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Groups in which some primary subgroups are weakly s-supplemented

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Abstract. A subgroup H of a group G is called weakly s-supplemented in G if there is a subgroup T such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. The influence of primary weakly s-supplemented subgroups on the structure of finite groups is investigated. An open question promoted by Skiba is studied and some known results are generalized.

1. Introduction

All groups considered in this paper are finite. The notions and notations not introduced are standard and the reader is referred to [1], [2], [3] if necessary.

A subgroup H of a group G is said to be a permutable subgroup (cf. [1]) of G or a quasinormal subgroup of G (cf. [4]) if H is permutable with all subgroups of G. The permutability of subgroups plays an important role in the study of the structure of finite groups and was generalized extensively. Recall that a subgroup H of a group G is called s-permutable (or s-quasinormal) in G if H permutes with every Sylow subgroup of G(cf. [5]). Let H be a subgroup of G. H_{sG} denotes the subgroup of H generated by all those subgroups of H which are s-permutable in G. In [6], the following definitions are introduced.

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Definition 1.1 ([6]). Let H be a subgroup G. H is called weakly s-supplemented in G if there is a subgroup T such that $G = HT$ and $H \cap T \leq H_{sG}$, and if T is subnormal in G then H is called weakly s-permutable in G.

By using this idea, Skiba [6] proved the following nice result.

Theorem 1.2. *Let $\mathfrak{F}$ be a saturated formation containing all supersolvable groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) not having a supersolvable supplement in G are weakly s-permutable in G. Then $G \in \mathfrak{F}$.*

The above theorem generalized many known results. In connection with this, the following question was proposed by A. Skiba.

Question 1 ([6, Question 6.4])**.** *Let $\mathfrak{F}$ be a saturated formation containing all supersolvable groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are weakly s-supplemented in G. Is then $G \in \mathfrak{F}$?*

We have given an example in [7] to show that the answer of this question is negative in general. But, in the following theorem, we will prove in many case the question has positive answer.

For convenience, if $m = p^\alpha$ is a p-number, let $\iota(m)$ denote $\log_p m = \alpha$ and if P is a p group we use $\iota(P)$ instead of $\iota(|P|)$.

Theorem 1.3. *Let $\mathfrak{F}$ be a saturated formation containing all supersolvable groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) having no supersolvable supplement in G are weakly s-supplemented in G. If one of the following holds:*

(i) $\Phi(P) \neq P'$;

(ii) $|D| \leq |P'|$;

(iii) $|P'| < |D|$ *and* $(\iota(P/P'), \iota(|D|/|P'|)) = 1$ *or* $(\iota(P), \iota(|P : D|)) = 1$;

then $G \in \mathfrak{F}$

2. Elementary properties

Lemma 2.1 ([6, Lemma 2.10]). *Let G be a group and $H \leq K \leq G$. Then*

(i) *Suppose that H is normal in G. Then K/H is weakly s-supplemented in G/H if and only if K is weakly s-supplemented in G.*

(ii) *If H is weakly s-supplemented in G, then H is weakly s-supplemented in K.*

(iii) *Suppose that H is normal in G. Then the HE/H is weakly s-supplemented in G/H for every weakly s-supplemented in G subgroup E satisfying $(|H|, |E|) = 1$.*

Lemma 2.2 ([8, Lemma 2.8]). *Let $\mathfrak{F}$ be a saturated formation and P be a normal p-subgroup of G. Then $P \subseteq Z_\infty^{\mathfrak{F}}(G)$ if and only if $P/\Phi(P) \subseteq Z_\infty^{\mathfrak{F}}(G/\Phi(P))$.*

Lemma 2.3 ([6, Lemma 2.7]). *If H is s-permutable in a group G and H is a p-group for some prime p, then $O^p(G) \leq N_G(H)$.*

Lemma 2.4 ([3, Lemma 3.8.7]). *Let G be a p-solvable group. If $O_{p'}(G) = 1$ and $O_p(G) \leq H \leq G$, then $O_{p'}(H) = 1$*

The following Lemma is well known.

Lemma 2.5. *Let N be a nilpotent normal subgroup of G. If $N \cap \Phi(G) = 1$ then $N \leq Soc(G)$, that is, $N = N_1 \times N_2 \times \cdots \times N_r$, where $N_1, N_2, \ldots, N_r$ are minimal normal subgroups of G.*

Lemma 2.6 ([9]). *Let G be a nonabelian simple group and H a subgroup of G. If $|G : H| = p^a$, where p is a prime. Then one of the following holds:*

(i) *$G = A_n$, $H \cong A_{n-1}$, where $n = p^a$;*

(ii) *$G = PSL_n(q)$, $|G : H| = (q^n - 1)/(q - 1) = p^a$;*

(iii) *$G = PSL_n(11)$, $H \cong A_5$;*

(iv) *$G = M_{23}$ and $H \cong M_{22}$ or $G = M_{11}$ and $H \cong M_{10}$;*

(v) *$G = PSU_4(2)$,the index of H in G is 27.*

Lemma 2.7. *Let G be a group, p the minimal prime divisor of the order of G and P a Sylow p-subgroup of G. If every maximal subgroup of P having no supersolvable supplement in G is weakly s-supplemented in G then G is p-nilpotent.*

Proof. It can be obtained directly from [7, Theorem C]. □

Lemma 2.8 ([2, III, 5.2 and IV, 5.4]). *Suppose that p is a prime and G is a minimal non-p-nilpotent group. Then*

(i) *G has a normal Sylow p-subgroup P and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*

(ii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(iii) *If P is abelian or $p > 2$, then $\exp(P) = p$.*

(iv) *If P is non-abelian and $p = 2$, then $\exp(P) = 4$.*

Lemma 2.9. *Let G be a group, p the minimal prime divisor of the order of G and P a Sylow p-subgroup of G. If every subgroup of P of order p or 4(when P is a non abelian 2-group) having no supersolvable supplement in G is weakly s-supplemented in G then G is p-nilpotent.*

