $\beta=0<\gamma$;*

(2d) *either $SO(3)$ or $SU(2)$ if $\beta>0$ and $\gamma<0$;*

(2e) *the Heisenberg group H_3 if $\beta=\gamma=0$.*

(3) $\mathfrak{g}_4: [\xi,e]=-e+(2\varepsilon-\beta)\varphi e$, $[\xi,\varphi e]=-\beta e+\varphi e$, $[e,\varphi e]=2\xi$, *with* $\varepsilon=\pm 1$.

In this case, G is

(3a) *the identity component of $O(1,2)$ or $\widetilde{SL}(2,\mathbb{R})$ if $\beta\neq\varepsilon$;*

(3b) *$\widetilde{E}(2)$ if $\beta=\varepsilon=1$;*

(3c) *$E(1,1)$ if $\beta=\varepsilon=-1$.*

• *If G is non-unimodular, then the Lie algebra of G is one of the following:*

(4) $\mathfrak{g}_5,\mathfrak{g}_6: [\xi,e]=[\xi,\varphi e]=0$, $[e,\varphi e]=2\xi+\delta e$, *with* $\delta\neq 0$.

(5) $\mathfrak{g}_7: [\xi,e]=-[\xi,\varphi e]=-\beta(e+\varphi e)$, $[e,\varphi e]=2\xi+\delta(e+\varphi e)$, *with* $\delta\neq 0$.

Notations $\mathfrak{g}_2 - \mathfrak{g}_7$ for Lie algebras listed in Theorem 4.4 refer to the classification of all three-dimensional Lorentzian Lie groups, obtained in [4].

In the symmetric case, such a paracontact homogeneous three-manifold is either flat or of constant sectional curvature -1. These cases are included in the classification given in Theorem 4.4 above. In fact, in case (2a) with $\beta = \gamma = 2$, unimodular Lie groups $O(1,2)$ or $\widetilde{SL}(2,\mathbb{R})$ have constant sectional curvature -1, while in case (2b) with $\beta = 2$, the unimodular Lie group $\widetilde{E}(2)$ is flat.

Tensor $h = (1/2)\mathcal{L}_\xi\varphi$ of all examples listed in Theorem 4.4 can be easily deduced from the above Lie brackets. Moreover, the curvature and the Ricci tensor of any left-invariant Lorentzian structure over a three-dimensional Lie group was completely described in [5]. In particular, describing the Ricci operator with respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3\} = \{\xi, e, \varphi e\}$, we get:

For case (1):

$$\begin{cases} he = \gamma\varphi e, \\ h\varphi e = -\gamma e, \end{cases} \qquad Q = \begin{pmatrix} -2-2\gamma^2 & 0 & 0 \\ 0 & 2-2\beta & \gamma(2-2\beta) \\ 0 & -\gamma(2-2\beta) & 2-2\beta \end{pmatrix}. \tag{4.4}$$

For case (2):

$$\begin{cases} he = -\dfrac{1}{2}(\beta-\gamma)e, \\ h\varphi e = \dfrac{1}{2}(\beta-\gamma)\varphi e, \end{cases}$$

$$Q = \begin{pmatrix} -2+\frac{1}{2}(\beta-\gamma)^2 & 0 & 0 \\ 0 & \frac{1}{2}((2-\gamma)^2-\beta^2) & 0 \\ 0 & 0 & \frac{1}{2}((2-\beta)^2-\gamma^2) \end{pmatrix}. \tag{4.5}$$

For case (3):

$$\begin{cases} he = \varepsilon e + \varphi e, \\ h\varphi e = -e - \varepsilon\varphi e, \end{cases} \qquad Q = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4+2\eta(2-\beta)-2\beta & 2(1+\eta-\beta) \\ 0 & -2(1+\eta-\beta) & -2\beta+2\eta\beta \end{pmatrix} \tag{4.6}$$

For case (4):

$$h = 0, \quad Q = \begin{pmatrix} -2 & 0 & 0 \\ 0 & \delta^2+2 & 0 \\ 0 & 0 & \delta^2+2 \end{pmatrix} \tag{4.7}$$

For case (5):

$$\begin{cases} he = \beta(e+\varphi e), \\ h\varphi e = -\beta(e+\varphi e), \end{cases} \qquad Q = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2-2\beta & 2\beta \\ 0 & -2\beta & 2+2\beta \end{pmatrix} \tag{4.8}$$

Thus, in all the above cases, $\xi = e_1$ is a Ricci eigenvector. Hence, by Theorem 3.1, ξ is harmonic. Indeed, we can prove the following stronger result.

