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On conformally flat (α,β)-metrics with constant flag curvature

By GUANGZU CHEN (Shanghai), QUN HE (Shanghai)
and ZHONGMIN SHEN (Indianapolis)

This paper is dedicated to Professor L. Kozma

Abstract. In this paper, we study conformally flat (α,β)-metrics in the form $F=\alpha\phi(\beta/\alpha)$, where α is a Riemannian metric and β is a 1-form on a C^∞ manifold M. We prove that conformally flat (α,β)-metrics with constant flag curvature on a manifold of dimension $n>2$ must be either a locally Minkowski metric or a Riemannian metric.

1. Introduction

In Riemannian geometry, the conformal properties of Riemannian metrics have been well studied by many geometers. There are many important local and global results in Riemannian conformal geometry. For example, the Poincaé metric on B^n is conformally flat Riemannian metric of constant sectional curvature $K=1$. More generally, the conformal properties of a Finsler metric deserve extra attention. The Weyl theorem states that the projective and conformal properties of a Finsler space determine the metric properties uniquely ([14], [18]). The study of conformal geometry is a recent popular trend in Finsler geometry. Two Finsler metrics F and $\tilde{F}$ on a manifold M are said to be *conformally related* if there is a scalar function $c(x)$ on M such that $\tilde{F}=e^{c(x)}F$. A Finsler metric which is

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The second author is the corresponding author.

conformally related to a Minkowski metric is called *conformally flat Finsler metric.* The *conformal transformation* between F and $\tilde{F}$ is defined by $L : F \to \tilde{F}$, $\tilde{F} = e^{c(x)}F$.

In conformal Finsler geometry, it is one hot issue to study the conformal transformation. The famous Matsumoto's problem states how many different ways are there essentially to realize conformal equivalence of a Finsler manifold to a Berwald manifold? In other words, whether or not are there two Berwald manifolds which are conformally equivalent (but not homothetic) to each other ([17])? The Matsumoto's problem is closely related to the theory of generalized Berwald manifolds, especially Wagner spaces. C. Vincze answers this problem and shows that the conformal equivalence between two Berwald manifolds must be homothetic unless they are Riemannian [20]. Recently, the first author, X. Cheng and Y. Zou characterize the conformal transformations between two (α, β)-metrics. We prove that if both conformally related (α, β)-metrics F and $\tilde{F}$ are Douglas metrics, then the conformal transformation between them is a homothety ([3]). It is well-known that the set of Berwald metrics is included in the set of Douglas metrics. Our theorem can be deemed to be the generalization of C. Vincze' result in part.

There is the other hot issue how to characterize conformally flat Finsler metrics (conformally Berwald metrics). M. Hashuiguchi and Y. Ichijyō defined a conformally invariant linear connection in a Finsler space with an (α, β)-metric and gave a condition that a Randers metric is conformally flat based on their connection ([11]). Later, S. Kikuchi found a conformally invariant Finsler connection and gave a necessary and sufficient condition for a Finsler metric to be conformally flat by a system of partial differential equations under an extra condition ([13]). By using Kikuchi's conformally invariant Finsler connection, S.-I. Hojo, M. Matsumoto and K. Okubo studied conformally Berwald Finsler spaces and its applications to (α, β)-metrics ([10]). In fact, a Finsler manifold is a conformally Berwald manifold if and only if it is a Wagner manifold ([9]). In 2006, C. Vincze gets a structure theorem for conformally Berwald Randers manifolds and gives examples of Wagnerian Finsler manifolds ([21]). But the local structure of conformally flat Finsler metrics (conformally Berwald metrics) is unknown in general. In [12], L. Kang has proved that any conformally flat Randers metric of scalar flag curvature is projectively flat and classified completely such metrics. The first author and X. Cheng classify conformally flat weak Einstein polynomial (α, β)-metrics and show also that there is no non-trivial conformally flat

(α, β)-metrics with isotropic S-curvature. However, it is unfortunate that the local structure of conformal flat Finsler metrics is still unknown, even if conformal flat (α, β)-metrics.

For a Finsler manifold (M, F), the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is a function of tangent planes $P \subset T_xM$ and non-zero vectors $y \in P$. This quantity tells us how curved the space is at a point. When F is Riemannian, $\mathbf{K} = \mathbf{K}(P)$ is independent of $y \in P \setminus \{0\}$, which is just the sectional curvature. Thus the flag curvature is an analogue of sectional curvature in Riemannian geometry. A Finsler metric F is said to be of *constant flag curvature* if the flag curvature $\mathbf{K} =$ constant. It is one of important problems in Finsler geometry to study and characterize Finsler metrics of constant flag curvature [5], [8], [19]. In [15], B. Li and the third author finish the classification of the projectively flat (α, β)-metrics with constant flag curvature.

