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On generalized metric spaces and their associated Finsler spaces II. g-Landsberg spaces of scalar curvature K

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Dedicated to Professor Lajos Tamássy on his 70th birthday

The present paper is the continuation of "On generalized metric spaces and their associated Finsler spaces I. Fundamental relations" (Publ. Math., Debrecen, Vol. 45 (1994), 187–203). So we follow the notation and terminology of the previous paper.

In §5 we introduce the notion of g-C-reducible condition and obtain

[**A**] In a g-C-reducible M_n space there exists a scalar τ such that
$C_{hj} - \dfrac{F}{n+1} T_{hj} = \tau h_{hj}$ (Theorem 5.9).

[**B**] In a g-C-reducible and g-Landsberg space M_n, for $n > 3$ the Douglas tensor ${D_h}^i{}_{jk}$ vanishes (Theorem 5.13).

[**C**] In a g-C-reducible and g-Landsberg space M_n, for $n > 3$ if the scalar G vanishes, then the space is a g-Berwald space (Theorem 5.17).

In §6 we consider a space M_n of scalar curvature K and obtain

[**D**] A g-Landsberg space of scalar curvature K is g-C-reducible (Theorem 6.8).

[**E**] A g-Landsberg space M_n of scalar curvature K is for $n > 3$ projectively flat (Theorem 6.9).

[**F**] In a g-Landsberg space of scalar curvature K, the curvature tensor ${S_h}^i{}_{jk}$ has the form $S_{hijk} = F^{-2} S(h_{hj}h_{ik} - h_{hk}h_{ij})$, where S is a constant (Theorem 6.11).

The purpose of §7 is to prove

[**G**] The Finsler space associated with a g-Landsberg space of scalar curvature K is semi-C-reducible (Theorem 7.1).

[**H**] If the associated Finsler space of a g-Landsberg space of scalar curvature K is $S3$-like, then the space is an $RccM_n$ space (Theorem 7.7).

§5. A g-Landsberg space, g-C-reducible condition and a g-Berwald space

In this section we refer to some special M_n spaces.

5.1. A g-Landsberg space.

Definition. If the tensor $P^i{}_{jk}$ vanishes, then the space M_n is called a *g-Landsberg* space.

Theorem 5.1. ([3], Theorems 4.6, 4.7 and 4.8). *A space M_n is a g-Landsberg space, if and only if any one of the following conditions holds*:

$$(a)\quad P^i{}_{jk}=0,\quad (b)\quad D_j{}^i{}_k=0,\quad (c)\quad P_h{}^i{}_{jk}=0,\quad (d)\quad C_j{}^i{}_{k/0}=0.$$

Remark. In a g-Landsberg space, the connection $B\Gamma(G)$ is h-metrical, that is, $g_{ij/\!/k}=g_{ij/k}=0$ from (1.12)(a). However $g_{ij/\!/k}=0$ does not mean that the space is a g-Landsberg space (cf. (1.12)(b)).

From the Bianchi identity ([3],(2.18)(b))

$$P_j{}^i{}_{kl}+C_j{}^i{}_{l/k}-C_j{}^i{}_rP^r{}_{kl}-j|k=0,$$

we have the following

Lemma 5.2. *In a g-Landsberg space we have the following relations*:

$$(5.1)\qquad \begin{aligned} &(a)\quad C_{jil/k}-j|k=0, && (b)\quad C_{jl/k}-j|k=0,\\ &(c)\quad C_{j/k}-j|k=0, && (d)\quad C_{jk/0}=0,\quad C_{j/0}=0.\end{aligned}$$

Remark. The condition (a) is equivalent to the condition $P^i{}_{jk}=0$.

Using (1.16) and (1.17) we get

Lemma 5.3. *In a g-Landsberg space we have*

$$(5.2)\qquad \begin{aligned} &(a)\quad E_h{}^i{}_{jk}=0,\quad H_h{}^i{}_{jk}=K_h{}^i{}_{jk},\quad H^i{}_{jk}=R^i{}_{jk},\\ &(b)\quad H_{hijk}+H_{ihjk}+H^r{}_{jk}g_{hi(r)}=0,\\ &(c)\quad F_h{}^i{}_{jk}=G_h{}^i{}_{jk}=C_h{}^i{}_{j/k},\quad G_{jk}:=G_i{}^i{}_{jk}=C_{j/k},\\ &(d)\quad G_{hijk}+G_{ihjk}=g_{hi(k)/j},\quad G_h{}^0{}_{jk}=C_{hj/k}.\end{aligned}$$

From the Bianchi identity ([3],(2.20)(c))

$$-S_h{}^i{}_{kl/j}=P_h{}^i{}_{jk/(l)}+P_h{}^i{}_{rk}C_j{}^r{}_l+S_h{}^i{}_{lr}P^r{}_{jk}-k|l,$$

we obtain the following

Lemma 5.4. *In a g-Landsberg space we have* $S_h{}^i{}_{jk/l} = 0$.

Definition. A generalized metric space M_n whose associated Finsler space $F_n^*(g)$ is a Landsberg space ($P^{*i}{}_{jk} = 0$) is called an LM_n space (abbreviation).

From (2.12)(b) and Proposition 2.3 we infer

Theorem 5.5. *A g-Landsberg space is an* LM_n *space if and only if the condition* $C_{ij/k} = 0$ *holds.*

5.2. g-C-reducible condition ([33]).

Definition. A generalized metric space M_n is called *g-C-reducible* if the following condition holds:

$$C_{hik} = F^{-1} l_i C_{hk} + \frac{1}{n+1}(C_h h_{ik} + C_i h_{hk} + C_k h_{hi}). \tag{5.3}$$

Proposition 5.6. *The g-C-reducible condition is characterized by*

$$g_{hi(k)} = F^{-1}(l_i C_{hk} + l_h C_{ik}) + \frac{2}{n+1}(C_h h_{ik} + C_i h_{hk} + C_k h_{hi}). \tag{5.4}$$

Proof. As $g_{hi(k)} = C_{hik} + C_{ihk}$, (5.3) gives (5.4). Conversely, if we put $U_{hik} := C_{hik} - \{F^{-1} l_i C_{hk} + \frac{1}{n+1}(C_h h_{ik} + C_i h_{hk} + C_k h_{hi})\} = U_{kih}$, the condition (5.4) can be rewritten $U_{hik} + U_{ihk} = 0$. Then we see

$$(U_{hik} + U_{ihk}) + (U_{ikh} + U_{kih}) - (U_{khi} + U_{hki}) = 2U_{hik} = 0.$$