Proof. By Lemma 2.1, one can verify that the hypotheses are subgroups closed. Thus if G is not p-nilpotent then we can assume that G is a minimal non-p-nilpotent group, and hence, by Lemma 2.8, $G = P \rtimes Q$, where Q is a cyclic q-group for some prime q, $P/\Phi(P)$ is a chief factor of G and $\exp P = p$ or 4 (when P is a noncyclic 2-group). Let $a \in P \setminus \Phi(P)$ and $H = \langle a \rangle$. Then $|H| = p$ or 4 (when P is a nonabelian 2-group). Thus H either has a supersolvable supplement in G or is weakly s-supplemented in G. Assume $|H| = 2$. If H has a complement T in G, then T is p-nilpotent since G is minimal non-p-nilpotent group. Since $|G : T| = |HT : T| = |H| = p$, $T \trianglelefteq G$. This induces that G is p-nilpotent, a contradiction. Thus G is the only supplement of H in G. If H has a supersolvable supplement in G, then G is supersolvable and so is p-nilpotent since p is minimal. If H is weakly s-supplemented in G then H is s-permutable in G and hence $H\Phi(P)/\Phi(P)$ is s-permutable in $G/\Phi(P)$. It follows from Lemma 2.3 that $H\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$ and so $P/\Phi(P) = H\Phi(P)/\Phi(P)$ is cyclic of order p. Hence $|G : Q\Phi(P)| = p$ and so $Q\Phi(P) \trianglelefteq G$. This implies that $G/\Phi(P)$ is cyclic and so is G, a contradiction. If $|H| = 4$, considering the subgroup $H\Phi(P)/\Phi(P)$ in $G/\Phi(P)$, a contradiction can be also obtained by a similar argument. Therefore the lemma holds. □

3. Proof of Theorem 1.3

Lemma 3.1. *Let P be a normal p-subgroup of G with $P \cap \Phi(G) = 1$. Assume that D is a subgroup of P with $1 < D < P$. If every subgroup of order $|D|$ of P having no supersolvable supplement in G is weakly s-supplement in G, then $P = P_1 \times P_2 \times \cdots \times P_r$, where $P_1, P_2, \ldots, P_r$ are all minimal normal in G of same order and $\iota(D) = m\iota(P_i)$, that is, $|D| = |P_i|^m$ for some positive integer m, $i = 1, 2, \ldots, r$.*

PROOF. By Lemma 2.5, $P = P_1 \times P_2 \times \cdots \times P_r$, where $P_1, P_2, \ldots, P_r$ are all minimal normal in G. Assume that there is a P_i such that $|D| < |P_i|$. Then P_i has a proper subgroup H of order $|D|$. Moreover, by the property of p-groups we can choose H to be normal in some Sylow p-subgroup G_p of G containing P. By the hypotheses, H either has a supersolvable supplement in G or is weakly s-supplement in G. Let T be a supplement of H in G. Then $G = HT = P_iT$ and $P_i \cap T \neq 1$ since H is proper in P_i. As P_i is abelian, $P_i \cap T \trianglelefteq P_iT = G$. But P_i is minimal normal in G, so $P_i \cap T = P_i$ and thereby $T = G$. Thus G is the only supplement of H in G. If H has a supersolvable supplement in G, then G is supersolvable and $|P_i| = p$, which contradicts that $|D| < |P_i|$. If H is weakly s-supplement in G then $H = H \cap G$ is s-permutable in G. By Lemma 2.3, $O^p(G) \leq N_G(H)$. It follows that $H \trianglelefteq G = G_pO^p(G)$. This is nonsense for P_i is minimal normal in G. Thus $|D| \geq |P_i|$ for any i.

If $|P_i| = |D|$ for any i, then we can see that the conclusion holds. Assume $|P_i| < |D|$ for some i. Without loss of generality, we can assume that $i = 1$. Clearly $P/P_1 = P_2P_1/P_1 \times \cdots \times P_rP_1/P_1$ and $P/P_1 \cap \Phi(G/P_1) = 1$ by [1, A,(9.11)]. By Lemma 2.1, one can verify that the hypotheses still hold on G/P_1. Thus $|P_2P_1/P_1| = \cdots = |P_rP_1/P_1|$ and $|D|/|P_1| = |P_2P_1/P_1|^{m_1}$ for some positive integer m_1. It follows that $|P_2| = \cdots = |P_r|$ and $|D|/|P_1| = |P_2|^{m_1}$. In particularly, $|P_2| < |D|$ and the hypotheses also hold on G/P_2. If $r \geq 3$, then $|P_1| = |P_3| = \cdots = |P_r|$. It follows that $|P_1| = |P_2| = |P_3| = \cdots = |P_r|$ and $|D| = |P_1|^{m_1+1} = |P_1|^m$ where $m = m_1 + 1$. If $r = 2$ then $P/P_1 \cong P_1P_2/P_1$ is minimal normal in G/P_1. Since the hypotheses hold on G/P_1, we can find a contradiction as above. Thus the lemma holds. □

Corollary 3.2. *Let P be a normal p-subgroup of G with $P \cap \Phi(G) = 1$. Assume that D is a subgroup of P with $1 < D < P$ and every subgroup of order $|D|$ of P having no supersolvable supplement in G is weakly s-supplement in G. If $(\iota(P), \iota(D)) = 1$ or $(\iota(|P : D|), \iota(D)) = 1$ then $P \subseteq Z_\infty^{\mathfrak{U}}(G)$.*

PROOF. By Lemma 3.1, $P = P_1 \times P_2 \times \cdots \times P_r$, where $P_1, P_2, \ldots, P_r$ are all minimal normal in G, and $\iota(P_i)$, $i = 1, \ldots, r$, is a common divisor of $\iota(P), \iota(D)$ and $\iota(|P : D|)$. Thus if $(\iota(P), \iota(D)) = 1$ or $(\iota(|P : D|), \iota(D)) = 1$ then $P \subseteq Z_\infty^{\mathfrak{U}}(G)$. □

Lemma 3.3. *Let P be a normal p-subgroup of G and D a subgroup of P with $1 < D < P$. Assume that every subgroup of order $|D|$ or $2|D|$ (when P is a nonabelian 2-group) of P having no supersolvable supplement in G is weakly s-supplement in G. If $P' < P \cap \Phi(G)$ or $|D| \leq |P'|$, then $P \subseteq Z_\infty^{\mathfrak{U}}(G)$.*

PROOF. Assume that the lemma does not hold and choose P to be a counter example with minimal order. We prove the lemma via the following steps.

