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Sublattices of verbal subgroups

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Abstract. The problem of classification of group varieties is still open. We consider four classes of verbal subgroups of a free group F of rank 2: $\{VN\text{-verbal}\} \subseteq \{P\text{-verbal}\} \subseteq \{R\text{-verbal}\} \subseteq \{M\text{-verbal}\}$. The subgroups in each class define specific properties in corresponding varieties, namely, VN-varieties have their 2-generator groups virtually nilpotent; P-varieties satisfy positive laws; R-varieties are restrained; and M-varieties contain no $\mathfrak{A}_p\mathfrak{A}$ as a subvariety. It is shown that each of these classes of verbal subgroups forms a sublattice of the lattice of subgroups in F. Three questions are posed.

1. Introduction

The problem of classification of group varieties attracted attention of many authors. We make a step in this direction by distinguishing four sublattices of group varieties according to their properties defined by 2-variable laws they satisfy.

Let $F = \langle x, y\rangle$ be a free group of rank 2 and $\mathcal{F}$ a free semigroup on the set $\{x, y\}$. We denote $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$. The normal closure of $\langle x\rangle$ in F may appear denoted as one of the following

$$\langle x^F\rangle = \langle x^{\langle y\rangle}\rangle = \langle x^{y^i} : i \in \mathbb{Z}\rangle.$$

By V we denote any verbal subgroup in F and by $\hat{F^n}$ – the verbal subgroup defining the variety of locally finite groups of exponent dividing n. The fact that the class of these groups is actually a variety is a consequence of Zelmanov's solution of the restricted Burnside problem. Writing $\gamma_1(F) = F$, $\gamma_c = [\gamma_{c-1}(F), F]$

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for $c > 1$, we have that F/V is virtually nilpotent if and only if $V \supseteq \gamma_c(\hat{F^k})$ for some $c, k \in \mathbb{N}$. We call such a verbal subgroup V a VN-verbal subgroup. The set of the VN-verbal subgroups in F, denoted briefly by $\{VN\text{-verbal}\}$, forms a sublattice of the lattice of all subgroups because the inclusions $V_1 \supseteq \gamma_c(\hat{F^k})$ and $V_2 \supseteq \gamma_d(\hat{F^\ell})$ imply $V_1 \cap V_2 \supseteq \gamma_m(\hat{F^n})$ for $m = \max(c, d)$, $n = lcm(k, \ell)$.

We show that each VN-verbal subgroup V has the following properties:

- P-property: $V \cap \mathcal{F}\mathcal{F}^{-1} \neq 1$,
- R-property: $F'V/V$ is finitely generated,
- M-property: $V \nsubseteq F''(F')^p$.

To each of these properties there is associated the set of verbal subgroups satisfying it. We call these respectively

$$P\text{-verbal}, \quad R\text{-verbal}, \quad \text{and} \quad M\text{-verbal subgroups.}$$

We denote corresponding sets of verbal subgroups respectively:

$$\{P\text{-verbal}\}, \{R\text{-verbal}\}, \{M\text{-verbal}\}.$$

They also determine three types of varieties $\mathrm{var}(F/V)$:

- a P-variety: satisfies a positive law,
- an R-variety: G' is finitely generated for each two-generator group G in it,
- an M-variety: has no subvariety of the form $\mathfrak{A}_p\mathfrak{A}$ for any prime p.

The above property of the R-varieties is much stronger since every its finitely generated group G has G' finitely generated [8, Proposition 9]. This fact follows also from [1, Lemma 1] as

Proposition 1. *Let $\mathfrak{V}$ be an R-variety. Then every finitely generated group in $\mathfrak{V}$ has finitely generated commutator subgroup.*

We show that each of these properties is defined by a binary law. The following inclusions hold for the respective sets of verbal subgroups and corresponding types of varieties:

$$\{VN\text{-verbal}\} \subseteq \{P\text{-verbal}\} \subseteq \{R\text{-verbal}\} \subseteq \{M\text{-verbal}\},$$
$$\{VN\text{-varieties}\} \subseteq \{P\text{-varieties}\} \subseteq \{R\text{-varieties}\} \subseteq \{M\text{-varieties}\}.$$

Moreover every subset shown here is a sublattice of the lattice of all subgroups in F or the lattice of all varieties, as the case may be. We now discuss various inclusion problems.

2. *P*-verbal subgroups

Definition 1. A verbal subgroup V is called P-verbal if

$$V \cap \mathcal{F}\mathcal{F}^{-1} \neq 1.$$

Thus a verbal subgroup V is P-verbal if and only if the group F/V satisfies a positive law $u(x,y) \equiv v(x,y)$ for some words u and v in $\mathcal{F}$.