In this paper, we mainly focus on studying the conformally flat (α, β)-metrics with constant flag curvature. The condition "conformally flat" is very different from the condition "projectively flat". In [2], S. Bácsó and X. Cheng prove that if the conformal transformation $L : F \to \tilde{F}, \tilde{F} = e^{c(x)}F$ preserves the geodesics, then it must be a homothety, that is, $c =$ constant. It implies that conformally flat Finsler metrics are not projectively flat in general. We get the following

Theorem 1.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be a conformally flat (α, β)-metric on a manifold M of dimension $n > 2$. If F is of constant flag curvature, then it is either a locally Minkowski metric or a Riemannian metric.*

2. Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M and G^i be the geodesic coefficients of F, which are defined by

$$G^i = \frac{1}{4} g^{il} \left\{ \left[F^2\right]_{x^k y^l} y^k - \left[F^2\right]_{x^l} \right\}, \tag{2.1}$$

where $g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}(x, y)$ and $(g^{ij}) := (g_{ij})^{-1}$. For any $x \in M$ and $y \in T_xM \backslash \{0\}$, the *Riemann curvature* $\mathbf{R}_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m} \frac{\partial G^m}{\partial y^k}. \tag{2.2}$$

For a tangent plane $P \subset T_xM$ containing y, let

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(\mathbf{R}_y(u), u)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - [\mathbf{g}_y(y, u)]^2},$$

where $\mathbf{g}_y(u,v) := g_{ij}u^iv^j$ and $u \in P$ such that $P = \mathrm{span}\{y,u\}$. It is well-known that a Finsler metric is of scalar flag curvature if and only if

$$R^i{}_k = \mathbf{K}(x,y)F^2 h^i{}_k,$$

where $h^i{}_k = \delta^i{}_k - F^{-2}g_{kq}y^qy^i$.

In Finsler geometry, there are some important geometric quantities which have many important influences on the geometric structures of Finsler metrics and vanish in Riemannian case. We call them *non-Riemannian quantities.* For a non-zero vector $y \in T_xM$, the *Cartan torsion* $\mathbf{C}_y = C_{ijk}dx^i \otimes dx^j \otimes dx^k$: $T_xM \otimes T_xM \otimes T_xM \to \mathbf{R}$ is defined by

$$C_{ijk} := \frac{1}{4}[F^2]_{y^iy^jy^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

The *mean Cartan torsion* $\mathbf{I}_y = I_i(x,y)dx^i : T_xM \to \mathbf{R}$ is defined by

$$I_i := g^{jk}C_{ijk}.$$

It is obvious that $\mathbf{C} = 0$ if and only if F is Riemannian. Also, according to Deicke's theorem, a Finsler metric is Riemannian if and only if the mean Cartan torsion $\mathbf{I} = 0$.

We consider the following tensor $\chi = \chi_i dx^i$ on TM for a Finsler metric F on a manifold M defined by

$$\chi_i := \frac{1}{2}\left(\frac{\partial^2\Pi}{\partial x^m \partial y^i}y^m - \Pi_{x^i} - 2G^m\frac{\partial^2\Pi}{\partial y^m\partial y^i}\right), \tag{2.3}$$

where $\Pi := \frac{\partial G^m}{\partial y^m}$. When $F = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, its geodesic coefficients $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are quadratic in y. Further, $\Pi = \Gamma^m_{jm}y^j = y^j\frac{\partial}{\partial x^j}(\ln\sqrt{\det(a_{ml}(x))})$ is a 1-form by (2.1). Thus $\chi_i = 0$, i.e., $\chi = \chi_i dx^i$ is a non-Riemannian quantity. We can easily check that

Lemma 2.1 ([16]). *If a spray G^i is of isotropic curvature, then $\chi_i = 0$. Especially, if a Finsler metric F is of isotropic flag curvature, i.e., G^i induced by F is of isotropic curvature, then $\chi_i = 0$.*

By the definition, an (α,β)-metric is a Finsler metric expressed in the following form

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha<b_0,\ x\in M$. It is proved ([8]) that $F=\alpha\phi(\beta/\alpha)$ is a positive definite Finsler metric if and only if the function $\phi=\phi(s)$ is a C^∞ positive function on an open interval $(-b_0,b_0)$ satisfying

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\quad |s|\le b<b_0.$$

Let G^i and G^i_α denote the geodesic coefficients of F and α, respectively. Denote

$$r_{ij}:=(b_{i|j}+b_{j|i}),\quad s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i}),$$
$$s^i{}_j:=a^{il}s_{lj},\quad s_i:=b^js_{ji},$$

where $(a^{ij}):=(a_{ij})^{-1}$ and $b_{i|j}$ denote the covariant derivative of β with respect to α. Let $b:=\|\beta\|_\alpha$ denotes the norm of β with respect to α. Then we have