(1) $P \nsubseteq \Phi(G)$ *and* $P/P\cap\Phi(G) = P_1/P\cap\Phi(G) \times P_2/P\cap\Phi(G) \times \cdots P_r/P\cap\Phi(G)$*, where* $P_i/P\cap\Phi(G)$*,* $i = 1, 2, \ldots, r$*, is minimal normal in* $G/P\cap\Phi(G)$*.*

If $P \subseteq \Phi(G)$, then for any subgroup H of P, G is the only supplement of H in G. If there is a subgroup H of order $|D|$ (or $2|D|$ when P is a non-abelian 2-group) has a supersolvable supplement in G then G is supersolvable and $P \subseteq Z_\infty^{\mathfrak{U}}(G)$. Now assume that every such subgroup H has no supersolvable supplement in G. Then H is weakly s-supplement in G and hence $H = H \cap G$ is s-permutable in G. It follows from [10, Lemma 3.1] that $P \subseteq Z_\infty^{\mathfrak{U}}(G)$. This contradiction shows that $P \nsubseteq \Phi(G)$. Clearly, $(P/P\cap\Phi(G))\cap\Phi(G/P\cap\Phi(G)) = 1$. Hence by Lemma 2.5, $P/P\cap\Phi(G) = P_1/P\cap\Phi(G)\times P_2/P\cap\Phi(G)\times\cdots P_r/P\cap\Phi(G)$, where $P_i/P\cap\Phi(G)$, $i = 1, 2, \ldots, r$, is minimal normal in $G/P\cap\Phi(G)$.

(2) $|D| > p$.

Assume that $|D| = p$. Then the hypotheses hold on P_i for any $i \in \{1, \ldots, r\}$. If $r > 1$, then $P_i \subseteq Z_\infty^{\mathfrak{U}}(G)$ for any i and so $P \subseteq Z_\infty^{\mathfrak{U}}(G)$. Assume that $r = 1$. Then $P/P\cap\Phi(G)$ is a G-chief factor. If $P\cap\Phi(G) = 1$, then $P\cap\Phi(G) \subseteq Z_\infty^{\mathfrak{U}}(G)$ clearly holds. If $P\cap\Phi(G) \neq 1$ then the hypotheses hold on $P\cap\Phi(G)$ and hence $P\cap\Phi(G) \subseteq Z_\infty^{\mathfrak{U}}(G)$. Thus $P\cap\Phi(G) \subseteq Z_\infty^{\mathfrak{U}}(G)$ always holds. Let H be any cyclic subgroup of order p or 4 (if P is a nonabelian 2-group). If $H(P\cap\Phi(G)) = P$ then $P/P\cap\Phi(G) \cong H/H\cap\Phi(G)$ is cyclic. Since $P\cap\Phi(G) \subseteq Z_\infty^{\mathfrak{U}}(G)$, we see that $P \subseteq Z_\infty^{\mathfrak{U}}(G)$. Assume that $H(P\cap\Phi(G)) < P$ for any such subgroup H. Let T be any supplement of H in G. We claim that $T = G$. If $T < G$, then $T(P\cap\Phi(G)) < G$. Since $PT = HT = G$ and $P/P\cap\Phi(G)$ is an abelian minimal normal subgroup of $G/P\cap\Phi(G)$, $T(P\cap\Phi(G))/P\cap\Phi(G)$ is a complement of $P/P\cap\Phi(G)$. But $G/(P\cap\Phi(G)) = HT/(P\cap\Phi(G))$, so $P = H(P\cap\Phi(G))$, a contradiction. Thus our claim holds. Consequently, G is the only supplement of H in G. Hence H is s-permutable in G if H has no supersolvable supplement in G by the hypotheses. By [10, Lemma 3.1], $P \subseteq Z_\infty^{\mathfrak{U}}(G)$, a contradiction. Hence (2) holds.

(3) $|P'| < |D|$.

Assume that $|D| \leq |P'|$. Since $P' \leq P\cap\Phi(G) < P_i$, $|D| < |P_i|$, $i = 1, 2, \ldots, r$. To prove $P \subseteq Z_\infty^{\mathfrak{U}}(G)$, it is sufficient to prove that $P_i \subseteq Z_\infty^{\mathfrak{U}}(G)$ for every i. If $P_i' = P'$, then $|D| \leq |P_i'|$. If $P_i' < P'$, then $P_i' < P_i\cap\Phi(G)$ since $P' \leq \Phi(P) \leq \Phi(G)$. Therefore, the hypotheses still hold on (G, P_i). If $P_i < P$, then by induction on $|P|$, we have that $P_i \subseteq Z_\infty^{\mathfrak{U}}(G)$. Now, assume that $P_i = P$. Then $P/P\cap\Phi(G)$ is a chief factor of G.

Suppose that $P' = P\cap\Phi(G)$. Then P/P' is a chief factor of G. Let H be a subgroup of order $|D|$ (or $2|D|$ when P is an nonabelian 2-group) of P. We claim that G is the only supplement of H in G. Clearly, if $H \leq P' \leq \Phi(G)$, then $HT = G$

if and only if $T = G$. Assume that $H \nsubseteq P'$ and T is a supplement of H in G. If $T < G$, then TP' is still a proper subgroup of G. Since $G = HT = PT$, we have that $(P/P')(TP'/P') = G/P'$. As P/P' is minimal normal in G/P' and TP'/P' is a proper subgroup in G/P', $(P/P')\cap(TP'/P') = 1$ and TP'/P' is a complement of P/P' in G/P'. But $G = HT$, so TP'/P' is a supplement of HP'/P' in G/P'. This induces that $HP'/P' = P/P'$ and so $H = P$, a contradiction. Thus our claim holds. It follows that all subgroups of order $|D|$ (and $2|D|$ when P is an nonabelian 2-group) in P having no supersolvable supplement in G are s-permutable in G and hence $P \subseteq Z^{\mathfrak{U}}_{\infty}(G)$ by [10, Lemma 3.1].