Theorem 4.5. *The Reeb vector field of any three-dimensional homogeneous paracontact metric manifold defines a harmonic map into the unit tangent sphere bundle. In particular, all three-dimensional homogeneous paracontact metric manifolds are H-paracontact.*

Proof. We already concluded by (4.4)–(4.8) that $\xi = e_1$ is a Ricci eigenvector. So, all the above examples are H-paracontact, and it suffices to check the additional condition $\mathrm{tr}[R(\nabla.\xi,\xi)\cdot] = 0$. Note that in case (4), $h = 0$ and the conclusion follows from Theorem 4.1.

Consider a three-dimensional paracontact metric manifold (M,φ,ξ,η,g) and a local φ-basis $\{\xi, e, \varphi e\}$. Taking into account $h\varphi = -\varphi h$ and the first equation in (2.1), one has

$$\begin{aligned} \mathrm{tr}[R(\nabla.\xi,\xi)\cdot] &= R(\nabla_e\xi,\xi)e - R(\nabla_{\varphi e}\xi,\xi)\varphi e \\ &= R(\xi,\varphi e)e - R(\xi,\varphi he)e - R(\xi,e)\varphi e - R(\xi,he)\varphi e. \end{aligned} \tag{4.9}$$

The curvature tensor of three-dimensional left-invariant paracontact metric structures listed in Theorem 4.4 can be deduced either by direct calculation, or by comparison with the more general formulae obtained in [5] for the curvature of three-dimensional Lorentzian Lie groups.

For case (1), we find

$$R(\xi,e)e = -(2+\gamma^2)\xi, \qquad R(\xi,e)\varphi e = 2\gamma(1+\beta)\xi,$$
$$R(\xi,\varphi e)\varphi e = (2+\gamma^2)\xi, \qquad R(\xi,\varphi e)e = 2\gamma(1+\beta)\xi.$$

Then, taking into account the description of h given in (4.4), from (4.9) we get

$$\begin{aligned} \mathrm{tr}[R(\nabla.\xi,\xi)\cdot] &= R(\xi,\varphi e)e - \gamma R(\xi,e)e - R(\xi,e)\varphi e - \gamma R(\xi,\varphi e)\varphi e \\ &= 2\gamma(1+\beta)\xi + \gamma(2+\gamma^2)\xi - 2\gamma(1+\beta)\xi - \gamma(2+\gamma^2)\xi = 0. \end{aligned}$$

The calculations for the remaining cases are similar to the above one. It suffices to apply (4.9), using the description of tensor h given in equations (4.5), (4.6) and (4.8), and the following curvature equations:

For case (2): $R(\xi,e)e = \frac{1}{4}(4\beta - \beta^2 - 4 + 3\gamma^2 - 4\gamma - 2\beta\gamma)\xi, \quad R(\xi,e)\varphi e = 0,$

$$R(\xi,\varphi e)\varphi e = \frac{1}{4}(4 - 4\gamma + \gamma^2 - 3\beta^2 + 4\beta + 2\beta\gamma)\xi, \quad R(\xi,\varphi e)e = 0.$$

For case (3): $R(\xi,e)e = (1+2\varepsilon-2\varepsilon\beta)\xi, \quad R(\xi,e)\varphi e = 2(1+\varepsilon-\beta)\xi,$

$$R(\xi,\varphi e)\varphi e = (3+2\varepsilon-2\varepsilon\beta)\xi, \quad R(\xi,\varphi e)e = 2(1+\varepsilon-\beta)\xi.$$

For case (5): $R(\xi,e)e = -(1+2\beta)\xi, \quad R(\xi,e)\varphi e = 2\beta\xi,$

$$R(\xi,\varphi e)\varphi e = (1-2\beta)\xi, \quad R(\xi,\varphi e)e = 2\beta\xi.$$

In all the above cases, a straightforward calculation yields $\mathrm{tr}[R(\nabla_{\cdot}\xi,\xi)\cdot] = 0$. So, ξ defines a harmonic map into the unit tangent sphere bundle. $\square$

Remark 4.6. The above Theorem 4.5 formally agrees on the similar result valid for three-dimensional homogeneous contact Riemannian manifolds [23]. However, the paracontact metric case provides many more examples of harmonic maps than its contact Riemannian counterpart, since the classification of homogeneous paracontact metric three-manifolds is much richer.