Lemma 2.2 ([8])**.** *The geodesic coefficients of G^i are related to G^i_α by*

$$G^i=G^i_\alpha+\alpha Qs^i{}_0+\{-2Q\alpha s_0+r_{00}\}\{\Psi b^i+\Theta\alpha^{-1}y^i\},\tag{2.4}$$

where $s^i{}_0:=s^i{}_jy^j$, $s_0:=s_iy^i$, $r_{00}:=r_{ij}y^iy^j$ and

$$Q:=\frac{\phi'}{\phi-s\phi'},\quad \Theta:=\frac{Q-sQ'}{2\Delta},\quad \Psi:=\frac{Q'}{2\Delta}$$

and

$$\Delta:=1+sQ+(b^2-s^2)Q'.$$

3. Conformally flat (α,β)-metrics with $\chi_i=0$

Let $F=\alpha\phi(s), s=\beta/\alpha$ be an (α,β)-metric on a manifold M, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form on M. Assume that F is conformally related to a Finsler metric $\tilde F$ on M, that is, there is a scalar function $c(x)$ on M such that $F=e^{c(x)}\tilde F$. It is easy to see that $\tilde F=\tilde\alpha\phi(\tilde\beta/\tilde\alpha)$ is also an (α,β)-metric, where $\tilde\alpha=e^{-c(x)}\alpha$, $\tilde\beta=e^{-c(x)}\beta$. Write $\tilde\alpha=\sqrt{\tilde a_{ij}(x)y^iy^j}$, $\tilde\beta=\tilde b_i(x)y^i$. Then $\tilde a_{ij}=e^{-2c(x)}a_{ij}$, $\tilde b_i=e^{-c(x)}b_i$. Further, denote $c_i:=\frac{\partial c(x)}{\partial x^i}$ and $c^i:=\tilde a^{ij}c_j$, we have ([5])

$$b_{j|k}=e^{c(x)}\Big(\tilde b_{j||k}-\tilde b_kc_j+\tilde b_mc^m\tilde a_{jk}\Big).\tag{3.1}$$

Here $\tilde b_{j||k}$ denote the covariant derivative of $\tilde b_j$ with respect to $\tilde\alpha$. Note that $a^{ij}(x,y)=e^{-2c(x)}\tilde a^{ij}(x,y)$. We have $\tilde b:=\|\tilde\beta\|_{\tilde\alpha}=b$.

Under the assumption that F is conformally flat, we can compute the curvature tensor χ_i of F. Why is the χ_i? Firstly, the Lemma 2.1 implies

$$\{F \mid F \text{ is of constant flag curvature.}\} \subset \{F \mid F \text{ satisfies } \chi_i = 0.\}$$

Thus the non-Riemanian curvature tensor χ_i is useful to characterize the Finsler metrics of constant flag curvature. Secondly, computing the curvature tensor χ_i is simpler than computing the Riemannian curvature directly.

Now, write $c_{ij} := \frac{\partial^2 c(x)}{\partial x^i \partial x^j}$, $y_i := \tilde{a}_{ij} y^j$, $\tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j$, $c^i := \tilde{a}^{ij} c_j$, $f := \tilde{b}^i c_i$, $p_i := c_{ij}\tilde{b}^j$, $p_0 := p_i y^i$, $c_{i0} := c_{ij} y^j$, $c_0 := c_i y^i$, $c_{00} := c_{ij} y^i y^j$. We can obtain the following

Proposition 3.1. *Let $F = \alpha\phi(s), s = \beta/\alpha$ be a conformally flat (α, β)-metric on a manifold M, that is, there exists a locally Minkowski metric $\tilde{F} = \tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha})$ such that $F = e^{c(x)}\tilde{F}$, where $c = c(x)$ is a scalar function on M. Then the curvature tensor χ_i of F is determined by*

$$\begin{aligned}
\chi_i = \frac{1}{2}\Big\{ & \big[A_1 \tilde{\alpha} f c_0 + A_2(\tilde{\alpha} f)^2 + A_3 c_0^2 + N' \tilde{\alpha} p_0 + M' c_{00} \\
& + M'(1+sQ)\tilde{\alpha}^2 \|\nabla c\|_{\tilde{\alpha}}\big] \frac{\tilde{b}_i}{\tilde{\alpha}} + \big[B_1 \tilde{\alpha} f c_0 + B_2 (\tilde{\alpha} f)^2 + B_3 c_0^2 \\
& + (N - sN')\tilde{\alpha} p_0 - sM' c_{00} - sM'(1+sQ)\tilde{\alpha}^2 \|\nabla c\|_{\tilde{\alpha}}\big] \frac{y_i}{\tilde{\alpha}^2} \\
& + [C_1 c_0 + C_2 \tilde{\alpha} f] c_i - \tilde{\alpha} N p_i \Big\},
\end{aligned} \tag{3.2}$$

where

$$M := n + \frac{\Phi}{2\Delta^2}(s + \tilde{b}^2 Q), \qquad N := -\frac{\Phi}{2\Delta^2}(1 + sQ),$$

