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Rings whose unit graphs are planar

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Abstract. The unit graph of a ring R is the simple graph $G(R)$ with vertex set R, where two distinct vertices x and y are adjacent if and only if $x + y$ is a unit of R. In this paper, we completely characterize the rings whose unit graphs are planar.

1. The result

Throughout, rings are associative with $1 \neq 0$. The group of units of a ring R is denoted by $U(R)$. This paper concerns the unit graph associated with a ring. Recall that the unit graph of a ring R, denoted $G(R)$, is the simple graph with vertex set R, where two distinct vertices x and y are adjacent if and only if $x + y \in U(R)$. The unit graph was first investigated in 1990 by GRIMALDI in [5] for $\mathbb{Z}_n$, the ring of integers modulo n. In 2010, ASHRAFI, *et al.* [2] generalized the unit graph $G(\mathbb{Z}_n)$ to $G(R)$ for an arbitrary ring R. The unit graph is also the topic of several other publications (see [1], [3] [6], [7], [8], [9], [10]).

The concentration is on the planarity of the unit graph of a ring. A graph is said to be planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. The planarity is an important invariant in graph theory. This work is motivated by the following result of ASHRAFI, *et al.* [2] who completely determined the finite commutative rings whose unit graphs are planar. We write $\mathbb{F}_p$ for the field of p elements and $R[t]$ for the polynomial ring over a ring R in the indeterminate t.

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Theorem 1.1 ([2]). *Let R be a finite commutative ring. Then $G(R)$ is planar if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_3, \mathbb{F}_4, \mathbb{Z}_5, \mathbb{Z}_3 \times \mathbb{Z}_3, B, \mathbb{Z}_3 \times B, \mathbb{F}_4 \times B, \mathbb{Z}_4, \frac{\mathbb{Z}_2[t]}{(t^2)}, \mathbb{Z}_4 \times B, \frac{\mathbb{Z}_2[t]}{(t^2)} \times B,$$

where B is a finite Boolean ring.

A natural question is to characterize the rings whose unit graphs are planar. This question is settled in this paper. We denote by $\mathrm{char}(R)$ the characteristic of a ring R and by $|X|$ the cardinal of a set X. Let $\mathbf{c} = |\mathbb{R}|$ be the cardinality of the continuum. Our main result is the following characterization of rings with planar unit graphs.

Theorem 1.2. *Let R be a ring. Then $G(R)$ is planar if and only if one of the following holds:*

(1) $|U(R)| \leq 3$ *and* $|R| \leq \mathbf{c}$.

(2) $|U(R)| = 4$, $\mathrm{char}(R) = 0$ *and* $|R| \leq \mathbf{c}$.

(3) $R \cong \mathbb{Z}_5$.

(4) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

2. The proof

We proceed with a series of lemmas. The first one is a quick consequence of Theorem 1.1.

Lemma 2.1. *Let R be a finite commutative ring. If $G(R)$ is planar, then $2 \leq \mathrm{char}(R) \leq 6$. Furthermore,*

(1) *If* $\mathrm{char}(R) = 2$, *then* $|U(R)| \leq 3$.

(2) *If* $\mathrm{char}(R) = 3$, *then* $|U(R)| \leq 4$.

(3) *If* $\mathrm{char}(R) = 4$, *then* $|U(R)| \leq 2$.

(4) *If* $\mathrm{char}(R) = 5$, *then* $|U(R)| \leq 4$.

(5) *If* $\mathrm{char}(R) = 6$, *then* $|U(R)| \leq 2$.

Let G be a simple graph. For a vertex v in G, the degree of v is the number of edges of G incident with v. For an integer $k > 0$, the graph G is called k-regular if the degree of each vertex of G is equal to k. The next lemma was proved in [2, Proposition 2.4] for a finite ring R and it can be shown by the same argument there.

Lemma 2.2. *Let R be a ring with $|U(R)| = k < \infty$. If $2 \notin U(R)$, then $G(R)$ is k-regular.*

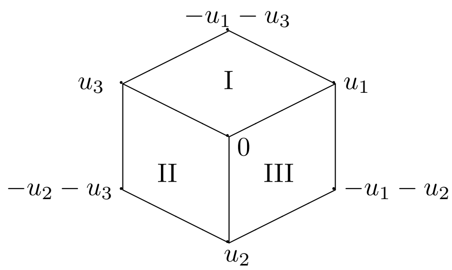
Let $K_{m,n}$ and K_n denote the complete bipartite graph with partitions of size m and n, and the complete graph of n vertices, respectively. A classical result of Kuratowski says that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ (see [11, Theorem 6.2.2]), where a subdivision of a graph G is a graph obtained from G by subdividing some of the edges, that is, by replacing the edges by paths having at most their endvertices in common. A quick consequence of Kuratowski's Theorem is that if the maximal degree of a graph is less than 3, then this graph must be planar. If a planar graph is finite, then the minimal degree of vertex is at most five. For an infinite graph, however, the situation is quite different. In fact, there exists a k-regular planar infinite graph for any positive integer k (see [4]). Of course, any subgraph of a planar graph is clearly planar.