We end this section by pointing out the following example, which obviously does not have any contact Riemannian analogue.

Example 4.7. **A nonSasakian paracontact metric manifold satisfying (2.6).**
Consider the three-dimensional left-invariant paracontact metric structure listed in case (3a) of Theorem 4.4 in the special case when $\beta = \varepsilon + 1$, that is,

$$[\xi,e] = -e + (\varepsilon - 1)\varphi e, \quad [\xi,\varphi e] = -(\varepsilon+1)e + \varphi e, \quad [e,\varphi e] = 2\xi, \text{ with } \varepsilon = \pm 1.$$

We already proved in equation (4.6) that $he = \varepsilon e + \varphi e$ and $h\varphi e = -e - \varepsilon\varphi e$. Therefore, $h \neq 0$ and so, this paracontact metric structure is not para-Sasakian. On the other hand, calculating the curvature tensor (or equivalently, using the formulas proved in [5] for the curvature), we easily get

$$R(e,\varphi e)\xi = 0,$$

$$R(\xi,e)\xi = -(2\varepsilon\beta - 1 - 2\varepsilon)e - 2(1+\varepsilon-\beta)\varphi e = -e = -\eta(\xi)e,$$

$$R(\xi,\varphi e)\xi = 2(1+\varepsilon-\beta)e - (3+2\varepsilon-2\varepsilon\beta)\varphi e = -\varphi e = -\eta(\xi)\varphi e,$$

from which it follows at once that equation (2.6) holds, since R and η are tensors. Thus, this paracontact metric manifold is an explicit example of a paracontact non-para-Sasakian metric manifold, satisfying (2.6).

5. Paracontact infinitesimal harmonic transformations and Ricci solitons

Let (M^n, g) be a semi-Riemannian manifold and $f : x \mapsto x'$ a point transformation in (M, g). If $\nabla(x)$ denotes the Levi–Civita connection at x and $\nabla'(x)$ is obtained bringing back $\nabla(x')$ to x by f^{-1} [27], the *Lie difference* at x is defined as $\nabla'(x) - \nabla(x)$. The map f is said to be *harmonic* if $\operatorname{tr}(\nabla'(x) - \nabla(x)) = 0$.

Consider now a vector field V on M and the local one-parameter group of infinitesimal point transformations f_t generated by V. The Lie derivative $L_V\nabla$ then corresponds to $\nabla'(x) - \nabla(x)$, where $\nabla'(x) = f_t^*(\nabla(x'))$, and V generates a group of harmonic transformations if and only if

$$\operatorname{tr}(L_V\nabla) = 0.$$

In this case, V is said to be an *infinitesimal harmonic transformation* [21].

Infinitesimal harmonic transformations also occur as critical points for a suitable energy functional. In fact, if g^c denotes the *complete lift metric* of g to TM, which is of neutral signature (n, n), a vector field V on M defines a harmonic section $V : (M, g) \to (TM, g^c)$ if and only if V is an infinitesimal harmonic transformation [21]. For this reason, infinitesimal harmonic transformations are also called 1-*harmonic vector fields*, because this harmonicity property is equivalent to the vanishing of the linear part of the tension field of the local one-parameter group of infinitesimal point transformations [16]. A vector field V is an infinitesimal harmonic transformation if and only if $\bar{\Delta}V = QV$ (see for example [10]), [12]).

We now consider a paracontact metric manifold $(M, \varphi, \xi, \eta, g)$. By Theorem 3.2 and equation $\varrho(\xi, \xi) = -2n + \operatorname{tr} h^2$, we get

$$\bar{\Delta}\xi = Q\xi \iff Q\xi = -2n\xi \iff \operatorname{tr} h^2 = 0 \text{ and } Q\xi \text{ is collinear to } \xi.$$

Thus, we have the following result.

Theorem 5.1. *Let $(M, \eta, \xi, g, \varphi)$ be a paracontact metric manifold. Then, the following assertions are equivalent:*

1) $Q\xi = -2n\xi$;

2) *ξ is an infinitesimal harmonic transformation (equivalently,* 1*-harmonic);*
3) *M is H-paracontact and* $\operatorname{tr} h^2 = 0$.

Note that the Reeb vector field of a contact *Riemannian* manifold is an infinitesimal harmonic transformation if and only if it is a Killing vector field [26].