Hence this gives (5.3). □

Lemma 5.7. *In a g-C-reducible* M_n *space we have*

$$\begin{aligned} &(a) \quad C_{hjk} - j|k = F^{-1} l_j C_{hk} - j|k, \\ &(b) \quad C_{hj(k)} - j|k = -C_{hjk} - j|k \\ &\qquad\qquad = -F^{-1} l_j C_{hk} - j|k \qquad \text{(cf. (1.7)}(d)), \\ &(c) \quad g_{hj(k)} - j|k = F^{-1} l_j C_{hk} - j|k. \end{aligned} \tag{5.5}$$

We shall give the following

Lemma 5.8. *In a g-C-reducible* M_n *space we have*

$$F\mathrm{p} \cdot C_{hij(k)} - j|k = F^{-1} h^*{}_{ik} C_{hj} + \frac{1}{n+1}(T_{ik} h_{hj} - T_{hj} h_{ik}) - j|k, \tag{5.6}$$

where $T_{ik} := F\mathrm{p}\cdot C_{i/(k)}$.

PROOF. Using the following calculations:

$$F\mathrm{p}\cdot F_{(k)} = F\mathrm{p}\cdot l_k = Fh^r_k l_r = 0,$$
$$F\mathrm{p}\cdot l_{i(k)} = h^*{}_{ik} = C_{ik} + h_{ik},$$
$$F\mathrm{p}\cdot C_{i(k)} = F\mathrm{p}\cdot (C_{i/(k)} + C_i{}^r{}_k C_r) = T_{ik} + \frac{F}{n+1}(C^2 h_{ik} + 2C_i C_k),$$
$$C_{j(k)} - j|k = (\log\sqrt{g})_{(j)(k)} - j|k = 0, \quad g := \det(g_{ij}),$$
$$F\mathrm{p}\cdot h_{hi(k)} = F\mathrm{p}\cdot g_{hi(k)} = \frac{2F}{n+1}(C_h h_{ik} + C_i h_{hk} + C_k h_{hi}),$$
$$F\mathrm{p}\cdot h_{hj(k)} - j|k = 0, \qquad C^2 := C^i C_i,$$

and after some further calculations, we obtain (5.6) from (5.3). □

As the identity: $g_{hi(j)(k)} - j|k = (C_{hij(k)} + C_{ihj(k)}) - j|k = 0$ holds, we see from (5.6)

$$F^2\mathrm{p}\cdot g_{hi(j)(k)} - j|k = h_{ik}(C_{hj} - \frac{2F}{n+1}T_{hj}) - h_{hj}(C_{ik} - \frac{2F}{n+1}T_{ik}) - j|k = 0.$$

Transvecting the above equation with h^{ik}, we have

$$(n-1)(C_{hj} - \frac{2F}{n+1}T_{hj}) - (C^i{}_i - \frac{2F}{n+1}T^i{}_i)h_{hj} = 0.$$

Hence we have

Theorem 5.9. *In a g-C-reducible M_n space there exists a scalar τ such that*

$$C_{hj} - \frac{2F}{n+1}T_{hj} = \tau h_{hj}, \qquad \tau := \frac{1}{n-1}(C^i{}_i - \frac{2F}{n+1}T^i{}_i). \tag{5.7}$$

To find the curvature tensor $S_h{}^i{}_{jk}$, we give the relation

$$\begin{aligned} S_{hijk} &:= g_{ir}S_h{}^r{}_{jk} = g_{ir}(C_h{}^r{}_{j(k)} + C_h{}^m{}_j C_m{}^r{}_k) - j|k \quad \text{(cf. [3],(2.12)(c))} \\ &= C_{hij(k)} - g_{ir(k)}C_h{}^r{}_j + C_h{}^r{}_j C_{rik} - j|k \\ &= C_{hij(k)} - C_h{}^r{}_j C_{irk} - j|k. \end{aligned}$$

As the tensor S_{hijk} is indicatric, we see

$$\begin{aligned} F^2 S_{hijk} &= F^2\mathrm{p}\cdot(C_{hij(k)} - C_h{}^r{}_j C_{irk}) - j|k \\ &= h_{ik}C_{hj} + \frac{F}{n+1}(T_{ik}h_{hj} - T_{hj}h_{ik}) \\ &\quad - \frac{F^2}{(n+1)^2}(C^2 h_{hj}h_{ik} + C_i C_k h_{hj} + C_h C_j h_{ik}) - j|k. \end{aligned}$$

Substituting $\dfrac{F}{n+1}T_{ik} = \dfrac{1}{2}(C_{ik} - \tau h_{ik})$, we have

Proposition 5.10. *In a g-C-reducible M_n space, the curvature tensor $S_h{}^i{}_{jk}$ has the form*

$$(5.8)\quad \begin{aligned} &(a)\quad S_{hijk} = \frac{F^{-2}}{2}(M_{hj}h_{ik} + M_{ik}h_{hj}) - j|k,\\ &(b)\quad M_{hj} := C_{hj} - \frac{F^2}{(n+1)^2}(C^2 h_{hj} + 2C_h C_j). \end{aligned}$$

5.3. A g-C-reducible and g-Landsberg space.

Let us consider the case that a space M_n satisfies the conditions $P^i{}_{jk} = 0$ and (5.3). Substituting (5.3) into (5.1)(a), we have

$$C_{hij/k} - j|k = F^{-1} l_i C_{hj/k} + \frac{1}{n+1}(C_{h/k}h_{ij} + C_{i/k}h_{hj} + C_{j/k}h_{hi}) - j|k = 0.$$

In virtue of (5.1)(b) and (c), we get

$$C_{h/k}h_{ij} + C_{i/k}h_{hj} - C_{h/j}h_{ik} - C_{i/j}h_{hk} = 0.$$

Transvecting the above equation with h^{ik}, we obtain

$$-(n-1)C_{h/j} + C^i{}_{/i}h_{hj} = 0, \quad \text{or} \quad -(n-1)G_{hj} + G^i{}_i h_{hj} = 0 \quad (\text{cf. } (5.2)(c)).$$

Hence we have from (5.2)(c)