Lemma 2.3. *Let R be a ring. If $G(R)$ is planar, then $|U(R)| < \infty$.*

Proof. Assume on the contrary that $|U(R)| = \infty$. Take $u_1 \in U(R)$ and $u_2 \in U(R)\backslash\{u_1, -u_1\}$. We show next that there is a contradiction.

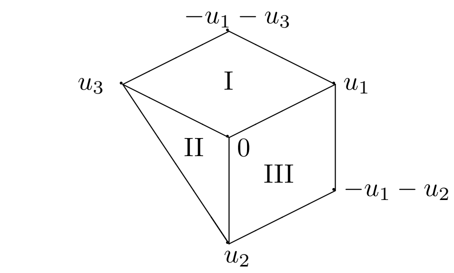
Case 1: $u_1 \neq -u_1 - u_2 \neq u_2$. In this case, we take $u_3 \in U(R)\backslash\{u_1, u_2, -u_1, -u_2, -u_1 - u_2\}$.

Subcase 1.1: $u_1 \neq -u_1 - u_3 \neq u_3$ and $u_2 \neq -u_2 - u_3 \neq u_3$. Then the following graph is a subgraph of $G(R)$:



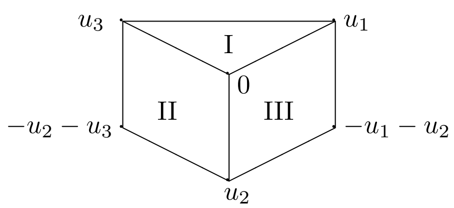
Now, take $v \in U(R)\backslash S$, where $S = \{u_1, u_2, u_3, -u_1, -u_2, -u_3, -u_1 - u_2, -u_1 - u_3, -u_2 - u_3, u_1 + u_2 - u_3, u_1 + u_3 - u_2, u_2 + u_3 - u_1\}$. Since $G(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, put v into region (I). Note that $-v - u_2$ is adjacent to both v and u_2. As $G(R)$ is planar, $-v - u_2$ must be one of the vertices 0, u_1, u_3, $-u_1 - u_3$. But this contradicts the choice of v.

Subcase 1.2: $u_1 \neq -u_1 - u_3 \neq u_3$ and $-u_2 - u_3 = u_2$ or u_3. Then the following graph is a subgraph of $G(R)$:



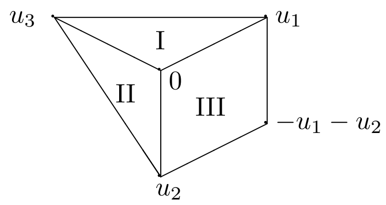
Now, take $v \in U(R)\backslash S$, where $S = \{u_1, u_2, u_3, -u_1, -u_2, -u_3, -u_1 - u_2, -u_1 - u_3, u_1 + u_2 - u_3, u_1 + u_3 - u_2\}$. Since $G(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, put v into region (I). Note that $-v - u_2$ is adjacent to both v and u_2. As $G(R)$ is planar, $-v - u_2$ must be one of the vertices 0, u_1, u_3, $-u_1 - u_3$. But this contradicts the choice of v.

Subcase 1.3: $-u_1 - u_3 = u_1$ or u_3, and $u_2 \neq -u_2 - u_3 \neq u_3$. Then the following graph is a subgraph of $G(R)$:



Now, take $v \in U(R)\backslash S$, where $S = \{u_1, u_2, u_3, -u_1, -u_2, -u_3, -u_1 - u_2, -u_2 - u_3, u_1 + u_2 - u_3, u_2 + u_3 - u_1\}$. Since $G(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, put v into region (I). Note that $-v - u_2$ is adjacent to both v and u_2. As $G(R)$ is planar, $-v - u_2$ must be one of the vertices 0, u_1, u_3. But this contradicts the choice of v.

Subcase 1.4: $-u_1 - u_3 = u_1$ or u_3, and $-u_2 - u_3 = u_2$ or u_3 (of course, it can't occur that $-u_1 - u_3 = u_3$ and $-u_2 - u_3 = u_3$). Then the following graph is a subgraph of $G(R)$:



Now, take $v \in U(R)\backslash S$, where $S = \{u_1, u_2, u_3, -u_1, -u_2, -u_3, -u_1 - u_2, u_1 + u_2 - u_3\}$. Since $G(R)$ is planar and v is adjacent to 0, v must be in one of the regions (I), (II) and (III). Without loss of generality, put v into region (I). Note that $-v - u_2$ is adjacent to both v and u_2. As $G(R)$ is planar, $-v - u_2$ must be one of the vertices 0, u_1, u_3. But this contradicts the choice of v.