Remark 5.2. In general, a harmonic vector field needs not to be 1-harmonic, nor conversely. This fact may be easily seen, for example, comparing the classifications of harmonic and 1-harmonic left-invariant vector fields over three-dimensional Lorentzian Lie algebras, given respectively in [6] and [12].

However, the above Theorem 5.1 yields that if the Reeb vector field of a paracontact metric manifold is 1-harmonic, then it is harmonic, while the converse does not hold, because of the additional condition $\operatorname{tr} h^2 = 0$.

We already proved in Corollary 3.4 that any paracontact (κ, μ)-space is H-paracontact. On the other hand, for a paracontact (κ, μ)-space one has $h^2 = (k+1)\varphi^2$ (see for example [14]), from which it easily follows that $\operatorname{tr} h^2 = 0$ if and only if $k = -1$. Hence, by the above Theorem 5.1, we have the following

Corollary 5.3. *The Reeb vector field of a paracontact (κ, μ)-space is an infinitesimal harmonic transformation if and only if $\kappa = -1$. Whenever $\kappa \neq -1$, the Reeb vector field of a paracontact (κ, μ)-space is harmonic but not* 1*-harmonic.*

Next, using the description of tensor h given in equations (4.4)–(4.8), we can easily deduce $\operatorname{tr} h^2$ for all three-dimensional left-invariant paracontact metric structures classified in Theorem 4.4. Taking into account Theorems 5.1 and 4.5, we then get the following result.

Corollary 5.4. *The Reeb vector field of a three-dimensional homogeneous paracontact metric manifold is an infinitesimal harmonic transformation if and only if the manifold is isometric to one of the following cases, as classified in Theorem 4.4:*

- *case* (2) *with $\beta = \gamma$;*
- *case* (3)*;*
- *case* (4)*;*
- *case* (5).

The above Corollary 5.4 is compatible with the results about left-invariant Killing and 1-harmonic vector fields on three-dimensional Lorentzian Lie groups obtained in [12] (see, in particular, Lemma 1, Theorem 6, Lemma 11 and Theorem 20 in [12]).

Remark 5.5. In the contact Riemannian case, the Reeb vector field is an infinitesimal harmonic transformation if and only if the contact Riemannian structure is K-contact [26].

Again by the description of tensor h given in the previous Section, it is easily seen that $h = 0$ (and so, the three-dimensional left-invariant paracontact metric structure is para-Sasakian, see Theorem 2.2 in [7]) if and only if we are either in case (2) with $\beta = \gamma$, in case (3), or in case (5) with $\beta = 0$. This corrects Theorem 4.3 in [7], as case (a) is not para-Sasakian.
Comparing this classification with the above Corollary 5.4, we see that in the following cases

- case (3);
- case (5) with $\beta \neq 0$,

ξ is an infinitesimal harmonic deformation, although the paracontact metric structure is not K-paracontact. Thus, *the class of paracontact metric structures, whose Reeb vector field is an infinitesimal harmonic transformation, is strictly larger than the one of K-paracontact structures.*

We can also exhibit a five-dimensional example of a paracontact, not K-paracontact metric manifold, whose characteristic vector field ξ is an infinitesimal harmonic transformation. Consider the simply connected Lie group, whose Lie algebra $\mathfrak{g} = \mathrm{Span}\{\xi, X_1, X_2, Y_1, Y_2\}$ is described by

$$[X_1, X_2] = 2X_2, \qquad [X_1, Y_1] = 2\xi, \qquad [X_2, Y_1] = -2Y_2,$$
$$[X_2, Y_2] = 2(Y_1 + \xi), \qquad [\xi, X_1] = -2Y_1, \qquad [\xi, X_2] = -2Y_2,$$

equipped with the left-invariant paracontact metric structure determined by the following conditions:

$$\varphi\xi = 0, \quad \varphi X_i = X_i, \quad \varphi Y_i = -Y_i, \quad \eta(X_i) = \eta(Y_i) = 0, \quad \eta(\xi) = 1$$

and

$$g(X_i, X_j) = g(Y_i, Y_j) = 0, \quad g(X_i, Y_j) = \delta_{ij},$$

for all $i, j = 1, 2$ (see [13, Example 4.8]). As proved in [13], this paracontact metric manifold is a paracontact (κ, μ)-space, with $\kappa = -1$ and $\mu = 2$. Hence, by Theorem 4.3, it is H-paracontact. Moreover, $h^2 = 0$, although $hX_1 = -Y_1 \neq 0$. Therefore, this paracontact metric manifold is not K-paracontact, but by Theorem 5.1 its Reeb vector field is an infinitesimal harmonic transformation.