Proposition 5.11. *In a g-C-reducible and g-Landsberg space, we have*

$$(5.9)\qquad G_{hj} = C_{h/j} = G h_{hj}, \qquad G := \frac{G^i{}_i}{n-1}.$$

From $S_h{}^i{}_{jk/l} = 0$ and (5.8), we see

$$M_{hj/l}h_{ik} + M_{ik/l}h_{hj} - M_{hk/l}h_{ij} - M_{ij/l}h_{hk} = 0.$$

Transvecting the above equation with h^{ik}, we get

$$(n-3)M_{hj/l} + M^i{}_{i/l}h_{hj} = 0.$$

Moreover, transvecting the above equation with h^{hj}, we get $2(n-2)M^i{}_{i/l} = 0$ and hence we obtain $(n-3)M_{hj/l} = 0$. On the other hand, for $n > 3$ (5.8)(b) and (5.9) lead us to

$$M_{hj/k} = C_{hj/k} - \frac{2F^2 G}{(n+1)^2}(C_h h_{jk} + C_j h_{hk} + C_k h_{hj}) = 0.$$

Proposition 5.12. *In a g-C-reducible and g-Landsberg space M_n, for $n > 3$ we have*

$$(5.10)\quad \begin{aligned} &(a)\quad C_{hj/k} = \frac{2F^2G}{(n+1)^2}(C_h h_{jk} + C_j h_{hk} + C_k h_{hj}),\\ &\qquad C^i{}_{i/k} = \frac{2F^2GC_k}{n+1},\\ &(b)\quad G_h{}^i{}_{jk} = C_h{}^i{}_{j/k} = \frac{G}{(n+1)^2}\{2Fl^i(C_h h_{jk} + C_j h_{hk} + C_k h_{hj})\\ &\qquad\qquad + (n+1)(h^i_h h_{jk} + h^i_j h_{kh} + h^i_k h_{hj})\}. \end{aligned}$$

On the other hand, from (2.12)(b) and (2.5)(a) we see

$$(5.11)\quad P^*{}_{hjk} := g^*{}_{hr}P^{*r}{}_{jk} = -g^*{}_{hr}A_j{}^r{}_k = -\frac{1}{2}C_{hj/k}.$$

Hence from the definition of the tensor $D^*{}_h{}^i{}_{jk}$ (cf. (6.3)(b)) we have

$$\begin{aligned} D^*{}_h{}^i{}_{jk} &= G_h{}^i{}_{jk} + 2F^{-1}l^iP^*{}_{hjk} - \frac{1}{n+1}(h^i_h G_{jk} + h^i_j G_{kh} + h^i_k G_{hj})\\ &= C_h{}^i{}_{j/k} - F^{-1}l^iC_{hj/k} - \frac{G}{n+1}(h^i_h h_{jk} + h^i_j h_{kh} + h^i_k h_{hj}) = 0. \end{aligned}$$

From Proposition 6.7 in §6 we obtain

Theorem 5.13. *In a g-C-reducible and g-Landsberg space M_n, for $n > 3$ the Douglas tensor $D_h{}^i{}_{jk}$ vanishes.*

Remark. In a Finsler geometry ($C_{ij} = 0$) it is known ([23],Theorem 1) that a C-reducible Landsberg space is a Berwald space ($G_h{}^i{}_{jk} = 0$). This fact implies $D_h{}^i{}_{jk} = 0$.

5.4. A g-Berwald space.

Definition. If the connection parameters $F_j{}^i{}_k$ of $C\Gamma(N)$ are independent of y^i, that is, $F_j{}^i{}_{kl} = 0$, then the space is called a *g-Berwald* space (an *affinely connected* space).

Theorem 5.14. ([3], Lemma 4.4, Theorem 4.5). *A space M_n is a g-Berwald space if any one of the following conditions holds*:

$$(a)\quad F_j{}^i{}_{kl} = 0,\qquad (b)\quad C_j{}^i{}_{k/l} = 0,\qquad (c)\quad g_{ij(k)/l} = 0.$$

Definition. A generalized metric space M_n whose associated Finsler space $F^*_n(g)$ is a Berwald space (${}^*\Gamma_j{}^i{}_{kl} = G_j{}^i{}_{kl} = 0$) is called a BM_n space (abbreviation).

Remark. In a g-Berwald space, we have

$$(a)\quad P^i{}_{jk}=0\quad (\text{cf. } (1.7)(b)),\qquad (b)\quad G_j{}^i{}_{kl}=0\quad (\text{cf. } (1.14)(c)).$$

Hence we see that a g-Berwald space is a g-Landsberg space from (a) and a BM_n space from (b).

From (2.16)(c) we see

Theorem 5.15. *A BM_n space is a g-Berwald space if the condition $C_{ij/k}=0$ holds.*

Recently S. Bácsó, F. Ilosvay and B. Kis proved the following theorem ([16],Theorem 1): *If a Landsberg space* ($P^{*i}{}_{jk}=0$) *satisfies the condition $D_h{}^i{}_{jk}=0$, then the space is a Berwald space* ($G_h{}^i{}_{jk}=0$). Hence we have

Theorem 5.16. *If an LM_n space satisfies the condition $D_h{}^i{}_{jk}=0$, then the space is a BM_n space.*

From (5.10)(b) and Theorem 5.14, we have

Theorem 5.17. *In a g-C-reducible and g-Landsberg space M_n, for $n>3$ if the scalar G vanishes, then the space is a g-Berwald space.*

§6. A generalized metric space of scalar curvature K

As the geodesics in $F_n^*(g)$ are geodesics in M_n, the notion of a space of scalar curvature is contained in the geometry of M_n. In this section we shall refer to this interesting property about which many results are known in Finsler geometry.

Definition. A space M_n $(n>2)$ whose associated Finsler space $F_n^*(g)$ is of scalar curvature K $(K\neq 0)$ is called a generalized metric space *of scalar curvature* K and called an M_nsc space (abbreviation). If the scalar K is constant, we call the space a generalized metric space *of constant curvature* K and call it an M_ncc space.