Case 2: $-u_1 - u_2 = u_1$ or u_2. Take $u_3 \in U(R)\backslash\{u_1, u_2, -u_1, -u_2\}$. A similar argument as in Case 1 yields a contradiction. □

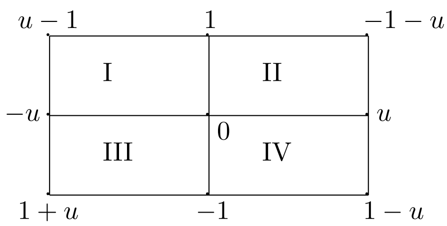
Lemma 2.4 is a self-strengthening of Lemma 2.3.

Lemma 2.4. *Let R be a ring. If $G(R)$ is planar, then $|U(R)| \leq 4$.*

Proof. Assume on the contrary that $|U(R)| \geq 5$. To get a contradiction, we proceed with two cases.

Case 1: $\mathrm{char}(R) = 0$. Then R contains $\mathbb{Z}$ as a subring. Since $|U(R)| < \infty$ by Lemma 2.3, $n \notin U(R)$ for all $\pm 1 \neq n \in \mathbb{Z}$. Take $\pm 1 \neq u \in U(R)$.

Subcase 1.1: $2u \neq -2$ and $2u \neq 2$. That is, $-1-u \neq 1+u$ and $u-1 \neq 1-u$. In this case, the following graph is a subgraph of $G(R)$:



Now, take $v \in U(R)\backslash\{1, -1, u, -u\}$. Since $G(R)$ is planar and v is adjacent to 0, either v is in one of the regions (I), (II), (III) and (IV), or v is one of the vertices $u-1$, $-1-u$, $1-u$ and $1+u$.

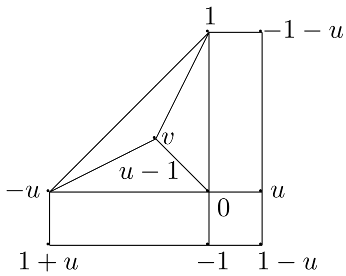
If v is in region (I), consider the vertices $1-v$ and $-u-v$. As $1-v$ is adjacent to both v and -1, we have $1 - v = -u$ or $1 - v = u - 1$. As $-u - v$ is adjacent to both v and u, we have $-u - v = 1$ or $-u - v = u - 1$. Thus, we must have a contradiction: If $1 - v = -u$ and $-u - v = 1$, then $2v = 0$, i.e. $2 = 0$; If $1 - v = u - 1$ and $-u - v = 1$, then $3 = 0$; If $1 - v = -u$ and $-u - v = u - 1$, then $3u = 0$, i.e. $3 = 0$; If $1 - v = u - 1$ and $-u - v = u - 1$, then $u = -1$.

If v is in region (II), consider the vertices $1 - v$ and $u - v$. Arguing as above, we have $1 - v = -1 - u$ or $1 - v = u$, and $u - v = 1$ or $u - v = -1 - u$. This clearly leads to a contradiction.

If v is in region (III), consider the vertices $-1 - v$ and $-u - v$. Then we have $-1 - v = -u$ or $-1 - v = 1 + u$, and $-u - v = 1 + u$ or $-u - v = -1$. This also leads to a contradiction.

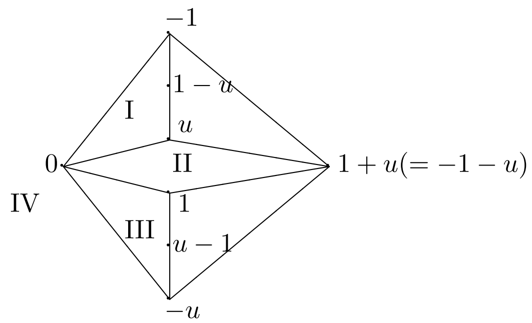
If v is region in (IV), consider the vertices $-1 - v$ and $u - v$. Then we have $-1 - v = u$ or $-1 - v = 1 - u$, and $u - v = -1$ or $u - v = 1 - u$, and this also leads to a contradiction.