$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (\tilde{b}^2 - s^2)(1 + sQ)Q''$$

and

$$\begin{aligned}
A_1 := & (1+sQ)(2s^2\Theta N'' + 2s^2\Psi M'' - sN'' + 4s\Psi M' + 2s\Theta N' - 2\Theta N + M'' \\
& - 2\Theta M' - 2\Psi\tilde{b}^2 M'' - 2\Theta\tilde{b}^2 N''),
\end{aligned}$$

$$A_2 := -(1+sQ)(2\Psi\tilde{b}^2 N'' + 2\Psi N - N'' - 2s\Psi N' - 2s^2\Psi N'' + 2\Psi M'),$$

$$A_3 := (1+sQ)(2s^2\Theta M'' - 2\Theta\tilde{b}^2 M'' - sM'' + 4s\Theta M' - M'),$$

$$\begin{aligned}
B_1 := & (1+sQ)(-2s^3\Psi M'' - 2s^3\Theta N'' - 2s^2\Theta N' - 6s^2\Psi M' + s^2 N'' + 2s\Theta N \\
& + 2s\Theta M' + 2s\tilde{b}^2\Psi M'' - sM'' + 2s\tilde{b}^2\Theta N'' + sN' - M' + 2\tilde{b}^2\Psi M' - N),
\end{aligned}$$

$$B_2 := -sA_2,$$

$$B_3 := (1+sQ)(3sM' + 2s\tilde{b}^2\Theta M'' + 2\tilde{b}^2\Theta M' - 6s^2\Theta M' + s^2M'' - 2s^3\Theta M''),$$

$$C_1 := -2(1+sQ)[s+\Theta(\tilde{b}^2 - s^2)]M',$$

$$C_2 := -(1+sQ)(2\tilde{b}^2\Psi M' - M' - N - 2s^2\Psi M' + sN').$$

Proof. It is well-known that (α,β)-metric $\tilde{F}$ is a locally Minkowski metric if and only if $\tilde{\alpha}$ is flat and $\tilde{b}_{j||k} = 0$ ([1]). By Lemma 2.2, we have

$$\begin{aligned} G^i = {} & \{\Theta(1+sQ)\tilde{\alpha}f + [1-\Theta(s+\tilde{b}^2Q)]c_0\}y^i \\ & - \frac{\tilde{\alpha}^2c^i}{2}(1+sQ) + \tilde{\alpha}\tilde{b}^i(\Psi\tilde{\alpha}f + \Theta c_0)(1+sQ), \end{aligned} \tag{3.3}$$

$$\Pi = \frac{\partial G^m}{\partial y^m} = Mc_0 + N\tilde{\alpha}f. \tag{3.4}$$

Further, we need to compute the following

$$\Pi_{x^i} = Mc_{i0} + N\tilde{\alpha}p_i, \tag{3.5}$$

$$\Pi_{y^i} = (M'c_0 + N'\tilde{\alpha}f)s_{y^i} + Mc_i + N\frac{y_i}{\tilde{\alpha}}f, \tag{3.6}$$

$$\begin{aligned} \Pi_{y^iy^m} = {} & (M'c_0 + N'\tilde{\alpha}f)s_{y^iy^m} + \left[(M''c_0 + N''\tilde{\alpha}f)s_{y^i} + M'c_i + N'\frac{y_i}{\tilde{\alpha}}f\right]s_{y^m} \\ & + M's_{y^i}c_m + N\frac{\tilde{a}_{im}}{\tilde{\alpha}}f + \left(N'fs_{y^i} - N\frac{y_i}{\tilde{\alpha}^2}f\right)\frac{y_m}{\tilde{\alpha}}. \end{aligned} \tag{3.7}$$

By a series of direct computations, we have

$$s_{y^i} = \frac{\tilde{b}_i\tilde{\alpha} - sy_i}{\tilde{\alpha}^2}, \tag{3.8}$$

$$s_{y^iy^m} = \left(-\frac{\tilde{b}_i}{\tilde{\alpha}^3} + 3s\frac{y_i}{\tilde{\alpha}^4}\right)y_m - \frac{\tilde{b}_my_i}{\tilde{\alpha}^3} - \frac{s\tilde{a}_{im}}{\tilde{\alpha}^2}, \tag{3.9}$$

$$s_{y^i}\tilde{b}^i = \frac{\tilde{b}^2 - s^2}{\tilde{\alpha}}, \quad s_{y^i}c^i = \frac{\tilde{\alpha}f - sc_0}{\tilde{\alpha}^2}, \tag{3.10}$$