From the above definition an M_nsc space is characterized by

$$(6.1)\qquad (a)\quad H^i{}_k=F^2Kh^i_k \qquad (\text{cf. } [17]).$$

From (1.10) we have

$$
(6.1)\quad
\begin{aligned}
&(b)\quad H^i{}_{jk} = F(Kl_j + \frac{1}{3}K_j)h^i_k - j|k,\\
&(c)\quad H_h{}^i{}_{jk} = Kg^*{}_{hj}\delta^i_k + \frac{1}{3}\{h^*{}_{hj}l^iK_k + h^i_h l_j K_k\\
&\qquad\qquad + h^i_k(K_{hj} + l_hK_j + 2K_hl_j)\} - j|k,\\
&(d)\quad H_j = F\{(n-1)Kl_j + \frac{n-2}{3}K_j\},\\
&(e)\quad H_{hj} = (n-1)Kg^*{}_{hj}\\
&\qquad\qquad + \frac{1}{3}\{(n-2)(K_{hj} + l_hK_j) + (2n-1)K_hl_j\},
\end{aligned}
$$

where $K_j := FK_{(j)},\ K_{hj} := F\mathrm{p}\cdot K_{j(h)} = FK_{j(h)} + K_hl_j = K_{jh}$.

6.1. An M_ncc space.

It is well known (e.g. [29],[25]) that in a Finsler space of scalar curvature K, the scalar K is constant if it is independent of y.

From Proposition 1.2 the following is evident.

Lemma 6.1. *An M_nsc space is an M_ncc space if $K_j = 0$ or $K_{hj} = 0$ holds.*

Moreover we know the following

Theorem 6.2. *A generalized metric space M_n reduces to one of constant curvature K if and only if the tensors $H_h{}^i{}_{jk}, H^i{}_{jk}$ and $H^i{}_k$ have any one of the following forms:*

$(a)\quad H^i{}_{jk} = K(y_j\delta^i_k - j|k)$ (cf. [29],p. 133),

$(b)\quad H_h{}^i{}_{jk} = A_{hj}\delta^i_k - A_{jk}\delta^i_h - j|k$ (cf. [35]),

$(c)\quad H_h{}^i{}_{jk} = \dfrac{1}{n+1}(\delta^i_kH_{j(h)} - \delta^i_hH_{jk} - j|k)$ (cf. [18],Theorem 2),

$(d)\quad H^i{}_k = \dfrac{1}{n-1}(H_0\delta^i_k - H_ky^i)$ (cf. [36],[37]),

where $A_{hj} := \dfrac{1}{n^2-1}(nH_{hj} + H_{jh})$.

Theorem 6.3. (e.g. [10]). *A generalized metric space of scalar curvature K reduces to one of constant curvature K if and only if any one of the following conditions holds*:

$$(a)\quad K_{/0}=0,\quad (b)\quad K_{/\!/j}=K_{/j}-F^{-1}P^h{}_jK_h=0 \quad \text{(cf. [26],Theorem 1)},$$

$$(c)\quad H_{hj}=H_{jh}\quad or\quad H_i{}^i{}_{jk}=0\quad or\quad \mathrm{p}\cdot H^i{}_{jk}=0,$$

$$(d)\quad G_{hj/\!/k}-j|k=0\quad or\quad H_{hj(k)}-j|k=0 \qquad \text{(cf. [18],Theorem 1)},$$

$$(e)\quad \mathrm{p}\cdot G_h{}^i{}_{jk/\!/0}=0.$$

Theorem 6.4. ([30], Theorem 4.3). *If a generalized metric space of scalar curvature K satisfies the condition*

$$F_h{}^i{}_{jk}:=H_h{}^i{}_{jk}-(g_{hj}L^i{}_k+L_{hj}\delta^i{}_k-h^i_hL_{jk}-j|k)=0 \quad \text{(cf. [21])},$$

$$L_{hj}:=\frac{1}{(n-1)(n-2)}\{(n-1)H_{hj}-\frac{1}{2}g^{ik}H_{ik}g_{hj}+l^il_j(H_{ih}-H_{hi})\},$$

then the space is one of constant curvature K.

6.2. Projective(geodesic) change in M_n.

It is well known that in a metric space, a *path* (autoparallel curve) is coincident with the *geodesic* (extremal curve). From Finsler geometry, we know the following results:

Theorem 6.5. ([35],[25]). *A generalized metric space M_n is projectively flat if the Weyl tensor $W^i{}_{jk}$ and the Douglas tensor $D_h{}^i{}_{jk}$ vanish, where*

$$(6.2)\quad \begin{aligned} &(a)\quad W^i{}_{jk}:=H^i{}_{jk}+\frac{1}{n+1}\{H_{jk}y^i+\frac{1}{n-1}(nH_k+H_{k0})\delta^i_j-j|k\},\\ &(b)\quad D_h{}^i{}_{jk}:=G_h{}^i{}_{jk}-\frac{1}{n+1}(l^iG_{hjk}+h^i_hG_{jk}+h^i_jG_{kh}+h^i_kG_{hj}),\end{aligned}$$

where $G_{hjk}:=F\mathrm{p}\cdot G_{jk(h)}=FG_{jk(h)}+l_jG_{kh}+l_kG_{hj}+l_hG_{jk}$.

Theorem 6.6. ([34],[25]). *A generalized metric space M_n $(n>2)$ is one of scalar curvature K if and only if the condition $W^i{}_{jk}=0$ holds.*

We shall show

Proposition 6.7. *The following relations hold*:

$$(a)\quad {W^i}_{jk} = 0 \text{ is equivalent to } {W^{*i}}_{jk} = 0, \text{ where}$$

$${W^{*i}}_{jk} := {H^i}_{jk} - \frac{1}{n-2}\{(H_j - F^{-1}Hl_j)h^i_k - j|k\},$$

$$(6.3)\qquad H := \frac{1}{n-1}{H^i}_i,$$

$$(b)\quad {D_h}^i{}_{jk} = 0 \text{ is equivalent to } \mathrm{p}\cdot {D_h}^i{}_{jk} =: {D^*}_h{}^i{}_{jk} = 0, \text{ where}$$

$${D^*}_h{}^i{}_{jk} = {G_h}^i{}_{jk} + 2F^{-1}l^i {P^*}_{hjk} - \frac{1}{n+1}(h^i_h G_{jk} + h^i_j G_{kh} + h^i_k G_{hj}).$$

Proof. For (a), from (6.1)(a) and (d) we see

$$(6.4)\qquad H_j = (n-2)FL_j + F^{-1}Hl_j, \quad L_j := Kl_j + \frac{1}{3}K_j.$$

Hence eliminating L_j in (6.1)(b), we have

$${H^i}_{jk} = \frac{1}{n-2}\{(H_j - F^{-1}Hl_j)h^i_k - j|k\}.$$

For (b), see [10], Theorem 4.4. □

6.3. A g-Landsberg space of scalar curvature K.

Now we shall refer to a g-Landsberg space of scalar curvature K.