By result of A. I. Mal'tsev [9], nilpotent groups and hence nilpotent-by-(finite exponent) groups satisfy positive laws, so we have the set inclusion

$$\{VN\text{-verbal}\} \subset \{P\text{-verbal}\}.$$

The inclusion is proper since the verbal subgroups defining infinite Burnside groups are P-verbal but not VN-verbal. Other examples of P-verbal but not VN-verbal subgroups are given by A. Yu. Ol'shanskii and A. Storozhev [12].

Theorem 1. *The set of P-verbal subgroups forms a sublattice in the lattice of all subgroups in F.*

Proof. Let V_1 and V_2 be P-verbal subgroups defining in F/V_i for $i = 1, 2$ the following positive laws $a(x,y) \equiv b(x,y)$ and $u(x,y) \equiv v(x,y)$ respectively. Since every positive law implies a balanced positive law, we shall assume that the laws $a(x,y) \equiv b(x,y)$ and $u(x,y) \equiv v(x,y)$ are balanced, that is the exponent sum of x (of y) in $a(x,y)$ and in $b(x,y)$ (resp. in $u(x,y)$ and in $v(x,y)$) is the same.

The join V_1V_2 provides each of these laws, so it suffices to show only that the intersection $V_1 \cap V_2$ yields a positive law. We consider the law

$$a(u(x,y), v(x,y)) \equiv b(u(x,y), v(x,y)). \tag{1}$$

This law is positive and by assumption on V_1 it is satisfied in F/V_1. In the group F/V_2 the law (1) has a form $a(u,u) \equiv b(u,u)$ and hence $u^k \equiv u^k$ for some integer k since the law $a(x,y) \equiv b(x,y)$ is assumed to be balanced. Thus the law (1) is satisfied in F/V_2 and hence is satisfied modulo $V_1 \cap V_2$, which finishes the proof. □

3. *R*-verbal subgroups

Definition 2. A verbal subgroup V is called R-verbal (R for restrained) if the commutator subgroup F' is finitely generated modulo V.

It follows from Proposition 1 that if V is R-verbal, then every finitely generated group in $\mathrm{var}(F/V)$ has finitely generated commutator subgroup.

Corollary 1. *Every verbal subgroup $V \subseteq F$ such that $V \nsubseteq F'$ is the R-verbal subgroup.*

Proof. It is known that F' is generated by commutators $[x^i, y^j]$, $i, j \in \mathbb{Z}$. If $V \nsubseteq F'$ then F/V has finite exponent, which implies that F' is finitely generated modulo V. □

It is shown in [7, Corollary 6.4] that each positive law define R-verbal subgroup, so we have the inclusions

$$\{VN\text{-verbal}\} \subset \{P\text{-verbal}\} \subseteq \{R\text{-verbal}\}.$$

By the example below, n-Engel laws define R-verbal subgroups. These subgroups are P-verbal for $n < 5$ [15], so the question whether the second inclusion is strict is related to the Problem 2.82 in [5], asking whether each variety of groups satisfying n-Engel law $[x, {}_n y] \equiv 1$ is defined by positive laws.

To prove that R-verbal subgroups form a lattice, we need to find an appropriate criterion for V to be R-verbal.

For $n \in \mathbb{N}$, we introduce an important subgroup E_n in $F = \langle x, y \rangle$, setting $E_0 = \langle x \rangle$, and for $n > 0$,

$$E_n := \langle x, x^y, x^{y^2}, \dots, x^{y^n} \rangle = \langle x, [x, y], [x, y^2], \dots, [x, y^n] \rangle \tag{2}$$

Lemma 1. *A verbal subgroup V is R-verbal if and only if the subgroup $\langle x^F \rangle$ is finitely generated modulo V.*

Proof. By definition, V is R-verbal if F' is finitely generated modulo V. We prove that the latter holds if and only if $\langle x^F \rangle$ is finitely generated modulo V. Indeed, if F' is finitely generated modulo V then, since $\langle x \rangle F' = \langle x^F \rangle$, the 'only if' part follows.

Assume now that $\langle x^F \rangle$ is finitely generated modulo V. Since $\langle x^F \rangle$ is normal, the conjugation by suitable y^i implies that there is $n \in \mathbb{N}$ such that $\langle x^F \rangle$ coincides modulo V with E_{n-1}, which we write as $\langle x^F \rangle \overset{V}{\equiv} E_{n-1}$.