$$s_{y^iy^m}\tilde{b}^m = \left(-\frac{\tilde{b}_i}{\tilde{\alpha}^2} + 3s\frac{y_i}{\tilde{\alpha}^3}\right)s - \frac{\tilde{b}^2y_i}{\tilde{\alpha}^3} - \frac{s\tilde{b}_i}{\tilde{\alpha}^2}, \tag{3.11}$$

$$s_{y^iy^m}c^m = \left(-\frac{\tilde{b}_i}{\tilde{\alpha}^3} + 3s\frac{y_i}{\tilde{\alpha}^4}\right)c_0 - \frac{y_if}{\tilde{\alpha}^3} - \frac{sc_i}{\tilde{\alpha}^2}. \tag{3.12}$$

Plugging (3.5)–(3.7) into (2.3) and by using (3.8)–(3.12), we get (3.2). □

It is obvious that $\tilde{b} =$ constant. If $\tilde{b} = 0$, then $F = e^{c(x)}\tilde{\alpha}$ is a Riemannian metric. So we always assume $b \neq 0$ in the following.

Lemma 3.2 ([7])**.** *For an (α, β)-metric $F = \alpha\phi(s)$, the mean Cartan torsion is given by*

$$I_i = -\frac{1}{2F}\frac{\Phi}{\Delta}(\phi - s\phi')h_i, \tag{3.13}$$

where

$$\Delta := 1 + sQ + (b^2 - s^2)Q', \qquad h_i := b_i - \alpha^{-1}sy_i.$$

According to Deicke's theorem, an (α, β)-metric is Riemannian if and only if $\Phi \equiv 0$. In fact, such identity is a differential equation in ϕ which is very difficult to solve. However, if $\phi = \phi(s)$ satisfies

$$Q - sQ' = 0,$$

it is easy to get

$$\phi(s) = a_1\sqrt{1 + a_2s^2},$$

where a_1 and a_2 are constants. Similarly, we have

Lemma 3.3. *If there is a neighborhood $U_s \subset [-b, b]$ such that $\forall s \in U_s$, $\phi(s)$ satisfies*

$$b^2Q + s = 0, \tag{3.14}$$

then

$$\phi(s) = a_0\sqrt{b^2 - s^2},$$

where a_0 is a constant.

Remark 3.4. If $\forall s \in U_s$, $\phi(s) = a_0\sqrt{b^2 - s^2}$, we have

$$\phi - s\phi' + (b^2 - s^2)\phi'' = 0.$$

Thus (α, β)-metric F is not a positive definite Finsler metric.

Further, we have

Lemma 3.5. *If $\phi(s)$ satisfies*

$$s(b^2 - s^2)Q' + Q(b^2 + s^2) + 2s = 0, \tag{3.15}$$

then

$$Q = \frac{k_1(b^2 - s^2) - 1}{s}.$$

where k_1 is a number independent of s.

PROOF. By maple program, we get the solution of (3.15)

$$Q=\frac{k_1(b^2-s^2)-1}{s}.$$

In special, if $k_1=\frac{1}{b^2}$, then we have

$$\frac{\phi'}{\phi-s\phi'}=-\frac{s}{b^2}.$$

Thus

$$\phi(s)=k_2\sqrt{b^2-s^2}.$$

where k_2 is a number independent on s. $\square$

The following lemma is trivial. One can verify it directly.

Lemma 3.6. *For any given $\phi(s)$, if $l(s)$ satisfies*

$$[l-sl']k=\frac{\phi-s\phi'}{\phi-s\phi'+(b^2-s^2)\phi''}[l-sl'+(b^2-s^2)l''],$$

where k is a constant, then

$$l(s)=k_1\int\frac{[(\phi-s\phi')\sqrt{b^2-s^2}\,]^k}{s^2\sqrt{b^2-s^2}}ds+k_2s,$$

where k_1, k_2 are constant.

In the following, considering conformally flat (α,β)-metric $F=\alpha\phi(s), s=\beta/\alpha$ with $\chi_i=0$ on a manifold M, we have

Proposition 3.7. *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$ be a conformally flat non Riemannian (α,β)-metric on a manifold M of dimension $n>2$. If F has vanishing χ_i, then it is a locally Minkowski metric.*