If v is one of the vertices $u - 1$, $-1 - u$, $1 - u$ and $1 + u$, we can assume that $v = u - 1$ (the other cases are similar). Note that 1 is adjacent to $-u$. So we have the following subgraph of $G(R)$:



As $-u - v$ is adjacent to both v and u, we must have $-u - v = 1$. As $1 - v$ is adjacent to both v and -1, we must have $1 - v = -u$. Thus, $2v = 0$, i.e. $2 = 0$, a contradiction.

Subcase 1.2: $2u = -2$, i.e. $-1 - u = 1 + u$. In this case, $u - 1 \neq 1 - u$, so the following graph is a subgraph of $G(R)$:



Take $v \in U(R) \backslash \{1, -1, u, -u\}$. Then either v is in one of the regions (I), (II), (III) and (IV) or $v \in \{1 - u, u - 1, 1 + u\}$.

If v is in region (I), consider the vertices $-1 - v$ and $u - v$. As $-1 - v$ is adjacent to both v and 1, we have $-1 - v = u$. As $u - v$ is adjacent to both v and $-u$, we have $u - v = -1$. It follows that $-2v = 0$, i.e. $2 = 0$, a contradiction.

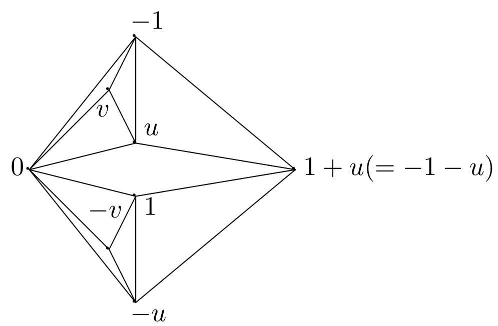
If v is in region (II), consider the vertices $1 - v$ and $u - v$. Arguing as above, we have $1 - v = u$ and $u - v = 1$, which gives $-2v = 0$, i.e. $v = 0$, a contradiction.

If v is in region (III), consider the vertices $1 - v$ and $-u - v$ and we have $1 - v = -u$ and $-u - v = 1$, giving $-2v = 0$, i.e. $v = 0$, a contradiction.

If v is in region (IV), consider the vertices $-1 - v$ and $-u - v$ and we have $-1 - v = -u$ and $-u - v = -1$, giving $-2v = 0$, i.e. $v = 0$, a contradiction.

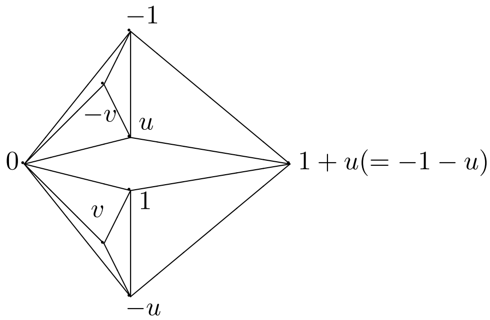
Now assume $v \in \{1 - u, u - 1, 1 + u\}$. If $v = 1 + u$, then 0 is adjacent to $1 + u$ and 1 is adjacent to u. This is impossible.

If $v = 1 - u$, then $G(R)$ has the following subgraph:



In this case, we consider the vertices $-1 - v$ and $u - v$. As $-1 - v$ is adjacent to both 1 and v, we have $-1-v=u$; as $u-v$ is adjacent to both $-u$ and v, we have $u-v=-1$. So $-2v=0$, i.e. $2=0$, a contradiction.

If $v=u-1$, $G(R)$ has the following subgraph:



In this case, we consider the vertices $1-v$ and $-u-v$. As $1-v$ is adjacent to both -1 and v, we have $1-v=-u$; as $-u-v$ is adjacent to both u and v, we have $-u-v=1$. So $-2v=0$, i.e. $2=0$, a contradiction.

Subcase 1.3: $2u=2$, i.e. $u-1=1-u$. In this case, $-1-u\neq 1+u$. By a similar process as Subcase 1.2, we also can get a contradiction.

Case 2: $\mathrm{char}(R)=n\geq 2$. Then R contains $\mathbb{Z}_n$ as a subring. Since $G(\mathbb{Z}_n)$ is planar, we have $n\leq 6$ by Lemma 2.1. We need two notations. For any $a\in R$, let $\mathbb{Z}_n[a]$ be the subring of R generated by $\mathbb{Z}_n\cup\{a\}$. Note that $G(\mathbb{Z}_n[a])$ is also planar. For $u\in U(R)$, let $o(u)$ be the order of u in the multiplicative group $U(R)$. Then $o(u)<\infty$ for all $u\in U(R)$ by Lemma 2.3.

Subcase 2.1: $n=6$. Take $\pm 1\neq u\in U(R)$. As $o(u)<\infty$, $\mathbb{Z}_6[u]$ is a finite commutative ring. So, by Lemma 2.1(5), $|U(\mathbb{Z}_6[u])|\leq 2$. But $\mathbb{Z}_6[u]$ has at least three units, a contradiction.