Assume that $P' < P \cap \Phi(G)$. Then by induction on P, $P \cap \Phi(G) \subseteq Z^{\mathfrak{U}}_{\infty}(G)$. Let N be a minimal normal subgroup of G contained in $P' < P \cap \Phi(G)$. Then N is of order p. By (2), $|D| > p$ and so the hypotheses still hold on $(G/N, P/N)$ by Lemma 2.1. Hence $P/N \subseteq Z^{\mathfrak{U}}_{\infty}(G/N)$. It follows directly from $|N| = p$ that $P \subseteq Z^{\mathfrak{U}}_{\infty}(G)$. This contradicts the choice of P and hence $|P'| < |D|$.

(4) $P' = 1$ *and* P *is abelian.*

Since $|D| \nleq |P'|$ by (3), $P' < P \cap \Phi(G)$ by the hypotheses. Then it can be verified that the hypotheses hold on $(G/P', P/P')$ by Lemma 2.1. If $P' \neq 1$, then $P/P' \subseteq Z^{\mathfrak{U}}_{\infty}(G/P')$ by induction. Since $P' \leq \Phi(P)$, It follows from Lemma 2.2 that $P \subseteq Z^{\mathfrak{U}}_{\infty}(G)$, which contradicts the choice of P. Thus (4) holds.

(5) *Let* N *be a minimal normal subgroup of* G *contained in* $P \cap \Phi(G)$. *Then* $|N| < |D|$

If $|D| < |N|$, then the hypotheses holds on (G, N) and hence $N \subseteq Z^{\mathfrak{U}}_{\infty}(G)$ by (1). It follows that $|N| = p$ and $|N| \leq |D|$, a contradiction.

Suppose that $|N| = |D|$. In this case, we claim that N is cyclic. Let L/N be a chief factor of G_p, where G_p is a Sylow p-subgroup of G. Then $|L| = p|N| = p|D|$. Let $\mho = \langle x^p \mid x \in L\rangle$. Then $\mho \leq \Phi(L) \leq N$. If $\mho = N$, then L is cyclic since $L/\Phi(L) = L/N$ is cyclic. It follows that N is cyclic. Assume that $\mho < N$. Clearly, $\mho \trianglelefteq G_p$ and hence there is a maximal subgroup M of N such that $\mho \leq M$ and $M \trianglelefteq G_p$. Choose an element $x \in L \setminus N$. Then $x^p \in \mho \leq M$ and $H = \langle M,\ x\rangle$ is of order $p|M| = |N| = |D|$. Let T be any supplement of H in G. Since P is abelian, $N \cap T \trianglelefteq PT = HT = G$. If $N \nsubseteq T$ then $N \cap T = 1$ by the minimality of N. Thus $|NT| = |N||T| \geq |HT| = G$ and thereby, $G = NT$, which is contrary to $N \subseteq \Phi(G)$. Hence $N \subseteq T$. Assume that H has a supersolvable supplement T in G. Then there is a cyclic subgroup R of N such that $R \trianglelefteq T$. Since P is abelian, $R \trianglelefteq PT = HT = G$. By the minimality of N, we have that $N = R$ is cyclic. Assume that H has no supersolvable supplement in G. Then H is weakly s-supplement in G by the hypotheses. It follows that $M = H \cap T \cap N$ is s-permutable in G. Since $M \trianglelefteq G_p$, $M \trianglelefteq G = G_pO^p(G)$. Again by the minimality of N, we have that $M = 1$ and so N is cyclic. Thus our claim holds. This implies that $|N| = |D| = p$. But $|D| > p$ by (2), a contradiction. Hence $|N| < |D|$ and (5) holds.

By the hypotheses, we can see that $P \cap \Phi(G) \neq 1$. In the following, N denotes always a minimal normal subgroup of G contained in $P \cap \Phi(G)$.

(6) *N is the unique minimal normal subgroup of G contained in P.*

Assume that this does not hold and P contains a minimal normal subgroup L of G different from N. Since $|N| < |D|$ by (5), by Lemma 2.1, every subgroup of order $|D|/|N|$ of P/N having no supersolvable supplement in G/N is weakly s-supplemented in G/N. If $P/N \cap \Phi(G/N) \neq 1$ then the hypotheses still hold on $(G/N, P/N)$ and so $P/N \subseteq Z_\infty^{\mathfrak{U}}(G/N)$. Hence L is of order p and $|L| < |D|$ since $|D| > p$ by (2). If $P/N \cap \Phi(G/N) = 1$, then by Lemma 3.1, $P/N = P_1/N \times P_2/N \times \cdots \times P_r/N$, where $P_1/N, P_2/N, \ldots, P_r/N$ are all minimal normal in G/N, $|P_1/N| = |P_2/N| = \cdots = |P_r/N|$ and $|D|/|N| = |P_1/N|^m$ for some positive integer m. In particularly, $|L| = |P_1/N| < |D|$. Thus $|L| < |D|$ holds in both case and hence the hypotheses hold on $(G/L, P/L)$ since, clearly, NL/L is contained in $\Phi(G/L)$. Thus $P/L \subseteq Z_\infty^{\mathfrak{U}}(G/L)$. In particularly, $N \cong NL/L$ is cyclic. If $P/N \subseteq Z_\infty^{\mathfrak{U}}(G/N)$, then $P \subseteq Z_\infty^{\mathfrak{U}}(G)$. Also, if L is cyclic, then $P \subseteq Z_\infty^{\mathfrak{U}}(G)$ since $P/L \subseteq Z_\infty^{\mathfrak{U}}(G/L)$. Assume that L is noncyclic and $P/N = P_1/N \times P_2/N \times \cdots \times P_r/N$, where $P_1/N, P_2/N, \ldots, P_r/N$ are all minimal normal in G/N and $|P_1/N| = |P_2/N| = \cdots = |P_r/N|$. Since $P/L \subseteq Z_\infty^{\mathfrak{U}}(G/L)$, any G-chief factor between L and P is cyclic. Thus if $r \neq 1$, then by [3, Theorem 1.6.8], there is some i such that $P_i/N \cong L$ is noncyclic and $P_j/N \cong Q/L$ is cyclic for any $j \neq i$, where Q/L is a G-chief factor contained in P. But $|P_i/N| = |P_j/N|$, a contradiction. Hence $r = 1$. This induces that P/N is minimal normal in G/N and $|P/N| = |P_1/N| < |D|/|N|$, a contradiction. Thus (6) holds.