PROOF. By assumption and (3.2), we have

$$\begin{aligned}&[A_1\tilde{\alpha}fc_0+A_2(\tilde{\alpha}f)^2+A_3c_0^2+N'\tilde{\alpha}p_0+M'c_{00}+M'(1+sQ)\tilde{\alpha}^2\|\nabla c\|^2_{\tilde{\alpha}}]\tilde{\alpha}\tilde{b}_i\\&\quad+[B_1\tilde{\alpha}fc_0+B_2(\tilde{\alpha}f)^2+B_3c_0^2+(N-sN')\tilde{\alpha}p_0-sM'c_{00}\\&\quad-sM'(1+sQ)\tilde{\alpha}^2\|\nabla c\|^2_{\tilde{\alpha}}]y_i+[C_1c_0+C_2\tilde{\alpha}f]\tilde{\alpha}^2c_i-\tilde{\alpha}^3Np_i=0.\end{aligned}\tag{3.16}$$

To simplify the computations, we take an orthonormal basis at x with respect to $\tilde{\alpha}$ such that

$$\tilde{\alpha}=\sqrt{\sum_{i=1}^n(y^i)^2},\quad\tilde{\beta}=\tilde{b}y^1.$$

Further, we take the following coordinate transformation in T_xM, $\psi : (s, u^A) \to (y^i)$:

$$y^1 = \frac{s}{\sqrt{\tilde{b}^2 - s^2}}\,\bar{\alpha}, \quad y^A = u^A,$$

where $\bar{\alpha} = \sqrt{\sum_{i=2}^{n}(u^A)^2}$ (see [6]). Here, our index conventions are

$$1 \le i, j, k, \cdots \le n, \quad 2 \le A, B, C, \cdots \le n.$$

We have

$$\tilde{\alpha} = \frac{\tilde{b}}{\sqrt{\tilde{b}^2 - s^2}}\,\bar{\alpha}, \quad \tilde{\beta} = \frac{\tilde{b}s}{\sqrt{\tilde{b}^2 - s^2}}\,\bar{\alpha}.$$

Further,

$$c_0 = \frac{c_1 s}{\sqrt{\tilde{b}^2 - s^2}}\,\bar{\alpha} + \bar{c}_0, \quad f = c_1\tilde{b},$$

$$p_0 = \frac{p_1 s}{\sqrt{\tilde{b}^2 - s^2}}\,\bar{\alpha} + \bar{p}_0, \quad c_{00} = \frac{c_{11}s^2}{\tilde{b}^2 - s^2}\,\bar{\alpha}^2 + \frac{2\bar{c}_{10}s}{\sqrt{\tilde{b}^2 - s^2}}\,\bar{\alpha} + \bar{c}_{00},$$

where

$$\bar{c}_0 := \sum_{A=2}^{n} c_A y^A, \quad \bar{p}_0 := \sum_{A=2}^{n} p_A y^A,$$

$$\bar{c}_{10} := \sum_{A=2}^{n} c_{1A} y^A, \quad \bar{c}_{00} := \sum_{A,B=2}^{n} c_{AB} y^A y^B.$$

For $i = A$, (3.16) is equivalent to the following two equations by the rational terms and irrational terms in y^A.

$$\begin{aligned}
&\Big[B_1 \frac{b^2 s c_1^2 \bar{\alpha}^2}{\tilde{b}^2 - s^2} + B_2 \frac{\tilde{b}^4 c_1^2 \bar{\alpha}^2}{\tilde{b}^2 - s^2} + B_3 \left(\frac{c_1^2 s^2 \bar{\alpha}^2}{\tilde{b}^2 - s^2} + \bar{c}_0^2\right) + (N - sN')\frac{\tilde{b}^2 s c_{11} \bar{\alpha}^2}{\tilde{b}^2 - s^2} \\
&\quad - sM' \left(\frac{c_{11} s^2 \bar{\alpha}^2}{\tilde{b}^2 - s^2} + \bar{c}_{00}\right) - sM'(1 + sQ)\frac{\tilde{b}^2 \|\nabla c\|_{\bar{\alpha}} \bar{\alpha}^2}{\tilde{b}^2 - s^2}\Big] y_A \\
&\quad + C_1 \frac{\tilde{b}^2 \bar{c}_0 \bar{\alpha}^2}{\tilde{b}^2 - s^2} c_A = 0,
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&[B_1 \tilde{b}^2 c_1 \bar{c}_0 + 2B_3 s c_1 \bar{c}_0 + (N - sN')\tilde{b}\bar{p}_0 - 2s^2 M' \bar{c}_{10}] y_A \\
&\quad + (C_1 c_1 s + C_2 \tilde{b}^2 c_1)\frac{\tilde{b}^2 \bar{\alpha}^2}{\tilde{b}^2 - s^2} c_A - N \frac{\tilde{b}^3 \bar{\alpha}^2}{\tilde{b}^2 - s^2} p_A = 0.
\end{aligned} \tag{3.18}$$