Subcase 2.2: $n=5$. Take $u\in U(R)\backslash U(\mathbb{Z}_5)$. Then $\mathbb{Z}_5[u]$ is a finite commutative subring of R. So, by Lemma 2.1(4), $|U(\mathbb{Z}_5[u])|\leq 4$. But $\mathbb{Z}_5[u]$ has at least five units, a contradiction.

Subcase 2.3: $n=4$. Take $\pm 1\neq u\in U(R)$. Then $\mathbb{Z}_4[u]$ is a finite commutative subring of R. So, by Lemma 2.1(3), $|U(\mathbb{Z}_4[u])|\leq 2$. But $\mathbb{Z}_4[u]$ has at least three units, a contradiction.

Subcase 2.4: $n=3$. Take $\pm 1\neq u\in U(R)$. As above, $\mathbb{Z}_3[u]$ is a finite commutative subring of R. So, by Lemma 2.1(2), we have $|U(\mathbb{Z}_3[u])|\leq 4$. In particular, $o(u)\leq 4$. If $o(u)=4$ and $u^2=-1$, then $\mathbb{Z}_3[u]$ contains at least 8 units: 1, -1, u, $-u$, $1+u$, $1-u$, $-1+u$ and $-1-u$, a contradiction. If $o(u)=4$ and $u^2\neq -1$, then 1, 2, u, u^2, u^3 are five distinct units of $\mathbb{Z}_3[u]$, a contradiction. If $o(u)=3$, then 1, 2, u, $2u$, u^2, $2u^2$ are six distinct units of $\mathbb{Z}_3[u]$, a contradiction.

Hence $o(u) = 2$, and in this case, $U(\mathbb{Z}_3[u]) = \{1, 2, u, 2u\}$. Note that the argument above already shows that $v^2 = 1$ for all $v \in U(R)$. So the group $U(R)$ is abelian. As $|U(R)| \geq 5$, take $v \in U(R)\backslash U(\mathbb{Z}_3[u])$. Consider the subring $\mathbb{Z}_3[u, v]$ of R generated by $\mathbb{Z}_3[u] \cup \{v\}$. Then $\mathbb{Z}_3[u, v]$ is a finite commutative ring containing at least 5 units: 1, 2, u, $2u$, v. This contradicts Lemma 2.1(2).

Subcase 2.5: $n = 2$. Let $H = U(R)$. For $u \in H$, $\mathbb{Z}_2[u]$ is a finite commutative ring. So, by Lemma 2.1(1), we have $|U(\mathbb{Z}_2[u])| \leq 3$. In particular, $o(u) \leq 3$. Thus, we have proved that $o(u) \leq 3$ for all $u \in H$.

If $H \cong S_3$, the symmetric group of degree 3, then the subring $\mathbb{Z}_2[H]$ of R generated by $\mathbb{Z}_2 \cup H$ is a finite ring containing exactly six units such that 2 is not a unit of $\mathbb{Z}_2[H]$. Hence, by [2, Proposition 2.4], $G(\mathbb{Z}_2[H])$ is 6-regular. In particular, $G(\mathbb{Z}_2[H])$ is not planar, and so $G(R)$ is not planar. This contradiction shows that H is not isomorphic to S_3. To finish the proof, we need the following claim. □

Claim. There exist $u, v \in H\backslash\{1\}$ such that $uv = vu$ and $\langle u \rangle \cap \langle v \rangle = \{1\}$.

Proof of Claim. As above, we have $|H| = 2^k 3^l$, where $k, l \geq 0$. Note that $|H| \geq 5$ by hypothesis. If $k = 0$ or $l = 0$, there is nothing to prove because any finite p-group has nontrivial center. If $k > 1$, consider a Sylow 2-subgroup P of H. Being a finite p-group, P contains a non-trivial central element, say u. As $|\langle u \rangle| \leq 3$ and $|P| \geq 2^k \geq 4$, we can take $v \in P\backslash\langle u \rangle$. Then $uv = vu$ and $\langle u \rangle \cap \langle v \rangle = \{1\}$. If $l > 1$, we can consider a Sylow 3-subgroup and a similar argument also shows the existence of such elements u and v. If $k = l = 1$, then $|H| = 6$. As $H \ncong S_3$, H is a cyclic group of order 6. But this is impossible, as every element of H has order less than or equal to 3. The completes the proof of Claim.