First, substituting (6.1)(b) and (c) into (5.2)(b) and after some arrangement, we obtain

$$(6.5)\qquad \begin{aligned} &3K(C_{hj}g_{ik} + C_{ij}g_{hk} + Fl_j g_{hi(k)}) + K_{hj}h_{ik} + K_{ij}h_{hk} + FK_j g_{hi(k)} \\ &+(l_iC_{hj} + l_hC_{ij})K_k + 2l_j(K_hh_{ik} + K_ih_{hk} + K_kh_{hi}) - j|k = 0. \end{aligned}$$

Transvecting (6.5) with l^j, we get

$$(6.6)\quad 3K\{Fg_{hi(k)} - (l_hC_{ik} + l_iC_{hk})\} + 2(K_hh_{ik} + K_ih_{hk} + K_kh_{hi}) = 0.$$

Moreover, on transvecting (6.6) with h^{hi}, we find

$$(6.7)\qquad 3FKC_k + (n+1)K_k = 0, \quad \text{or} \quad K_k = -\frac{3FK}{n+1}C_k.$$

Using (6.7), we can eliminate K_k in (6.6). As $K \neq 0$, we have

$$Fg_{hi(k)} - (l_hC_{ik} + l_iC_{hk}) - \frac{2F}{n+1}(C_hh_{ik} + C_ih_{hk} + C_kh_{hi}) = 0.$$

With Proposition 5.6 and Theorem 5.9 in mind, the above equation gives the following

Theorem 6.8. *A g-Landsberg space of scalar curvature K is g-C-reducible and there exists a scalar τ such that $C_{hj} - \frac{2F}{n+1} T_{hj} = \tau h_{hj}$.*

From Theorems 5.13, 6.5 and 6.6, we have

Theorem 6.9. *A g-Landsberg space M_n of scalar curvature K is for $n > 3$ projectively flat.*

Next, operating $F\mathrm{p} \cdot \dot{\partial}_i$ to (6.7), we have

$$3F[K_i C_k + K\{T_{ik} + \frac{F}{n+1}(C^2 h_{ik} + 2C_i C_k)\}] + (n+1)K_{ik} = 0.$$

Moreover, substituting (6.7) into the above equation, we obtain

$$K_{ik} = -3K\{\frac{F}{n+1} T_{ik} + \frac{F^2}{(n+1)^2}(C^2 h_{ik} - C_i C_k)\}. \tag{6.8}$$

Operating p$\cdot$ to (6.5), we get

$$\begin{aligned} &3K(C_{hj} h_{ik} - C_{ik} h_{hj}) + K_{hj} h_{ik} - K_{ik} h_{hj} \\ &+ \frac{2F}{n+1} K_j (C_h h_{ik} + C_i h_{hk} + C_k h_{hi}) - j|k = 0. \end{aligned}$$

Substituting K_k in (6.7) and K_{ik} in (6.8) into the above equation, we have $(K \neq 0)$

$$\begin{aligned} &(C_{hj} - \frac{F}{n+1} T_{hj} - \frac{F^2}{(n+1)^2} C_h C_j) h_{ik} \\ &\quad - (C_{ik} - \frac{F}{n+1} T_{ik} - \frac{F^2}{(n+1)^2} C_i C_k) h_{hj} - j|k = 0. \end{aligned}$$

On the other hand, we know $\frac{F}{n+1} T_{hj} = \frac{1}{2}(C_{hj} - \tau h_{hj})$. Then the above equation leads us to

$$(C_{hj} - \frac{2F^2}{(n+1)^2} C_h C_j) h_{ik} - (C_{ik} - \frac{2F^2}{(n+1)^2} C_i C_k) h_{hj} - j|k = 0.$$

Lastly transvecting the above equation with h^{ik}, we obtain

$$(n-1)(C_{hj} - \frac{2F^2}{(n+1)^2} C_h C_j) - ({C^i}_i - \frac{2F^2 C^2}{(n+1)^2}) h_{hj} = 0.$$

Thus we have

Theorem 6.10. *In a g-Landsberg space of scalar curvature K, there exists a scalar β such that*

$$C_{hj} - \frac{2F^2}{(n+1)^2} C_h C_j = \beta h_{hj}, \quad \beta := \frac{1}{n-1}\Big(C^i{}_i - \frac{2F^2 C^2}{(n+1)^2}\Big). \tag{6.9}$$

Theorem 6.11. *In a g-Landsberg space of scalar curvature K, the curvature tensor $S_h{}^i{}_{jk}$ has the form*

$$\begin{aligned} S_{hijk} &= F^{-2} S(h_{hj}h_{ik} - h_{hk}h_{ij}) \qquad \text{(called } S3\text{-type, cf. [27])}, \\ S &:= \beta - \frac{F^2C^2}{(n+1)^2} = \frac{1}{n^2-1}\{(n+1)C^i{}_i - F^2C^2\}, \end{aligned} \tag{6.10}$$

where S is a constant.

Proof. In view of (5.8) and (6.9), we see

$$M_{hj} = C_{hj} - \frac{F^2}{(n+1)^2}(C^2 h_{hj} + 2C_hC_j) = \Big(\beta - \frac{F^2C^2}{(n+1)^2}\Big) h_{hj}.$$

Putting $M_{hj} =: S h_{hj}$ we have $S = \beta - \frac{F^2C^2}{(n+1)^2}$, or exactly

$$S = \frac{1}{n-1}\Big(C^i{}_i - \frac{2F^2C^2}{(n+1)^2}\Big) - \frac{F^2C^2}{(n+1)^2} = \frac{1}{n^2-1}\{(n+1)C^i{}_i - F^2C^2\}.$$

On the other hand, we see

$$S_{/k} = \frac{1}{n^2-1}\{(n+1)C^i{}_{i/k} - 2F^2C^iC_{i/k}\}.$$

Substituting (5.10)(a) and (5.9), we obtain

$$S_{/k} = \frac{1}{n^2-1}(2F^2GC_k - 2F^2GC_k) = 0.$$

Accordingly, by means of the Ricci identity, we see from (6.1)(b)

$$S_{/j/k} - S_{/k/j} = 0 = -H^r{}_{jk}S_{(r)} = -F\Big(Kl_j + \frac{1}{3}K_j\Big)S_{(k)} - j|k,$$

and transvecting the above equation with y^j, we have $S_{(k)} = 0$. Therefore $S_{/k} = 0$ means that the scalar S is a constant. □