If denote $H := \langle [x, y], [x, y^2], \dots, [x, y^{n-1}] \rangle$, then in view of (2), $\langle x^F \rangle \overset{V}{\equiv} \langle x, H \rangle$ and then

$$\langle x^F \rangle \overset{V}{\equiv} \langle x \rangle H^{\langle x \rangle},$$

Now, since $H^{\langle x \rangle} \subseteq F' \subseteq \langle x^F \rangle$, we have by Dedekind's law

$$F' \overset{V}{\equiv} F' \cap \langle x\rangle H^{\langle x\rangle} = (F' \cap \langle x\rangle)H^{\langle x\rangle} = H^{\langle x\rangle}$$
$$= \langle [x,y]^{\langle x\rangle}, [x,y^2]^{\langle x\rangle}, [x,y^3]^{\langle x\rangle}, \dots, [x,y^n]^{\langle x\rangle}\rangle.$$

Since $\langle x^{\langle y\rangle}\rangle = \langle x^F\rangle$, the assumption that the subgroup $\langle x^F\rangle$ is finitely generated modulo V implies that $\langle x^{\langle y\rangle}\rangle$ is finitely generated and hence each subgroup of the form $\langle [x,y^i]^{\langle x\rangle}\rangle$ also is finitely generated modulo V. It follows that F' is finitely generated modulo V, which proves the 'if' part. □

Lemma 2. *The subgroup $\langle x^F\rangle$ is finitely generated modulo V if and only if there exists $n \in \mathbb{N}$ such that*

$$[x,\, _ny] \in E_{n-1}V. \tag{3}$$

Proof. Let $\langle x^F\rangle$ be finitely generated modulo V. Then, as above, the conjugation by suitable y^i implies that for some $n \in \mathbb{N}$, $\langle x^F\rangle \overset{V}{\equiv} E_{n-1}$. Hence $[x,\, _ny] \in F' \subseteq \langle x^F\rangle \subseteq E_{n-1}V$, which gives (3).

Conversely, let (3) hold. To show that $\langle x^F\rangle$ is finitely generated modulo V, it suffices to prove that

$$\langle x^F\rangle \overset{V}{\equiv} E_{n-1} = \langle x, x^y, \dots, x^{y^{n-1}}\rangle.$$

It is shown in [7, Corollary 5.4] that $E_n = \langle x, [x,y], \dots, [x,\, _ny]\rangle$. So if $[x,\, _ny] \in E_{n-1}V$ then by (2) $[x,y^n] \in E_{n-1}V$ and hence

$$x^{y^n} \in E_{n-1}V. \tag{4}$$

All inclusions below are meant modulo V, so we write (4) modulo V as:

$$x^{y^n} \in \langle x, x^y, x^{y^2}, \dots, x^{y^{n-1}}\rangle. \tag{5}$$

Substitution $y \to y^{-1}$ gives $x^{y^{-n}} \in \langle x, x^{y^{-1}}, x^{y^{-2}}, \dots, x^{y^{-(n-1)}}\rangle$. Now, conjugation by y^{n-1} gives $x^{y^{-1}} \in \langle x, x^y, x^{y^2}, \dots, x^{y^{n-1}}\rangle$ and by induction $x^{y^{-i}} \in \langle x, x^y, x^{y^2}, \dots, x^{y^{n-1}}\rangle \overset{(5)}{\subseteq} E_{n-1}$ for all $i > 0$.

Similarly, by conjugating (5) by y we obtain

$$x^{y^{n+1}} \in \langle x^y, x^{y^2}, \dots, x^{y^n}\rangle \overset{(5)}{\subseteq} E_{n-1}.$$

Repeated conjugation gives by induction $x^{y^i} \in E_{n-1}$ for all $i \geq 0$ and implies that $\langle x^F\rangle \overset{V}{\equiv} E_{n-1}$ is finitely generated, which finishes the proof. □

By Lemmas 1, 2, the condition (3) allows us to formulate the following

Criterion. A verbal subgroup V is R-verbal if and only if

$$\exists n \in \mathbb{N}, \quad [x,\ _n y] \overset{V}{\equiv} u(x,y), \qquad u(x,y) \in E_{n-1}. \tag{6}$$

Example. If F/V satisfies the n-Engel law $[x,\ _n y]{\equiv}1$ then V is R-verbal.