Now by the Claim, take $u, v \in H\backslash\{1\}$ such that $uv = vu$ and $\langle u \rangle \cap \langle v \rangle = \{1\}$. Then the subring $\mathbb{Z}_2[u, v]$ of R generated by $\mathbb{Z}_2 \cup \{u, v\}$ is a finite commutative ring, containing at least four distinct units 1, u, v, uv. This contradicts Lemma 2.1(1).

The proof is now complete. □

Our last lemma is about the genus of a simple graph. A surface is said to be of genus g if it is topologically homeomorphic to a sphere with g handles. A graph G that can be drawn without crossing on a compact surface of genus g, but not on one of genus $g - 1$, is called a graph of genus g. The genus of a graph G is denoted by $\gamma(G)$. Not that a graph is planar if and only if it has genus zero.

Lemma 2.5 ([12, Corollaries 6.14, 6.15])**.** *Suppose that a simple graph G is connected with $p \geq 3$ vertices and q edges. Then $\gamma(G) \geq \frac{q}{6} - \frac{p}{2} + 1$. Furthermore, if G has no triangles, then $\gamma(G) \geq \frac{q}{4} - \frac{p}{2} + 1$.* □

Now we are ready to prove our main result.

Proof of Theorem 1.2. ($\Longrightarrow$). Suppose that $G(R)$ is planar. Then R embeds in $\mathbb{R}\times\mathbb{R}$ as sets, so $|R|\le \mathbf{c}$. By Lemma 2.4, $|U(R)|\le 4$. If $|U(R)|=3$, we are done. So we can assume that $|U(R)|=4$, and we can further assume $n:=\mathrm{char}(R)>0$. Then R contains $\mathbb{Z}_n$ as a subring. Being a subgraph of $G(R)$, $G(\mathbb{Z}_n)$ is planar, so $2\le n\le 6$ by Lemma 2.1. Take $\pm 1\ne u\in U(R)$. Then $\mathbb{Z}_n[u]$ is a finite commutative subring of R, and hence $G(\mathbb{Z}_n[u])$ is planar. If $n=4$ or $n=6$, then $\mathbb{Z}_n[u]$ contains at least three units; this is impossible by Lemma 2.1(3,4). So $n\ne 4$ and $n\ne 6$. Next we prove that $n\ne 2$. Assume that $n=2$. Then, for any $1\ne u\in U(R)$, $\mathbb{Z}_2[u]$ is a finite commutative subring of R, and hence $o(u)\le 3$ by Lemma 2.1(1). If $o(u)=3$, take $v\in U(R)\backslash\{1,u,u^2\}$ and we see 1, u, u^2, v, uv are five distinct units of R, contradicting that $|U(R)|=4$. Hence $o(u)\le 2$ for all $u\in U(R)$. So $U(R)$ is a commutative multiplicative group. Take $1\ne u\in U(R)$ and $v\in U(R)\backslash\{1,u\}$. Then $\mathbb{Z}_2[u,v]$ is a finite commutative subring of R containing four units 1, u, v, uv. But this is impossible by Lemma 2.1(1). Hence $n\ne 2$. Thus, we have proved that $n=3$ or $n=5$.

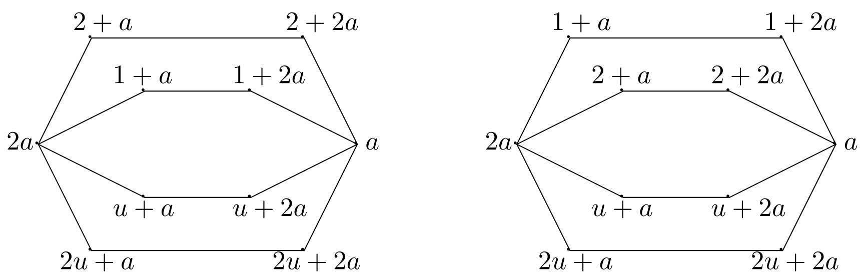
Suppose $n=3$. We prove that $R\cong\mathbb{Z}_3\times\mathbb{Z}_3$. Take $\pm1\ne u\in U(R)$. Then $\mathbb{Z}_3[u]$ is a finite commutative subring of R, and $U(\mathbb{Z}_3[u])=\{1,2,u,2u\}$ (as $|U(R)|=4$). If $R\ne\mathbb{Z}_3[u]$, take $a\in R\backslash\mathbb{Z}_3[u]$ and consider the subring $\mathbb{Z}_3[u,a]$ of R generated by $\mathbb{Z}_3\cup\{u,a\}$. Note that

$$a\longleftrightarrow 1+2a\longleftrightarrow 1+a\longleftrightarrow 2a\longleftrightarrow u+a\longleftrightarrow u+2a\longleftrightarrow a$$

and

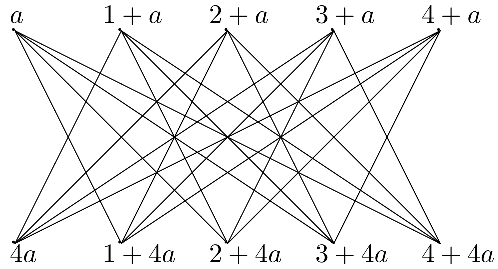
$$a\longleftrightarrow 2+2a\longleftrightarrow 2+a\longleftrightarrow 2a\longleftrightarrow 2u+a\longleftrightarrow 2u+2a\longleftrightarrow a$$

are two 6-cycles in $G(\mathbb{Z}_3[u,a])$. By symmetry, essentially there are two ways to draw the subgraph below:



For the subgraph on the left, as $u + 2 + 2a$ is adjacent to both $1 + a$ and $2u + a$, the planarity of $G(R)$ ensures that $u + 2 + 2a = a$. On the other hand, as $u + 2 + a$ is adjacent to both $1 + 2a$ and $2u + 2a$, the planarity of $G(R)$ ensures that $u + 2 + a = 2a$. So, it follows that $a = -a$, i.e., $2a = 0$ or $a = 0$, a contradiction. For the subgraph on the right, as $u + 1 + 2a$ is adjacent to both $2 + a$ and $2u + a$, the planarity of $G(R)$ ensures that $u + 1 + 2a = a$. On the other hand, as $u + 1 + a$ is adjacent to both $2 + 2a$ and $2u + 2a$, the planarity of $G(R)$ ensures that $u + 1 + a = 2a$. So, it follows that $a = -a$, i.e., $2a = 0$ or $a = 0$, a contradiction. Therefore, $R = \mathbb{Z}_3[u]$ with $\mathbb{Z}_3[u] \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Suppose $n = 5$. We prove that $R \cong \mathbb{Z}_5$. We see that R contains $\mathbb{Z}_5$ as a subring. Assume on the contrary that $R \neq \mathbb{Z}_5$. Take $a \in R \backslash \mathbb{Z}_5$. The following graph H is a subgraph of $G(\mathbb{Z}_5[a])$, and hence of $G(R)$:



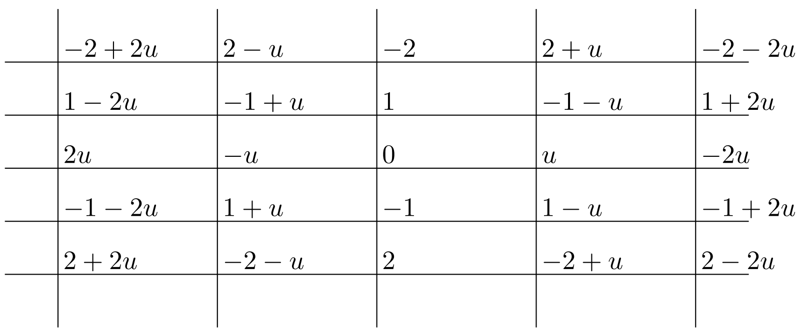
Note that H has 10 vertices and 20 edges with no triangles. So $\gamma(H) \geq 1$ by Lemma 2.5. This shows that H is not planar, giving the contradiction that $G(R)$ is not planar.

($\Longleftarrow$). We have $|R| \leq \mathbf{c}$. If $R \cong \mathbb{Z}_5$ or $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $G(R)$ is planar by Theorem 1.1. If $|U(R)| \leq 2$, then the maximal degree of $G(R)$ is at most two, so $G(R)$ must be planar.

Suppose that $|U(R)| = 3$. Then we easily see that $2 = 0$ in R. So $G(R)$ is 3-regular by Lemma 2.2. Let $U(R) = \{u_1, u_2, u_3\}$. For a given $r \in R$, r is adjacent to $u_i - r$ $(i = 1, 2, 3)$. If $u_1 - r$ is adjacent to one of $u_i - r$ $(i = 2, 3)$, say $u_2 - r$, then $(u_1 - r) + (u_2 - r) = u_1 + u_2$ is a unit of R, so it must be that $u_1 + u_2 = u_3$. Thus $u_1 - r$ is also adjacent to $u_3 - r$ and $u_2 - r$ is adjacent to $u_3 - r$. Hence, the vertices r, $u_1 - r$, $u_2 - r$, $u_3 - r$ form a complete graph K_4. As $G(R)$ is 3-regular, $G(R)$ must be a disjoint union of copies of K_4, so $G(R)$ is planar. Therefore, we can let the neighborhoods of $u_1 - r$ be r, a, b, where $a, b \notin \{u_2 - r, u_3 - r\}$. We may assume $u_1 - r + a = u_2$ and $u_1 - r + b = u_3$. Then $u_2 - r + a = u_1$ and $u_3 - r + b = u_1$. This means that a is adjacent to $u_2 - r$