Theorem 3.3, evidently implies the following

Theorem 6.12. *A g-Berwald space of scalar curvature K is an $RccM_n$ space, which satisfies $C_i = 0$ and $C_{hij} = F^{-1}l_iC_{hj}$.*

§7. Semi-C-reducibility

In this section we refer to the tensor C_{hj} which vanishes in a Finsler geometry. The purpose of this section is to prove the following

Theorem 7.1. *The Finsler space associated with a g-Landsberg space of scalar curvature K is semi-C-reducible, that is,*

$$
\begin{aligned}
C^*{}_{hjk} = {} & \frac{p}{n+1}(C^*{}_h h^*{}_{jk} + C^*{}_j h^*{}_{hk} + C^*{}_k h^*{}_{hj}) \\
& + \frac{q}{(C^*)^2} C^*{}_h C^*{}_j C^*{}_k, \quad (C^*)^2 := g^{*ij} C^*{}_i C^*{}_j,
\end{aligned}
\tag{7.1}
$$

where $p + q = 1, \quad p \neq 0, \quad q \neq 0.$

Definition. A Finsler space that satisfies the condition (7.1) (without asterisk mark *) is called *semi-C-reducible*.

Examples. There are many semi-C-reducible Finsler spaces: e.g.

(1°) A Finsler space with (α, β)-metric is semi-C-reducible ([24],[28]).

(2°) If an $R3$-like Finsler space satisfies the condition $\mathrm{p} \cdot P^i{}_{jk/l} - k|l = 0$, then the space is semi-C-reducible or satisfies $F^{-1}P_{hij/0} + FKC_{hij} = 0$, where K is some scalar ([20],Proposition 5.5).

(3°) If an $R3$-like Finsler space satisfies the condition $^*P^i{}_{jk} = 0$, then the space is semi-C-reducible or $S3$-like ([38],Theorem 4.3).

(4°) If a Finsler space with property $\mathcal{H}$ satisfies the condition $^*P^i{}_{jk} = 0$ or $\mathrm{p} \cdot P^i{}_{jk/l} - k|l = 0$, then the space is semi-C-reducible under some condition ([31],Theorems 4.2 and 4.4).

See Appendix.

7.1. Tensors m_i, m_{hj} and m_{hjk}.

First, in a g-Landsberg space of scalar curvature K, we recall the following relations:

$$3C^*{}_{ijk} = C_{ijk} + C_{jki} + C_{kij} + \frac{1}{2}(C_{ij(k)} + C_{jk(i)} + C_{ki(j)}), \tag{3.1}$$

$$C_{hik} = F^{-1} l_i C_{hk} + \frac{1}{n+1}(C_h h_{ik} + C_i h_{hk} + C_k h_{hi}), \tag{5.3}$$

$$C_{hj} - \frac{2F}{n+1} T_{hj} = \tau h_{hj}, \tag{5.7}$$

$$C_{hj} - \frac{2F^2}{(n+1)^2} C_h C_j = \beta h_{hj}, \tag{6.9}$$

$$S = \beta - \frac{F^2 C^2}{(n+1)^2} \qquad (S\text{: constant}). \tag{6.10}$$

To save the complicated calculations we introduce new notations:

$$m_i := \frac{F}{n+1} C_i, \qquad m_{hj} := F\mathrm{p} \cdot m_{h/(j)} = \frac{F}{n+1} T_{hj},$$

then we see that the above equations are expressed by

$$(7.2) \quad \begin{aligned} &(a) \quad C_{hj} = 2m_h m_j + \beta h_{hj}, \qquad (b) \quad C_{hj} = 2m_{hj} + \tau h_{hj}, \\ &(c) \quad m_{hj} = m_h m_j + \frac{\beta - \tau}{2} h_{hj}, \\ &(d) \quad S = \beta - m^2, \quad m^2 := m^i m_i, \\ &(e) \quad m_{hjk} := F\mathrm{p} \cdot C_{hjk} = m_h h_{jk} + m_j h_{hk} + m_k h_{hj}. \end{aligned}$$

7.2. Metric tensors in $F_n^*(g)$.

We see

$$(7.3) \quad \begin{aligned} {g^*}_{hj} &= g_{hj} + C_{hj} = g_{hj} + 2m_h m_j + \beta h_{hj}, \\ {h^*}_{hj} &= (1+\beta) h_{hj} + 2m_h m_j. \end{aligned}$$

We set $g^{*hk} = g^{hk} + A m^h m^k + B h^{hk}$ with unknown coefficients A and B, and substituting (7.3) into the definition $g^{*hk} {g^*}_{hj} = \delta_j^k$, we obtain

$$(7.4) \quad \begin{aligned} &B + \beta + B\beta = 0, \qquad B = -\frac{\beta}{1+\beta}, \quad 1 + B = \frac{1}{1+\beta}, \\ &A(1 + \beta + 2m^2) + 2(1 + B) = 0, \\ &\qquad A = -\frac{2}{a(1+\beta)}, \quad a := 1 + \beta + 2m^2. \end{aligned}$$

Hence we have

Lemma 7.2. *In a g-Landsberg space of scalar curvature K we have*

$$(7.5) \quad \begin{aligned} g^{*hk} &= g^{hk} - \frac{2}{a(1+\beta)} m^h m^k - \frac{\beta}{1+\beta} h^{hk}, \\ h^{*hk} &= \frac{1}{1+\beta} (h^{hk} - \frac{2}{a} m^h m^k). \end{aligned}$$

Remark. The scalar β cannot satisfy $1 + \beta = 0$. In fact, if $1 + \beta = 0$ holds in (7.3), we have ${h^*}_{hj} = 2m_h m_j$. The last equation means that the rank of the matrix $({h^*}_{hj})$ must be 1. As the rank of the matrix $({h^*}_{hj})$ is $n - 1$, the relation $1 + \beta = 0$ cannot hold for $n > 2$. Moreover, the last equation of (7.4) is rewritten as $Aa(1 + \beta) = -2$. Accordingly we see that the scalar a cannot vanish.