(7) $\Phi(P) = 1$.

Assume that $\Phi(P) \neq 1$. Then $N \subseteq \Phi(P)$ by (6). If $P/N \subseteq Z_\infty^{\mathfrak{U}}(G/N)$, then $P \subseteq Z_\infty^{\mathfrak{U}}(G)$ by Lemma 2.5. Suppose that $P/N \nsubseteq Z_\infty^{\mathfrak{U}}(G/N)$. Then $\Phi(G/N) \cap P/N = 1$ and $P/N = P_1/N \times P_2/N \times \cdots \times P_r/N$ by Lemma 3.1. Moreover, $|P_1/N| = |P_2/N| = \cdots = |P_r/N|$ and $|D|/|N| = |P_1/N|^m$ for some positive integer m. Assume $\exp P = p$. Then $\Phi(P) = \mho_1(P) = 1$, where $\mho_1(P) = \{a^p \mid a \in P\} < P$. This contradicts $\Phi(P) \neq 1$. Assume $\exp P > p^2$. Then $\exp(P/N) > p$ and so $\Phi(P/N) \neq 1$. This contradicts that $\Phi(G/N) \cap P/N = 1$. Thus $\exp P = p^2$. If D is maximal in P, then $p = |P : D| = |P/N : D/N| = \frac{|P_1/N|^r}{|P_1/N|^m}$. Hence $r = m + 1$ and $|P_1/N| = |P_2/N| = \cdots = |P_r/N| = p$. This induces that $P/N \subseteq Z_\infty^{\mathfrak{U}}(G/N)$, a contradiction. Thus $|P : D| \geq p^2$. We claim that N is cyclic. Otherwise, there must be a subgroup H of order $|D|$ such that $H \cap N \neq 1$ since $\exp P = p^2$ and

$|P:D| \geq p^2$. Moreover, we can choose that $H \cap N \trianglelefteq G_p$, where G_p is some Sylow p-subgroup of G. Let T be any supplement of H in G. Then $P \cap T \trianglelefteq PT = HT = G$. Clearly $P \cap T \neq 1$ so $N \subseteq T$ as N is the only minimal normal subgroup of G contained in P. If T is supersolvable, then N has a cyclic subgroup R, which is normal in T. But P is abelian, so R is normal in G and hence $N = R$ is cyclic. Assume that H is weakly s-supplemented in G. Then $N \cap H \leq T \cap H \leq H_{sG}$ and so $H \cap N = H_{sG} \cap N$ is s-permutable in G. This induces that $O^p(G) \subseteq N_G(N \cap H)$ and so $N \cap H \trianglelefteq G$, which contradicts the minimality of N. Thus N is cyclic and our claim holds.

Since $\Phi((G/N) \cap (P/N)) = 1$, we have that $N = \Phi(P)$. Let $P = \langle a_1 \rangle \times \cdots \langle a_k \rangle \times \langle a_{k+1} \rangle \times \cdots \times \langle a_n \rangle$, where $|a_1| = \cdots = |a_k| = p^2$ and $|a_{k+1}| = \cdots = |a_n| = p$. Then $k = 1$ since $N = \Phi(P)$ is of order p. It follows that $\Omega = \Omega_1(P) = \{a \mid a^p = 1\}$ is a maximal subgroup of P. Clearly, $\Omega \cap \Phi(G) = P \cap \Phi(G) \neq 1$. Note that D is not maximal in P. Hence $|D| < |\Omega|$ and so $\Omega \subseteq Z_\infty^{\mathfrak{U}}(G)$ by induction. But P/Ω is of order p, so $P \subseteq Z_\infty^{\mathfrak{U}}(G)$. This contradiction shows that (7) holds.

The final contradiction

Since $\Phi(P) = 1$, P is an elementary abelian p-group. In particularly, N is complemented in P. Assume that N is noncyclic. Then there exists a subgroup H of order $|D|$ such that $1 < H \cap N < N$ and $H \cap N \trianglelefteq G_p$, where G_p is a Sylow p-subgroup of G. As above argument, one can find a contradiction. Thus N is cyclic. Let H_1/N be any subgroup of P/N of order $|D|$ and H be a complement of N in H_1. Then H is of order $|D|$. If H_1/N has no supersolvable supplement in G/N, then H has no supersolvable supplement in G and so is weakly s-supplement in G by hypotheses. Let T be any supplement of H in G with $H \cap T \leq H_{sG}$. Since $N \leq T$ by above argument, T/N is a supplement of H_1/N in G/N and $(H_1/N) \cap (T/N) = (H_1 \cap T)/N = (H \cap T)N/N \leq H_{sG}N/N$. Hence H_1/N is weakly s-supplemented in G/N. If $P/N \cap \Phi(G/N) \neq 1$, then the hypotheses hold on $(G/N, P/N)$ and so $P/N \subseteq Z_\infty^{\mathfrak{U}}(G/N)$. It follows from $|N| = p$ that $P \subseteq Z_\infty^{\mathfrak{U}}(G)$. If $P/N \cap \Phi(G/N) = 1$, then $P/N = Q_1/N \times Q_2/N \times \cdots \times Q_s/N$, where Q_i/N is minimal normal in G, $i = 1, \cdots, s$, $|Q_1/N| = |Q_2/N| = \cdots = |Q_s/N|$ and $|D| = |Q_1/N|^{m'}$ for some integer m' by Lemma 3.1. But we have that $P/N = P_1/N \times P_2/N \times \cdots \times P_r/N$, so $|Q_1/N| = |P_1/N|$ by [3, Theorem 1.6.8]. Thus $|P_1/N|^m = |D|/p = |P_1/N|^{m'}/p$ for some integers m and m'. This implies that $m' = m+1$ and $|P_1/N| = p$. Therefore $P/N \subseteq Z_\infty^{\mathfrak{U}}(G/N)$ and then, $P \subseteq Z_\infty^{\mathfrak{U}}(G)$ since N is cyclic. The final contradiction completes the proof. □