Using the above criterion we prove the following

Theorem 2. *The set of R-verbal subgroups forms a sublattice of the lattice of all subgroups of F.*

Proof. Let U and V be the R-verbal subgroups of F. Then by (6) there exist $k, m \in \mathbb{N}$, and words $u(x,y) \in E_{k-1}$, $v(x,y) \in E_{m-1}$ such that

$$\text{(i)} \quad [x,\ _k y] \overset{U}{\equiv} u(x,y), \qquad \text{(ii)} \quad [x,\ _m y] \overset{V}{\equiv} v(x,y). \tag{7}$$

It is clear that the join UV provides both of these laws, hence by (6), UV is R-verbal. We prove now that the intersection $U \cap V$ yields a law of the form (6), namely:

$$[x,\ _{k+m} y] \overset{U \cap V}{\equiv} w(x,y), \quad \text{for some } w(x,y) \in E_{k+m-1}.$$

Construction of the law

In (7)(i) we put $[x,\ _m y]$ for x, and also in (7)(i) we put v for x to get the following two laws satisfied in F/U:

$$[x,\ _{k+m} y] \overset{U}{\equiv} u([x,\ _m y], y) \quad \text{and} \quad [v,\ _k y] \overset{U}{\equiv} u(v,y). \tag{8}$$

The laws (8) imply in F/U a law of the form $[x,\ _{k+m} y] \overset{U}{\equiv} w(x,y)$:

$$[x,\ _{k+m} y] \overset{U}{\equiv} u(\,[x,\ _m y], y) \cdot \overbrace{(u(v,y))^{-1}[v,\ _k y]}^{\in U}. \tag{9}$$

The law (7)(ii) $[x,\ _m y] \overset{V}{\equiv} v$ also implies two laws. For the first we take k-repeated commutator on both sides with y, and for the second we put each side of (7)(ii) for x into $u(x,y)$. So we get:

$$[x,\ _{m+k} y] \overset{V}{\equiv} [v,\ _k y], \quad u(\,[x,\ _m y], y) \overset{V}{\equiv} u(v,y).$$

These two laws imply in F/V the law

$$[x,\ _{m+k} y] \overset{V}{\equiv} \underbrace{u([x,\ _m y], y) \cdot (u(v,y))^{-1}}_{\in V}[v,_k y],$$

which coincides with (9), hence is satisfied modulo $U \cap V$, and has a form $[x,\ _{k+m}y] \equiv w(x,y)$. So to finish the proof we have to check that

$$u([x,\ _{m}y],y)(u(v,y))^{-1}[v,\ _{k}y] \in E_{k+m-1}.$$

We shall consider the factors in the order: $[v,\ _{k}y]$, $u(v,y)$, $u([x,\ _{m}y],y)$.

Since $v(x,y) \in E_{m-1} = \langle\, x,\ [x,y],\ [x,\ _{2}y],\ \dots,\ [x,\ _{m-1}y]\,\rangle$, by means of the commutator identity $[ab,y] = b^{-1}[a,y]b[b,y]$, we conclude that

$$[v,\ _{k}y] \in E_{(m-1)+k}. \tag{10}$$

Since $u(x,y)\in E_{k-1}$, we get $u(v,y)\in\langle v,[v,y],[v,\ _{2}y],\dots,[v,\ _{k-1}y]\rangle$. So by (10),

$$u(v,y) \in E_{(m-1)+(k-1)} \subseteq E_{m+k-1}.$$

For the third factor we have $u(x,y) \in E_{k-1}$ and hence $u([x,\ _{m}y],y) \in$

$$\langle [x,\ _{m}y],[[x,\ _{m}y],y],[[x,\ _{m}y],\ _{2}y],\dots [[x,\ _{m}y],\ _{k-1}y]\rangle \subseteq E_{m+k-1}.$$

Thus the law (9) of the form $[x,\ _{k+m}y] \overset{U\cap V}{\equiv} w$, has $w \in E_{k+m-1}$, and by (6) defines the R-verbal subgroup $U \cap V$, which finishes the proof. □

4. M-verbal subgroups

Definition 3. A verbal subgroup $V \subseteq F$, for $F = \langle x,y\rangle$ is called M-verbal if for all primes p

$$V \nsubseteq F''(F')^p, \quad \text{i.e. } \operatorname{var}(F/V) \nsupseteq \mathfrak{A}_p\mathfrak{A}.$$

The name M-verbal is chosen because F/V satisfies so called Milnor identity defined by F. Point [13], that is, a law not holding in any of the varieties $\mathfrak{A}_p\mathfrak{A}$ [14, Proposition 1.1].