Then (3.17) implies that there is a scalar function $d := d(s)$ such that

$$B_3\bar{c}_0^2 - sM'\bar{c}_{00} = d\bar{\alpha}^2.$$

Plugging it into (3.17) yields

$$\Big[B_1\frac{b^2sc_1^2}{\tilde{b}^2-s^2} + B_2\frac{\tilde{b}^4c_1^2}{\tilde{b}^2-s^2} + B_3\frac{c_1^2s^2}{\tilde{b}^2-s^2} + (N-sN')\frac{\tilde{b}^2sc_{11}}{\tilde{b}^2-s^2} - s^3M'\frac{c_{11}}{\tilde{b}^2-s^2} \\ - sM'(1+sQ)\frac{\tilde{b}^2\|\nabla c\|_{\bar{\alpha}}}{\tilde{b}^2-s^2} + d\Big]y_A + C_1\frac{\tilde{b}^2\bar{c}_0}{\tilde{b}^2-s^2}c_A = 0,$$

Contracting (**??**) with y^A yields $q(s)\bar{\alpha}^2 = \bar{q}(s)\bar{c}_0^2$, where

$$q(s) := B_1\frac{b^2sc_1^2}{\tilde{b}^2-s^2} + B_2\frac{\tilde{b}^4c_1^2}{\tilde{b}^2-s^2} + B_3\frac{c_1^2s^2}{\tilde{b}^2-s^2} + (N-sN')\frac{\tilde{b}^2sc_{11}}{\tilde{b}^2-s^2} \\ - s^3M'\frac{c_{11}}{\tilde{b}^2-s^2} - sM'(1+sQ)\frac{\tilde{b}^2\|\nabla c\|_{\bar{\alpha}}}{\tilde{b}^2-s^2} + d,$$

$$\bar{q}(s) := C_1\frac{\tilde{b}^2}{\tilde{b}^2-s^2}$$

We claim that $\bar{c}_0 = 0$. Suppose $\bar{c}_0 \neq 0$. When $n > 2$, if $q(s) \neq 0$ and $\bar{q}(s) \neq 0$, noting that s is independent on y^A, then

$$1 \geq \mathrm{Rank}(c_Ac_B) = \mathrm{Rank}(\delta_{AB}) \geq 2,$$

which is impossible. Hence $q(s) = \bar{q}(s) = 0$, then we have $C_1 = 0$, i.e.,

$$(1+sQ)[s+\Theta(\tilde{b}^2-s^2)]M' = 0. \tag{3.19}$$

Noting that F is a positive definite (α,β)-metric, we have

$$\Delta = \frac{\phi[\phi - s\phi' + (\tilde{b}^2-s^2)\phi'']}{(\phi-s\phi')^2} > 0, \qquad 1+sQ = \frac{\phi}{\phi-s\phi'} > 0.$$

Then (3.19) becomes to $[s(\tilde{b}^2-s^2)Q' + Q(\tilde{b}^2+s^2) + 2s]M' = 0$, $\forall s \in [-\tilde{b},\tilde{b}]$.

By above equation, we can prove $M' = 0$ $\forall s \in [-\tilde{b},\tilde{b}]$. If not, there must be a neighborhood $U_s \subset [-\tilde{b},\tilde{b}]$ such that $M' \neq 0, \forall s \in U_s$. Then $s(\tilde{b}^2-s^2)Q' + Q(\tilde{b}^2+s^2) + 2s = 0$, $\forall s \in U_s$, by Lemma 3.5, we have

$$Q = \frac{k_1(\tilde{b}^2-s^2)-1}{s}.$$

Plugging it into the formulation of M, we get $M = k_1\tilde{b}^2(n-1)$, $\forall s \in U_s$. It is impossible. Thus $M' = 0$, $\forall s \in [-\tilde{b}, \tilde{b}]$.

Noting that the formulation of M, we have

$$n + \frac{\Phi}{2\Delta^2}(s + \tilde{b}^2 Q) = \bar{d}, \tag{3.20}$$

where $\bar{d}$ is a constant. Put $h(s) := s + \tilde{b}^2 Q$, we can obtain

$$h(s) = \frac{(\tilde{b}^2 - s^2)\phi' + s\phi}{\phi - s\phi'}.$$

Then we have

$$h(\tilde{b}) = \frac{\tilde{b}\phi(\tilde{b})}{\phi(\tilde{b}) - \tilde{b}\phi'(\tilde{b})} > 0, \quad h(-\tilde{b}) = \frac{-\tilde{b}\phi(-\tilde{b})}{\phi(-\tilde{b}) + \tilde{b}\phi'(-\tilde{b})} < 0$$

Thus there exists $\bar{b} \in (-\tilde{b}, \tilde{b})$ such that $\bar{b} + \tilde{b}^2 Q(\bar{b}) = 0$. Taking $s = \bar{b}$ in (3.20), we get $\bar{d} - n = 0$. Then (3.20) becomes

$$\Phi(s + \tilde{b}^2 Q) = 0.$$

By Lemma 3.2, Lemma 3.3 and Remark 3.4, F is a Riemannian metric. It is a contraction. Thus $\bar{c}_0 = 0$.