and b is adjacent to $u_3 - r$. Let c be the third neighborhood of $u_2 - r$. Then $u_2 - r + c = u_3$, so $u_3 - r + c = u_2$. This means that c is also a neighborhood of $u_3 - r$. Now consider the vertex a. Let the neighborhoods of a be $u_1 - r$, $u_2 - r$, x. Then $a + x = u_3$. As $b + x = b + u_3 - a = r + u_1 - a = u_1 - r + a = u_2$, x is adjacent to b. Similarly, x is adjacent to c. So, the vertices r, $u_1 - r$, $u_2 - r$, $u_3 - r$, a, b, c and x form a cube, which is 3-regular. As $G(R)$ is 3-regular, $G(R)$ must be a disjoint union of copies of a cube. As a cube is a planar graph, $G(R)$ is planar.

Finally, suppose that $|U(R)| = 4$ and $\mathrm{char}(R) = 0$. Then R contains $\mathbb{Z}$ as a subring. Take $\pm 1 \neq u \in U(R)$. As $|U(R)| = 4$, we have $U(R) = \{1, -1, u, -u\}$. By Lemma 2.2, both $G(\mathbb{Z}[u])$ and $G(R)$ are 4-regular. It follows that $G(R)$ is a disjoint union of $G(\mathbb{Z}[u])$. As shown below, $G(\mathbb{Z}[u])$ is planar, so $G(R)$ is planar.



Graph $G(\mathbb{Z}[u])$

□

We end the paper by giving some examples of rings with planar unit graphs.

Example 2.6. *Let $\mathbb{T}_2(\mathbb{Z}_2)$ be the 2×2 upper triangular matrix ring over $\mathbb{Z}_2$ and let B be the zero ring or a finite Boolean ring. Then $R = \mathbb{T}_2(\mathbb{Z}_2) \times B$ has a planar unit graph.*

A ring R is semilocal if $R/J(R)$ is semisimple Artinian, where $J(R)$ is the Jacobson radical of R. The next example gives a countable non-semilocal ring whose unit graph is planar. Let D be a ring and C be a subring of D. With addition and multiplication defined componentwise, $\mathcal{R}[D, C] := \{(d_1, \cdots, d_n, c, c, \cdots) : d_i \in D, c \in C,\ n \geq 1\}$ becomes a ring. For a bimodule M over a ring R, the trivial extension of R by M is the ring $R \propto M := \{(a, x) : a \in R,\ x \in M\}$ with addition defined componentwise and with multiplication defined by $(a, x)(b, y) = (ab, ay + xb)$.

Example 2.7. *Let $S = R \propto R/I$ where $R = \mathcal{R}[\mathbb{Z}_2, \mathbb{Z}_2]$ and $I = \mathcal{R}[\mathbb{Z}_2, 0]$. Then S is not semilocal, but $G(S)$ is planar.*

Proof. We easily see that $J(S) = \{(0, x) : x \in R/I\}$, so $|J(S)| = |R/I| = 2$, and $S/J(S) \cong R$ is Boolean. Since $S/J(S)$ is an infinite Boolean ring, S is not semilocal. As $|U(S)| = 2$, $G(S)$ is planar by Theorem 1.2. $\square$

Some other examples of rings with planar unit graphs can be constructed through polynomial rings. In [1], the authors determined the finite rings R with $G(R[t])$ planar. By Theorem 1.2, we now can characterize the rings R with $G(R[t])$ planar. Remark that, for a reduced ring R, $U(R[t]) = U(R)$ (we can't find a reference for this, but it can be easily proved).

Corollary 2.8. *Let R be a ring, and let $t_1, t_2, \dots, t_n$ be commuting indeterminates over R. Then $G(R[t_1, t_2, \dots, t_n])$ is planar if and only if R is reduced with $|R| \leq \mathbf{c}$ such that either $|U(R)| \leq 3$, or $|U(R)| = 4$ with $\mathrm{char}(R) = 0$.*

Proof. Without loss of generality, we can assume that $n = 1$.

($\Longleftarrow$). This is by Theorem 1.2 and the Remark above.

($\Longrightarrow$). As $G(R[t])$ is planar, R is reduced by [1, Proposition 6.1(ii)], and $|R[t]| \leq \mathbf{c}$. So $|R| \leq \mathbf{c}$. Moreover, by Theorem 1.2, either $|U(R[t])| \leq 3$, or $|U(R[t])| = 4$ with $\mathrm{char}(R) = 0$. Since R is reduced, $U(R[t]) = U(R)$. So the claim follows. $\square$

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