7.3. Tensor $C^*{}_{hjk}$ in $F_n^*(g)$.

Let us carry out the following calculation:

$$\begin{aligned} m_{ik} &= F\mathrm{p}\cdot(m_{i(k)} - C_i{}^r{}_k m_r) = F\mathrm{p}\cdot m_{i(k)} - m_{irk}m^r \\ &= F\mathrm{p}\cdot m_{i(k)} - (m^2 h_{ik} + 2m_i m_k). \end{aligned}$$

From (7.2)(c) we have

$$F\mathrm{p}\cdot m_{i(k)} = 3m_i m_k + \frac{2m^2+\beta-\tau}{2}h_{ik}.$$

From (7.2)(d) and (c), we see

$$\begin{aligned} \beta_k &:= F\mathrm{p}\cdot\beta_{(k)} = F\mathrm{p}\cdot(S+m^2)_{/(k)} = 2m^i F\mathrm{p}\cdot m_{i/(k)} \\ &= 2m^i m_{ik} = (2m^2+\beta-\tau)m_k, \\ F\mathrm{p}&\cdot h_{hj(k)} = F\mathrm{p}\cdot g_{hj(k)} = 2m_{hjk}. \end{aligned}$$

Using (7.2)(a) and the above, we shall carry out the following calculation:

$$\begin{aligned} F\mathrm{p}\cdot C_{hj(k)} &= 2(F\mathrm{p}\cdot m_{h(k)}m_j + m_h F\mathrm{p}\cdot m_{j(k)}) + \beta_k h_{hj} + \beta F\mathrm{p}\cdot h_{hj(k)} \\ &= 12 m_h m_j m_k + (2m^2+3\beta-\tau)m_{hjk}. \end{aligned}$$

As the term $F\mathrm{p}\cdot C_{hj(k)}$ is symmetric in the indices h, j, k, (3.1) gives

$$FC^*{}_{hjk} = m_{hjk} + \frac{1}{2}F\mathrm{p}\cdot C_{hj(k)}.$$

Hence we have

Lemma 7.3. *In a g-Landsberg space of scalar curvature K we have*

$$FC^*{}_{hjk} = 6m_h m_j m_k + b m_{hjk}, \qquad b := \frac{2m^2+3\beta-\tau+2}{2}. \tag{7.6}$$

Remark. If $b=0$ holds, then the space $F_n^*(g)$ is called $C2$-*like*. Hence we assume $b \neq 0$.

7.4. Torsion vectors in $F_n^*(g)$.

Using (7.5) and (7.6) we see

$$FC^*{}_j = Fh^{*hk}C^*{}_{hjk} = \frac{1}{1+\beta}\{6m^2 + (n+1)b - \frac{6m^2(2m^2+b)}{a}\}m_j.$$

Here, let us put

$$FC^*{}_j = \frac{D}{1+\beta}m_j, \qquad D := (n+1)b + \frac{6m^2(1+\beta-b)}{a}.$$

Moreover from (7.5) and $a - 2m^2 = 1 + \beta$, we find

$$FC^{*k} = Fh^{*hk}C^*{}_h = \frac{D}{(1+\beta)^2}(1 - \frac{2m^2}{a})m^k = \frac{D}{a(1+\beta)}m^k.$$

Hence we have

Lemma 7.4. *In a g-Landsberg space of scalar curvature K we have*

$$FC^*{}_j = \frac{D}{1+\beta}m_j, \qquad FC^{*k} = \frac{D}{a(1+\beta)}m^k, \tag{7.7}$$
$$F^2(C^*)^2 = \frac{D^2m^2}{a(1+\beta)^2}.$$

Remark. From (7.7) we see that if the scalar D vanishes, then $C^*{}_j = 0$, which leads to $C^*{}_{hjk} = 0$ by Deicke's Theorem. Hence we assume $D \neq 0$. When the vector m_i vanishes, we see $C_{hjk} = F^{-1}l_jC_{hk}$ and $C^*{}_{hjk} = 0$, that is, the space considered reduces to an $RccM_n$ space.

7.5. Proof of Theorem 7.1. From (7.3) and (7.7), we see

$$h_{hj} = \frac{1}{1+\beta}(h^*{}_{hj} - 2m_hm_j) = \frac{1}{1+\beta}(h^*{}_{hj} - \frac{2m^2}{a(C^*)^2}C^*{}_hC^*{}_j)$$

and

$$m_kh_{hj} = \frac{F}{D}(C^*{}_kh^*{}_{hj} - \frac{2m^2}{a(C^*)^2}C^*{}_hC^*{}_jC^*{}_k),$$
$$m_hm_jm_k = \frac{F(1+\beta)m^2}{Da(C^*)^2}C^*{}_hC^*{}_jC^*{}_k.$$

Hence (7.2)(e) gives

$$m_{hjk} = \frac{F}{D}(C^*{}_hh^*{}_{jk} + C^*{}_jh^*{}_{hk} + C^*{}_kh^*{}_{hj} - \frac{6m^2}{a(C^*)^2}C^*{}_hC^*{}_jC^*{}_k).$$

Finally (7.6) is rewritten

$$C^*{}_{hjk} = \frac{b}{D}(C^*{}_hh^*{}_{jk} + C^*{}_jh^*{}_{hk} + C^*{}_kh^*{}_{hj}) + \frac{6m^2(1+\beta-b)}{Da(C^*)^2}C^*{}_hC^*{}_jC^*{}_k,$$

and we have

$$p = \frac{(n+1)b}{D}, \quad q = \frac{6m^2(1+\beta-b)}{Da}, \qquad p + q = 1.$$

Thus the proof is complete. □

Remark. This theorem means that if a g-Landsberg space of scalar curvature K satisfies $m_j = 0$ ($C_j = 0$), then the space reduces to an $RccM_n$ space.