We also emphasize the fact that both in the above three-dimensional examples and in this five-dimensional example, with ξ being an infinitesimal harmonic transformation, the tensor h is two-step nilpotent.

The recent paper [27] showed that the vector field V determining a Riemannian Ricci soliton is necessarily an infinitesimal harmonic transformation. The same argument also applies to the semi-Riemannian case. A *Ricci soliton* is a semi-Riemannian manifold (M, g), admitting a vector field V and a real constant λ, such that

$$\varrho + \frac{1}{2} L_V g = \lambda g. \tag{5.1}$$

A Ricci soliton is said to be *shrinking*, *steady* or *expanding*, according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. An Einstein manifold, together with a Killing vector field, is a trivial solution of equation (5.1). Ricci solitons have been intensively studied in recent years, particularly because of their relationship with the *Ricci flow*. Examples and more details on Ricci solitons in semi-Riemannian settings may be found in [2], [8] and references therein.

In analogy to the contact metric case, by a *paracontact (metric) Ricci soliton* we shall mean a paracontact metric manifold $(M, \varphi, \xi, \eta, g)$, such that equation (5.1) holds for $V = \xi$. In this case, we necessarily have

$$Q\xi = \lambda\xi.$$

In fact, if $V = \xi$, then equation (5.1) yields

$$\begin{aligned} 0 &= \varrho(\xi, X) + \frac{1}{2}\big(L_\xi g\big)(\xi, X) - \lambda g(\xi, X) \\ &= g(Q\xi, X) + \frac{1}{2} g(\nabla_\xi \xi, X) + \frac{1}{2} g(\nabla_X \xi, \xi) - \lambda g(\xi, X) \\ &= g(Q\xi, X) - \lambda g(\xi, X), \end{aligned}$$

for any vector field X, taking into account the fact that ξ is unit and geodesic (as it easily follows from (2.1)). On the other hand, if $(M, \varphi, \xi, \eta, g)$ is a paracontact Ricci soliton, then in particular ξ is an infinitesimal harmonic transformation. Hence, Theorem 5.1 yields that M is H-paracontact and $Q\xi = -2n\xi$. Thus, $\lambda = -2n$ and we have the following result.

Theorem 5.6. *A paracontact Ricci soliton is H-paracontact, and is necessarily expanding.*

In the contact Riemannian case, ξ is an infinitesimal harmonic transformation only when it is Killing. As a consequence, a contact Riemannian Ricci soliton is necessarily trivial, that is, an Einstein K-contact metric manifold [26]. The above Theorem 5.6 specifies that pseudo-Riemannian paracontact Ricci solitons must be found among H-paracontact manifolds. On the one hand, this does not exclude the existence of nontrivial paracontact Ricci solitons, on the other hand, we could not find examples of nontrivial paracontact Ricci solitons. This leads to state the following

Open Question: Do there exist nontrivial paracontact Ricci solitons?

References

[1] D. V. Alekseevski, V. Cortes, A. S. Galaev and T. Leistner, Cones over pseudo-Riemannian manifolds and their holonomy, *J. Reine Angew. Math.* **635** (2009), 23–69.

[2] M. Brozos-Vazquez, G. Calvaruso, E. Garcia-Rio and S. Gavino-Fernandez, Three--dimensional Lorentzian homogeneous Ricci solitons, *Israel J. Math.* **188** (2012), 385–403.

[3] D. E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Progress in Math. 203, 2nd ed., *Birkhäuser Boston*, 2010.

[4] G. Calvaruso, Homogeneous structures on three-dimensional Lorentzian manifolds, *J. Geom. Phys.* **57** (2007), 1279–1291, Addendum: J. Geom. Phys. **58** (2008), 291–292.

[5] G. Calvaruso, Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds, *Geom. Dedicata* **127** (2007), 99–119.

[6] G. Calvaruso, Harmonicity properties of invariant vector fields on three-dimensional Lorentzian Lie groups, *J. Geom. Phys.* **61** (2011), 498–515.

[7] G. Calvaruso, Homogeneous paracontact metric three-manifolds, *Illinois J. Math.* **55** (2011), 697–718.

[8] G. Calvaruso and A. Fino, Ricci solitons and geometry of four-dimensional non-reductive homogeneous spaces, *Canad. J. Math.* **64** (2012), 778–804.