Lemma 3.4. *Let G be a group and p the minimal prime divisor of the order of G. Assume that P is a Sylow p-subgroup of G and D is a nontrivial proper subgroup of P. If every subgroup of P of order $|D|$ or $2|D|$ (when P is a nonabelian*

2-group) having no supersolvable supplement in G is weakly s-supplemented in G, then G is p-solvable and the p-length of G is 1.

Proof. Assume the lemma does not hold and let G be a counter example of minimal order. Then G is not p-nilpotent. We proceed the proof via the following steps.

(1) $O_{p'}(G) = 1$

By Lemma 2.1, it can be verified that the hypotheses still hold on $G/O_{p'}(G)$ and if $O_{p'}(G) \neq 1$ then $G/O_{p'}(G)$ is p-solvable and the p-length is 1. It follows that G is p-solvable and the p-length of G is 1. So we can assume that $O_{p'}(G) = 1$.

(2) *Let N be a minimal normal subgroup of G. If N is a p-group, then $|N| = |D|$.*

If $|D| < |N|$, then N has a proper subgroup H of order $|D|$. Since N is minimal normal in G and N is abelian, G is the only supplement of H in G. If H has a supersolvable supplement in G, then G is supersolvable and so the p-length of G is 1, which contradicts the choice of G. Assume that every such subgroup H is weakly s-supplemented in G. Then H is s-permutable in G. Without loss of generality, we can assume that H is normal in P. Then $H \trianglelefteq \langle P, O^p(G)\rangle = G$, which contradicts the minimality of N. Thus $|N| \leq |D|$.

If $|N| < |D|$, then the hypotheses still hold on G/N. Therefore, G/N is p-solvable and the p-length is 1. It follows that G is p-solvable. Assume that G has another minimal normal subgroup L. Then, similarly, G/L is also p-solvable and its p-length is 1. This induces that $G \cong G/N \cap L$ is p-solvable and the p-length of it is 1, a contradiction. Thus N is the unique minimal normal subgroup of G. Since the class of all p-solvable groups with the p-length is 1 is a saturated formation, we have that $N \nsubseteq \Phi(G)$. It follows that $G = N \rtimes M$ for some maximal subgroup M of G and $N = O_p(G)$. Clearly, $O_{p'}(M) \neq 1$ and $|P \cap M| > |D|/|N|$. Let P_1 be a subgroup of $P \cap M$ of order $p|D|/|N|$ and $M_1 = P_1 O_{p'}(M)N$. Then $P_1 N$ is a Sylow p-subgroup of M_1, and every maximal subgroup H of $P_1 N$ is of order $|D|$. Thus, if H has no supersolvable supplement in M_1, then H is weakly s-supplemented in M_1 by Lemma 2.1, and then by Lemma 2.7, M_1 is p-nilpotent. But $O_p(G) = N \leq M_1$, so $O_{p'}(M_1) = 1$ by Lemma 2.4. Thus M_1 is a p-group. This contradiction shows that (2) holds.

(3) $O_p(G) = 1$

Assume that $O_p(G) \neq 1$. Suppose $\Phi(G) \cap O_p(G) \neq 1$ and let N be a minimal normal subgroup of G contained in $\Phi(G) \cap O_p(G)$. By (2), $|N| = |D|$. If $N < O_p(G)$ then $O_p(G) \subseteq Z_\infty^{\mathfrak{U}}(G)$ by Lemma 3.3. Hence N is cyclic and $|N| = |D| = p$. It follows from Lemma 2.9 that G is p-nilpotent, a contradiction. Assume $N =$

$O_p(G)$. Let L/N be a minimal normal subgroup of P/N. Then $|L| = p|N| = p|D|$ and N is maximal in L. It follows that $\mho_1(L) = \langle x^p \mid x \in L\rangle \leq \Phi(L) \leq N$. If $\mho_1(L) = N$, then $N = \Phi(L)$ and hence L is cyclic. This induces that N is cyclic and so $|D| = |N| = p$. In this case, by Lemma 2.9, G is p-nilpotent, a contradiction. Therefore, $\mho_1(L) < N$. Let M be a maximal subgroup of N such that $M \trianglelefteq P$ and $\mho_1(L) \leq M$. Choose $x \in L \setminus N$. Then $H = \langle M, x\rangle$ is a subgroup of order $|N| = |D|$. Let T be a supplement of H in G. Suppose $T < G$. Since $N \leq \Phi(G)$, $TN < G$. But $|G : TN| = |TL : TN| = p\frac{|T\cap N|}{|T\cap L|} \leq p$, so TN is maximal in G and $|G : NT| = p$. Thus NT is normal in G by the minimality of p. If N is not a Sylow p subgroup of NT, then the hypotheses still hold on NT and hence NT is p-nilpotent. Since $|G : NT| = p$ and $NT \trianglelefteq G$, G is p-nilpotent. If N is a Sylow p subgroup of NT, then D is maximal in P and so G is p-nilpotent by Lemma 2.7. This contradiction shows that G is the only supplement of H in G. If T is supersolvable, then $G = T$ is p-nilpotent, a contradiction. Assume that H has no supersolvable supplement in G. Then H is weakly s-supplemented in G by the hypotheses and consequently, $H = H \cap G$ is s-permutable in G. It follows that $M = H \cap N$ is s-permutable in G and so $M \trianglelefteq PO^p(G) = G$. By the minimality of N, we have that $M = 1$ and N is cyclic of order p. Still by Lemma 2.9, G is p-nilpotent, a contradiction.