Substituting $\bar{c}_A = 0$ into (3.17) yields

$$[B_1\tilde{b}^2 s + B_2\tilde{b}^4 + B_3 s^2 - sM'(1+sQ)\tilde{b}^2]c_1^2 + [(N - sN')\tilde{b}^2 - s^2 M']sc_{11} = 0. \tag{3.21}$$

By a direct computation,

$$\begin{aligned} B_1\tilde{b}^2 s + B_2\tilde{b}^4 + B_3 s^2 - sM'(1+sQ)\tilde{b}^2 = {} & -\frac{s(1+sQ)}{\Delta}(N''\tilde{b}^4 + \tilde{b}^2 sM'' + 2M'\tilde{b}^2 \\ & -\tilde{b}^2 s^2 N'' - \tilde{b}^2 sN' + \tilde{b}^2 N - s^3 M'' - 3s^2 M'). \end{aligned} \tag{3.22}$$

It is easy to check

$$(N - sN')\tilde{b}^2 - s^2 M' = N\tilde{b}^2 + sM - s(N\tilde{b}^2 + sM)',$$

$$N''\tilde{b}^4 + \tilde{b}^2 sM'' + 2M'\tilde{b}^2 = \tilde{b}^2(N\tilde{b}^2 + sM)'',$$

$$-\tilde{b}^2 s^2 N'' - \tilde{b}^2 sN' + \tilde{b}^2 N - s^3 M'' - 3s^2 M' = [s(N - sN')\tilde{b}^2 - s^3 M']'.$$

Let $l := N\tilde{b}^2 + sM$, then (3.21) can be reduced to

$$-\frac{1+sQ}{\Delta}[l - sl' + (\tilde{b}^2 - s^2)l'']c_1^2 + (l - sl')c_{11} = 0. \tag{3.23}$$

Finally, we show $c_1 = 0$. Suppose that $c_1 \neq 0$, let $k = c_{11}/c_1^2$, then (3.23) becomes

$$[l - sl']k = \frac{\phi - s\phi'}{\phi - s\phi' + (b^2 - s^2)\phi''}\,[l - sl' + (b^2 - s^2)l''].$$

By Lemma 3.6, we have

$$l(s) = k_1 \int \frac{[(\phi - s\phi')\sqrt{b^2 - s^2}\,]^k}{s^2\sqrt{b^2 - s^2}}\,ds + k_2 s.$$

Then

$$l' = k_1 \frac{[(\phi - s\phi')\sqrt{b^2 - s^2}\,]^k}{s^2\sqrt{b^2 - s^2}} + k_2.$$

Using the formulation of l, we have

$$k_1\Big[(\phi - s\phi')\sqrt{\tilde{b}^2 - s^2}\,\Big]^k = \Big[n - k_2 + \frac{\Phi}{\Delta^2}s - \Big(\frac{\Phi}{2\Delta^2}\Big)'(\tilde{b}^2 - s^2)\Big]s\sqrt{\tilde{b}^2 - s^2}.$$

Taking $s = 0$ in above equation, we obtain $k_1 = 0$. Thus $l = k_2 s$, i.e.,

$$sn - \frac{\Phi}{2\Delta^2}(\tilde{b}^2 - s^2) = k_2 s. \tag{3.24}$$

Further, let $s = \tilde{b}$ in (3.24), we get $k_2 = n$. Then (3.24) can be reduced to

$$\frac{\Phi}{2\Delta^2}(\tilde{b}^2 - s^2) = 0.$$

It implies that $\Phi \equiv 0$. By Lemma 3.2, F is a Riemannian metric. We get a contraction. Thus $c_1 = 0$.

Hence $c_i = 0$, that is $c(x) =$ constant. Then F is a Minkowski metric. $\square$

By Lemma 2.1 and Proposition 3.7, we can obtain

Corollary 3.8. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be a conformally flat (α, β)-metric on a manifold M of dimension $n > 2$. If F is of constant flag curvature, then it is either a locally Minkowski metric or a Riemannian metric.*

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GUANGZU CHEN
DEPARTMENT OF MATHEMATICS
TONGJI UNIVERSITY
SHANGHAI 200092
P.R. CHINA

E-mail: chenguangzu1@163.com

QUN HE
DEPARTMENT OF MATHEMATICS
TONGJI UNIVERSITY
SHANGHAI 200092
P.R. CHINA

E-mail: hequn@tongji.edu.cn

ZHONGMIN SHEN
DEPARTMENT OF MATHEMATICAL SCIENCES
INDIANA UNIVERSITY-PURDUE UNIVERSITY
INDIANAPOLIS, IN 46202-3216
USA

E-mail: zshen@math.iupui.edu

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