(6.7) tells us the following

Theorem 7.5. *A g-Landsberg space of constant curvature K is an $RccM_n$ space.*

Moreover, from (5.9) we see $G_{hj/\!/k} = G_{hj/k} = G_{/k} h_{hj}$. From the condition (d) of Theorem 6.3, we have

Theorem 7.6. *If the scalar G is constant, then the g-Landsberg space of scalar curvature K is an $RccM_n$ space.*

M. Matsumoto and C. Shibata showed ([24],(1.8)) that the curvature tensor $S_h{}^i{}_{jk}$ of a semi-C-reducible Finsler space is expressed by (without asterisk mark *)

$$(7.8)\qquad \begin{aligned} &(a)\quad S_{hijk} = \frac{F^{-2}}{2}(M_{hj}h_{ik} + M_{ik}h_{hj}) - j|k, \\ &(b)\quad M_{hj} := -\frac{p^2F^2C^2}{(n+1)^2}h_{hj} - \frac{2pF^2}{(n+1)^2}(nq+1)C_hC_j. \end{aligned}$$

Remark. A Finsler space with the tensor S_{hijk} of (7.8)(a) is called *S4-like*. If the condition $nq + 1 = 0$ holds, then the tensor S_{hijk} has the form (6.10) and the space is called *S3-like*.

In a g-Landsberg space of scalar curvature K, the condition $nq+1 = 0$ reduces to

$$(7.9)\qquad ab + 6m^2(1 + \beta - b) = 0,$$

and after some rearrangement we have

$$(7.10)\qquad 3\beta^2 - (14S + 3\tau + 11)\beta + 8S^2 + 2(2\tau + 3)S + \tau - 2 = 0,$$

where the scalars β , τ and the constant S exist exactly.

However, with (5.9) and (7.2)(d) in mind we have

$$(7.11)\qquad \beta_{/k} = (S + m^2)_{/k} = 2m^i m_{i/k} = \frac{2FG}{n+1}m_k, \qquad \beta_{/0} = 0.$$

Now, differentiating (7.10) by x^k we find

$$(4S - 3\beta + 1)\tau_{/k} = (14S + 3\tau - 6\beta + 11)\beta_{/k}.$$

Hence from (7.11) we see $(4S-3\beta+1)\tau_{/0}=0$. Put $4S-3\beta+1=a-6m^2=0$, then the condition (7.9) can be rewritten as $a(1+\beta)=0$, which cannot hold. Thus, from (5.7) and (5.10)(a) we obtain

$$\tau_{/0}=\frac{1}{n-1}(C^i{}_{i/0}-\frac{2F}{n+1}T^i{}_{i/0})=-\frac{2F}{n^2-1}T^i{}_{i/0}=0.$$

On the other hand, by means of the Ricci identity we find

$$C_{i/k/(j)}-C_{i/(j)/k}=-P_i{}^h{}_{kj}C_h-C_k{}^h{}_jC_{i/h}-P^h{}_{kj}C_{i/(h)}=-C_k{}^h{}_jC_{i/h}.$$

Transvecting the above equation with y^k we have $C_{i/(j)/0}=-C_{i/j}=-Gh_{ij}$. From $T_{ij/k}=FC_{i/(j)/k}+l_iC_{j/k}+l_jC_{i/k}$ we obtain

$$0=T^i{}_{i/0}=Fg^{ij}C_{i/(j)/0}=-(n-1)FG.$$

Thus we have $G=0$. Hence we have from Theorem 7.6

Theorem 7.7. *If the associated Finsler space of a g-Landsberg space of scalar curvature K is S3-like, then the space is an $RccM_n$ space.*

7.6. Appendix.

In the cases of (2°), (3°) and (4°), the original condition is expressed by (e.g. [20],(5.12),(5.19))

$$(a)\quad (n-1)C_kC_{hij}+h_{ij}(C_h{}^r{}_kC_r-C_hC_k)$$
$$+h_{hj}(C_i{}^r{}_kC_r-C_iC_k)-j|k=0\quad (^*C\text{-reducible}),$$

$$(b)\quad (n-2)C^2C^2C^2C_{hij}+(A-C^2C^2)C^2(C_hh_{ij}+C_ih_{hj}+C_jh_{hi})$$
$$+\{3C^2C^2-(n+1)A\}C_hC_iC_j=0,$$
$$A:=C_{hij}C^hC^iC^j\quad (\text{semi-}^*C\text{-reducible}).$$

It has been proved that (a) and (b) are equivalent to the semi-C-reducible condition ([32], Proposition 1.1). As C_{hij} satisfies $A=cC^2C^2$ with some scalar c, we find

$$(n-2)C^2C_{hij}+(c-1)C^2(C_hh_{ij}+C_ih_{hj}+C_jh_{hi})+\{3-(n+1)c\}C_hC_iC_j=0.$$

Accordingly the scalar p is not arbitrary and the scalar c decides some property. However as $p\neq 0$ and $q\neq 0$, we see that $c\neq 1$ and $c\neq \dfrac{3}{n+1}$.

Definition. A Finsler space is called

(1) a *Finsler space with (α,β)-metric* if the Finsler metric is given by $F(x,y)=L(\alpha,\beta)$, where L is p-homogeneous of degree 1 in the two variables $\alpha(x,y):=\sqrt{a_{ij}(x)y^iy^j}$ and $\beta(x,y):=b_i(x)y^i$.

(2) a *Finsler space with property* $\mathcal{H}$, if the condition $\mathcal{H}^i{}_{jk} = 0$ holds ([22],(2.3),[31]), where

$$\mathcal{H}^i{}_{jk} := Z^i{}_{jk} - \frac{1}{n-2}(Z_j h^i_k - Z_k h^i_j), \quad Z^i{}_{jk} := \mathrm{p} \cdot H^i{}_{jk}, \quad Z_j := Z^i{}_{ji}.$$

(3) a *$^*P$-Finsler* space, if the condition $^*P^i{}_{jk} := P^i{}_{jk} - \lambda C_j{}^i{}_k = 0$ holds (called a $^*P$-condition, cf. [19]).

(4) a *Finsler space with* $F_h{}^i{}_{jk} = 0$, if the condition $F_h{}^i{}_{jk} = 0$ holds ([30]).

(5) an *R3-like Finsler* space, if the condition $C_{hijk} = 0$ (the formally Weyl conformal curvature tensor vanishes as in Riemannian geometry) holds ([20], [38]), where

$$C_{hijk} := R_{hijk} - (L_{hj} g_{ik} + g_{hj} L_{ik} - j|k), \quad L_{hj} := \frac{1}{n-2}(R_h{}^i{}_{ji} - r g_{hj}).$$

Remark. Three special Finsler spaces: a Finsler space of scalar curvature K, an $R3$-like Finsler space and a Finsler space with $F_h{}^i{}_{jk} = 0$, have the property $\mathcal{H}$ ([22],Theorems 2.4, 2.5 and 2.6).

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