Suppose that $\Phi(G) \cap O_p(G) = 1$. Then $O_p(G)$ is abelian. Let N be a minimal normal subgroup of G contained in $O_p(G)$. Then N is of order $|D|$ by (2) and is complemented in G by $\Phi(G) \cap O_p(G) = 1$. Let $G = N \rtimes M$. Clearly, $O_p(M) \subseteq O_p(G)$. If $O_p(M) \neq 1$, then $O_p(M) \trianglelefteq MO_p(G) = G$. Let L be a minimal normal subgroup of G contained in $O_p(M)$. Then the order of L is $|D|$ by (2). If L is a Sylow p-subgroup of M, then $O_p(G) = NL$ is a Sylow p-subgroup of G. This is contrary to the choice of G. Thus the order of a Sylow p-subgroup of M is greater than $|D|$ and so, the hypotheses hold on M. Therefore, $G/N \cong M$ is p-solvable and the p-length of it is 1. By the same argument, we have that G/L is also p-solvable and the p-length of G/L is 1. Therefore, $G \cong G/L\cap N$ is p-solvable and the p-length of it is 1, which contradicts the choice of G. Hence $O_p(M) = 1$. Now assume that $O_{p'}(M) \neq 1$. Let $x \in P \cap M$ of order p and $P_1 = \langle N, x\rangle$. Then $|P_1| = p|D|$. Since $NO_{p'}(M) \trianglelefteq G$, $X = O_{p'}(M)P_1 = O_{p'}(G)NP_1$ is a subgroup of G and every maximal subgroup of P_1 is of order $|D|$. Hence by Lemma 2.7, X is p-nilpotent and so is $NO_{p'}(M)$. This induces that $O_{p'}(M)$ char $NO_{p'}(M) \trianglelefteq G$, which contradicts $O_{p'}(G) = 1$. Thus $O_{p'}(M) = O_p(M) = 1$ and in particularly, G is not solvable. If $p > 2$, then G is of odd order and so is solvable, a contradiction. Hence $p = 2$. Let R be a minimal subnormal subgroup of M. Then R is nonabelian and $p = 2$ is a divisor of $|R|$. Let $G_1 = NR$. Then the hypotheses still hold on G_1 since $|N| = |D| < |G_{1_p}|$, where G_{1_p} is a Sylow p-subgroup of G_1. If $G_1 < G$, then G_1 is p-solvable and so is R, a contradiction. Hence $G = G_1 = NR$. If the Sylow p-subgroup P is abelian, then $P \cap R \trianglelefteq P$ and so $(P \cap R)^G = (P \cap R)^R \leq R$. Since R is simple, we have that $R = (P \cap R)^G \trianglelefteq G$ and so $G = N \times R$. But N is minimal normal in G, so N is of order p. Thus G is p-nilpotent by Lemma 2.9 and $|N| = |D|$. Assume that P is nonabelian. Then every subgroup H of order $2|D| = 2|N|$ having no supersolvable supplement in G is weakly s-supplemented in G. By Lemma 2.1, it can be verified that every subgroup of order 2 of G/N having no supersolvable supplement in G/N is weakly s-supplemented in G/N. Since $R \cong RN/N = G/N$, every subgroup of order 2 of R having no supersolvable supplement in R is weakly s-supplemented in R. Let H be a subgroup of R of order 2 and T a supplement of H in R. If $T \neq R$, then $|R : T| = 2$ and so $T \trianglelefteq R$, which contradicts that R is simple. Hence $T = R$. If R supersolvable then G is solvable, a contradiction. Hence H is weakly s-supplemented in R. But R is the only supplement of H in R, so $H = H \cap R$ is s-permutable in R and hence $O_p(R) \neq 1$, which contradicts that R is simple. This contradiction shows that (3) holds.

(4) *G is simple.*

Let N be a minimal normal subgroup of G. By (1) and (3), N is a nonabelian pd-group. If $|P \cap N| \leq |D|$, then there is a subgroup P_1 of P with $N \cap P < P_1$ and $|P_1| = p|D|$. Let $X = NP_1$. Then every maximal subgroup of the Sylow p-subgroup $P \cap X$ of X is of order $|D|$. By Lemma 2.7, X is p-nilpotent and so is N, a contradiction. If $|P \cap N| > |D|$, then the hypotheses hold on N. If $N < G$, then N is p-solvable by the choice of G, a contradiction. Hence $N = G$ and G is simple.

The final contradiction

If $p > 2$, then G is solvable and so is abelian, a contradiction. Hence $p = 2$. Assume that P is abelian and H is a subgroup of P of order $|D|$. If H is weakly s-supplemented in G and let T be a supplement of H in G with $H \cap T \leq H_{sG}$. Then $T \neq G$ since H could not be s-permutable in G. Clearly, $P \cap T \neq 1$ since $D < P$. Hence $1 \neq (P \cap T)^G = (P \cap T)^{PT} = (P \cap T)^T \leq T < G$, which contradicts (4). Now consider that P is nonabelian. Then every subgroup of order $|D|$ or $2|D|$ having no supersolvable supplement in G is weakly s-supplemented in G. By Lemma 2.7, D is not maximal in P and so $2|D| < |P|$. If all subgroups of order $|D|$(or of order $2|D|$) have supersolvable supplements in G, then all maximal subgroups of P have supersolvable supplements in G. By Lemma 2.7, G is p-nilpotent, a contradiction. Hence there is a subgroup H_1 of order $|D|$ and a subgroup H_2 of order $2|D|$ are weakly s-supplemented in G. But G is simple, so both H_1 and H_2

are complemented in G. Hence there are subgroups T_1 and T_2 with $|G : T_1| = |D|$ and $|G : T_2| = 2|D|$. In view of Lemma 2.6, such a nonabelian simple group does not exist and our lemma holds. □