[9] G. Calvaruso and V. Martin-Molina, Paracontact metric structures on the unit tangent sphere bundle, *Ann. Mat. Pura Appl.*, DOI:10.1007/s10231-014-0424-4 (*to appear*).

[10] G. Calvaruso and D. Perrone, H-Contact semi-Riemannian manifolds, *J. Geom. Phys.* **71** (2013), 11-21.

[11] G. Calvaruso and D. Perrone, Metrics of Kaluza–Klein type on the anti-de Sitter space $\mathbb{H}^3_1$, *Math. Nachr.* **287** (2014), 885–902.

[12] E. Calvino-Louzao, J. Seoane-Bascoy, M. E. Vazquez-Abal and R. Vazquez-Lorenzo, One-harmonic invariant vector fields on three-dimensional Lie groups, *J. Geom. Phys.* **62** (2012), 1532–1547.

[13] B. Cappelletti Montano and L. Di Terlizzi, Geometric structures associated to a contact metric (κ, μ)-space, *Pacific J. Math.* **246** (2010), 257–292.

[14] B. Cappelletti Montano, I. Kupeli Erken and C. Murathan, Nullity conditions in paracontact geometry, *Diff. Geom. Appl.* **30** (2012), 665–693.

[15] S. H. Chun, J. H. Park and K. Sekigawa, H-contact unit tangent sphere bundles of Einstein manifolds, *Quart. J. Math.* **62** (2011), 59–69.

[16] C. T. J. Dodson, M. Trinidad Perez and M. E. Vazquez-Abal, Harmonic-Killing vector fields, *Bull. Belg. Math. Soc. Simon Stevin* **9** (2002), 481–490.

[17] S. Dragomir and D. Perrone, Harmonic Vector Fields: Variational Principles and Differential Geometry, *Elsevier, Science Ltd.*, 2011.

[18] O. Gil-Medrano and A. Hurtado, Spacelike energy of timelike unit vector fields on a Lorentzian manifold, *J. Geom. Phys.* **51** (2004), 82–100.

[19] S. Kaneyuki and F. L. Williams, Almost paracontact and parahodge structures on manifolds, *Nagoya Math. J.* **99** (1985), 173–187.

[20] S. Ivanov, D. Vassilev and S. Zamkovoy, Conformal paracontact curvature and the local flatness theorem, *Geom. Dedicata* **144** (2010), 79–100.

[21] O. Nouhaud, Transformations infinitesimales harmoniques, *C. R. Acad. Sc. Paris, Sér. A* **274** (1972), 573–576.

[22] B. O'Neill, Semi-Riemannian Geometry, *Academic Press, New York*, 1983.

[23] D. Perrone, Harmonic characteristic vector fields on contact metric three-manifolds, *Bull. Austral. Math Soc.* **67** (2003), 305–315.

[24] D. Perrone, Contact metric manifolds whose characteristic vector field is a harmonic vector field, *Diff. Geom. Appl.* **20** (2004), 367–378.

[25] D. Perrone, Stability of the Reeb vector field of H-contact manifolds, *Math. Z.* **263** (2009), 125–147.

[26] D. Perrone, Geodesic Ricci solitons on unit tangent sphere bundles, *Ann. Glob. Anal. Geom.* **44** (2013), 91–103.

[27] S. E. Stepanov and V. N. Shelepova, A remark on Ricci solitons, *Mat. Zametki* **86** (2009), 474–477, Translation in Math. Notes **86** (2009), 447–450 (in *Russian*).

[28] C. M. Wood, On the energy of a unit vector field, *Geom. Dedicata* **64** (1997), 319-330.

[29] G. Wiegmink, Total bending of vector fields on Riemannian manifolds, *Math. Ann.* **303** (1995), 325–344.

[30] S. Zamkovoy, Canonical connections on paracontact manifolds, *Ann. Glob. Anal. Geom.* **36** (2009), 37–60.

GIOVANNI CALVARUSO
DIPARTIMENTO DI MATEMATICA
E FISICA "E. DE GIORGI"
UNIVERSITY OF SALENTO
PROV. LECCE-ARNESANO
73100 LECCE (LE)
ITALY

E-mail: giovanni.calvaruso@unisalento.it

DOMENICO PERRONE
DIPARTIMENTO DI MATEMATICA
E FISICA "E. DE GIORGI"
UNIVERSITY OF SALENTO
PROV. LECCE-ARNESANO
73100 LECCE (LE)
ITALY

E-mail: domenico.perrone@unisalento.it

(Received January 31, 2014; revised November 5, 2014)