Theorem 3. *A verbal subgroup V is M-verbal if and only if it satisfies*

$$VF'' \cap \mathcal{F}\mathcal{F}^{-1} \neq 1, \tag{11}$$

that is, if and only if it yields a positive law in metabelian groups.

PROOF. By result of BELYAEV and SESEKIN [2] the wreath product $C_p wr C$ contains a free semigroup. Since $C_p wr C$ generates the product variety $\mathfrak{A}_p\mathfrak{A}$ of the variety of all abelian groups of exponent p by the variety of all abelian groups (see e.g. [10, 17.6 and Corollary 22.44]), it follows that the equality $F''(F')^p \cap \mathcal{F}\mathcal{F}^{-1}=1$ holds for every prime p. Hence (11) implies $V \nsubseteq F''(F')^p$.

Conversely, by result of J. Groves [4, Theorem C (ii)], the group $G := F/VF''$ is either nilpotent-by-finite or $\operatorname{var} G$ contains a subvariety $\mathfrak{A}_p\mathfrak{A}$ for some prime p. So if $V \nsubseteq F''(F')^p$ for all prime p, then F/VF'' must be virtually nilpotent, hence it satisfies a positive law and then (11) follows. □

Now, each R-verbal subgroup V is M-verbal. Indeed, if V is R-verbal then $(F/V)'$ is finitely generated. Since $F'/F''(F')^p$ is infinitely generated $V \nsubseteq F''(F')^p$ and hence $\operatorname{var}(F/V) \not\supseteq \mathfrak{A}_p\mathfrak{A}$.

$$\{VN\text{-verbal}\} \subset \{P\text{-verbal}\} \subseteq \{R\text{-verbal}\} \subseteq \{M\text{-verbal}\}.$$

If V defines a pseudo-abelian variety $\mathfrak{V}$, i.e. nonabelian variety in which every metabelian group is abelian, then $\mathfrak{V}$ does not contain any of $\mathfrak{A}_p\mathfrak{A}$ hence, by definition, V is M-verbal but need not be P-verbal. For example, it is shown in [6], that the pseudo-abelian relatively free groups F/V by A. YU. OL'SHANSKII [11] contain free non-abelian semigroups which do not satisfy any positive law, so that V is M-verbal but not P-verbal. Thus we have the following strict inclusions:

$$\{VN\text{-verbal}\} \subset \{P\text{-verbal}\} \subset \{M\text{-verbal}\}.$$

Theorem 4. *The set of M-verbal subgroups forms a sublattice of the subgroup lattice of F.*

PROOF. The property $VF'' \cap \mathcal{F}\mathcal{F}^{-1} \neq 1$ means that F/VF'' satisfies a positive law. Let V_1 and V_2 yield respectively the following positive laws modulo F''

$$a(x,y) \equiv b(x,y)f_1'' \text{ and } c(x,y) \equiv d(x,y)f_2'', \quad a,b,c,d \in \mathcal{F},\ f'' \in F''.$$

The join V_1V_2 provides each of these laws. To speak of $V_1 \cap V_2$ we can assume that laws $a(x,y) \equiv b(x,y)$ and $c(x,y) \equiv d(x,y)$ are balanced. Now we consider the following law

$$a(c, df_2'') \equiv b(c, df_2'') \cdot f_1''(c, df_2''). \tag{12}$$

This law is positive modulo F'', and by assumption, is satisfied modulo V_1. By assumption on V_2, there is $v_2 \in V_2$ such that $df_2'' = cv_2$. Then modulo V_2, (12) has a form $a(c,c) \equiv b(c,c) \cdot f_1''(c,c)$, and since $a \equiv b$ is balanced, it is trivial modulo V_2. So (12) is satisfied modulo $V_1 \cap V_2$. Hence by Theorem 3, the subgroup $V_1 \cap V_2$ is M-verbal, which finishes the proof. □

Question 1. Which R-verbal subgroups are not P-verbal?

Question 2. Which M-verbal subgroups are not R-verbal?

A group is *locally graded* if every non-trivial finitely generated subgroup has a proper subgroup of finite index. Considering the class of verbal subgroups V for which F/V is locally graded, we infer from [3, Theorem B] that the properties of being VN-, P-, and R-verbal coincide, since every R-verbal subgroup with a locally graded F/V, is VN-verbal.

By [3, Theorem A] this also holds for M-verbal subgroups if F/V belongs to the smaller class $\mathcal{S}$ defined in [3].

Question 3. Is an M-verbal subgroup VN-verbal if F/V is locally graded?

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