Proof of Theorem 1.3. We first prove that E satisfies Sylow tower property (see [11, p5]). In fact, by Lemma 2.1, it can be verified that the hypotheses still holds on E. If $E < G$ then $E \in \mathfrak{U}$ by induction and hence E satisfies Sylow tower property in this case. Now assume that $E = G$. Let p be the minimal prime divisor of $|G|$ and P a Sylow p-subgroup of G. It follows from Lemma 3.4 that G is p-solvable and the p-length of G is 1. Thus $G/O_{p'}(G)$ is p-closed. By Lemma 2.1, every subgroup of $G/O_{p'}(G)$ of order $|D|$ or $2|D|$ (when $PO_{p'}(G)/O_{p'}(G) \cong P$ is a nonabelian 2-group) having no supersolvable supplement in $G/O_{p'}(G)$ is weakly s-supplemented in $G/O_{p'}(G)$. If $\Phi(P) \neq P'$, then $P'O_{p'}(G)/O_{p'}(G) < \Phi(P)O_{p'}(G)/O_{p'}(G) \leq \Phi(G/O_{p'}(G)$. It follows from Lemma 3.3 that $PO_{p'}(G)/O_{p'}(G) \subseteq Z_\infty^{\mathfrak{U}}(G/O_{p'}(G))$. Assume that $P' = \Phi(P)$. If $|D| \leq |\Phi(P)|$ then $|D| \leq |P'|$. Again by Lemma 3.3, it holds that $PO_{p'}(G)/O_{p'}(G) \subseteq Z_\infty^{\mathfrak{U}}(G/O_{p'}(G))$. Assume that $|D| > |\Phi(P)|$. Then (iii) holds on P and so $(\iota(P/P'), \iota(|D|/|P'|)) = 1$ or $(\iota(P), \iota(|P : D|)) = 1$. By Corollary 3.2, it can be verified that $PO_{p'}(G)/P'O_{p'}(G) \subseteq Z_\infty^{\mathfrak{U}}(G/P'O_{p'}(G))$, and it follows from Lemma 2.2 that $PO_{p'}(G)/O_{p'}(G) \subseteq Z_\infty^{\mathfrak{U}}(G/O_{p'}(G))$. Since p is the minimal prime divisor of G, we have that $G/O_{p'}(G)$ is p-nilpotent and hence G is p'-closed. Since the hypotheses still hold on $O_{p'}(G)$, we have that $O_{p'}(G)$ satisfies Sylow tower property, and consequently G satisfies Sylow tower property.

Let q be the maximal prime divisor of $|E|$ and Q a Sylow q-subgroup of E. Then Q char $E \trianglelefteq G$ and as above argument, $Q \subseteq Z_\infty^{\mathfrak{U}}(G)$ since (i) or (ii) or (iii) holds on Q. We can also see that the hypotheses still holds on G/Q. Hence $G/Q \in \mathfrak{F}$ by induction on the order of G. Since $Q \subseteq Z_\infty^{\mathfrak{U}}(G)$ and $\mathfrak{U} \subseteq \mathfrak{F}$, we obtain that $G \in \mathfrak{F}$. Therefore the theorem holds. □

4. Remarks, examples and some corollaries

1. If D is minimal or maximal in P, then $(\iota(P), \iota(D)) = 1$ or $(\iota(P), \iota(|P : D|)) = 1$ and hence Theorems A and B in [7] are special cases of our results.

2. In Lemma 3.4 the minimality of p is necessary. In fact, if p is not the minimal prime divisor of the order of $|G|$, then we have the following counterexample.

Example 4.1. Let $A = Z_3 \rtimes Z_2 \cong S_3$, where Z_3 is a cyclic subgroup of order 3, Z_2 a cyclic subgroup of order 2 and S_3 is the symmetric group of degree 3. Let $B = A \wr Z_3$, the regular wreath product of A by Z_3. Put $G = O^2(B) = \langle x \mid o(x) = 3 \rangle$. Then $G \cong (Z_3 \times Z_3 \times Z_3) \rtimes A_4$, where A_4 is the alternative group of degree 4. Let P be a Sylow 3-subgroup of G. It can be proved that for any maximal subgroup H of P, H is complemented in G. So every maximal subgroup of P is weakly s-supplemented in G. But the p-length of G is not 1, where $p = 3$.

3. Clearly, if a subgroup H is normal, s-permutable or c-normal in G, then H is weakly s-supplement in G. Hence one can find the following special cases of Theorem 1.3 in the literature.

Corollary 4.2. ([12]) *Let G be a group of odd order. If all subgroups of G of prime order are normal in G, then G is supersolvable.*

Corollary 4.3. ([13]) *If the maximal subgroups of the Sylow subgroups of G are normal in G, then G is supersolvable.*

Corollary 4.4 ([14])**.** *If all subgroups of G of prime order or order 4 are c-normal in G, then G is supersolvable.*

Corollary 4.5 ([14])**.** *If the maximal subgroups of the Sylow subgroups of G are c-normal in G, then G is supersolvable.*

Corollary 4.6 ([15])**.** *If the maximal subgroups of the Sylow subgroups of G not having supersolvable supplement in G are normal in G, then G is supersolvable.*

Corollary 4.7 ([16])**.** *If the maximal subgroups of the Sylow subgroups of G not having supersolvable supplement in G are c-normal in G, then G is supersolvable.*

Corollary 4.8 ([17])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. If all minimal subgroups and all cyclic subgroups with order 4 of $G^{\mathfrak{F}}$ are c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 4.9 ([18])**.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with normal subgroup E such that $G/E \in \mathfrak{F}$. Assume that a Sylow 2-subgroup of G is abelian. If all minimal subgroups of E are permutable in G, then $G \in \mathfrak{F}$.*

Corollary 4.10 ([19])**.** *Let G be a solvable group. If all maximal subgroups of the Sylow subgroups of $F(E)$ are normal in G, then G is supersolvable.*

Corollary 4.11 ([18]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with a solvable normal subgroup E such that $G/E \in \mathfrak{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of E are weakly s-permutable in G, then $G \in \mathfrak{F}$.*

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