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Douglas–Randers manifolds with vanishing stretch tensor

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Abstract. In this paper, we prove that every Douglas–Randers metric with vanishing stretch curvature is a Berwald metric. It results that, a Douglas–Randers metric is R-quadratic if and only if it is a Berwald metric.

1. Introduction

The class of Randers metrics is among the simplest class of non-Riemannian Finsler metrics, which arises from many areas in mathematics, physics and biology [1]. A Randers metric is of the form $F = \alpha + \beta$ where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on M with $\|\beta\|_\alpha < 1$. Randers metrics were first studied by G. Randers, from the standpoint of general relativity [16]. These metrics were used in the theory of the electron microscope by Ingarden, who first named them Randers metrics. Recently, Bao–Robles–Shen showed that Randers metrics arise naturally from the navigation problem on a Riemannian manifold under the influence of an external force field [7]. The least time path from one point to another is a geodesic of a Randers metric. Randers metrics are computable and have very rich non-Riemannian curvature properties [9], [10], [18].

The geodesic curves of a Finsler metric F on a smooth manifold M are determined by the system of SODE $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions G^i are called the spray coefficients. A Finsler metric F is called a Berwald metric, if the G^i's are quadratic in $y \in T_xM$ for any $x \in M$. As a generalization of Berwald curvature Bácsó–Matsumoto [4] introduced the notion of Douglas

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metrics. They proved that a Randers metric $F = \alpha + \beta$ has vanishing Douglas curvature if and only if β is a closed 1-form [4].

There exists another extension of Berwald metrics. Let (M, F) be a Finsler manifold. The third order derivative of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is the Cartan torsion $\mathbf{C}_y$ on T_xM [6]. The rate of change of the Cartan torsion along geodesics is said to be Landsberg curvature. Finsler metrics with vanishing Landsberg curvature are called Landsberg metrics. Every Berwald metric is a Landsberg metric. In [12], Matsumoto proved that $F = \alpha + \beta$ is a Landsberg metric if and only if β is parallel. In [11], Hashiguchi-Ichijyō showed that for a Randers metric $F = \alpha + \beta$, if β is parallel, then F is a Berwald metric. Thus every Randers metric with vanishing Landsberg curvature is a Berwald metric.

As a generalization of Landsberg curvature, Berwald introduced the notion of stretch curvature and denoted it by $\mathbf{\Sigma}_y$ [8]. He showed that $\mathbf{\Sigma} = 0$ if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. Then, this curvature investigated by Matsumoto in [13].

In [5], Bácsó–Matsumoto proved that a Douglas metric with vanishing stretch curvature is R-quadratic if and only if its $\bar{\mathbf{E}}$-curvature vanishes. It is interesting to find some conditions under which Douglas metrics with vanishing stretch curvature reduce to Berwald metrics. We prove the following:

Theorem 1.1. *A Douglas–Randers manifold reduces to a Berwald manifold if and only if, its stretch tensor vanishes.*

According to Bácsó–Ilosvay–Kis [2] and Bácsó–Matsumoto [3], every Douglas metric with vanishing Landsberg curvature is a Berwald metric . Here we weaken their condition on the curvature and impose the Randers metric on the Finsler metric instead.

Throughout this paper, we use the Cartan connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by "|" and ", ", respectively.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on $TM_0 := TM \setminus \{0\}$; (ii) F is positive-homogeneous of degree 1; (iii) for each $y \in T_xM$, the quadratic form

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y+su+tv)\right]|_{s,t=0}, \quad u,v \in T_xM.$$

is positive definite.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u,v)\right]|_{t=0}, \quad u,v,w \in T_xM.$$

The family $\mathbf{C} := (\mathbf{C}_y)_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C}$=0 if and only if F is Riemannian.

For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y : T_xM \to \mathbb{R}$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Deicke's theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u,v,w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\},$$

$h_{ij} := g_{ij} - F^{-2}y_iy_j$ is the angular metric. F is called C-reducible if $\mathbf{M}_y = 0$. Matsumoto proved that every Randers metric $F = \alpha + \beta$ satisfies $\mathbf{M}_y = 0$. The converse is also true:

Lemma 2.1 ([14])**.** *A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, for all $y \in TM_0$.*

For $y \in T_xM_0$, define the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^iv^jw^k$ where $L_{ijk} := C_{ijk|s}y^s$. The family $\mathbf{L} := (\mathbf{L}_y)_{y \in TM_0}$ is called the Landsberg curvature. F is called a Landsberg metric if $\mathbf{L} = \mathbf{0}$.

The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where $J_i := I_{i|s}y^s$. A Finsler metric is said to be weakly Landsbergian if $\mathbf{J} = 0$.

For $y \in T_xM_0$, define the stretch curvature $\mathbf{\Sigma}_y : T_xM \otimes T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(u,v,w,z) := \Sigma_{ijkl}(y)u^iv^jw^kz^l$, where

$$\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k}). \tag{1}$$

A Finsler metric is said to be a stretch metric if $\mathbf{\Sigma} = 0$. Every Landsberg metric is a stretch metric.

Given a Finsler manifold (M,F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard local coordinate system (x^i,y^i) for TM_0 is given by $\mathbf{G}=y^i\frac{\partial}{\partial x^i}-2G^i\frac{\partial}{\partial y^i}$, where G^i are the spray coefficients. $\mathbf{G}$ is called the spray associated to F. For $y\in T_xM_0$, define $\mathbf{B}_y: T_xM\otimes T_xM\otimes T_xM\to T_xM$ and $\mathbf{D}_y: T_xM\otimes T_xM\otimes T_xM\to T_xM$ by $\mathbf{B}_y(u,v,w):=B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ and $\mathbf{D}_y(u,v,w):=D^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl}:=\frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l},\quad D^i{}_{jkl}:=B^i{}_{jkl}-\frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\left[\frac{2}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right].$$

The $\mathbf{B}$ and $\mathbf{D}$ are called the Berwald curvature and Douglas curvature, respectively. F is called a Berwald and Douglas metric if $\mathbf{B}=\mathbf{0}$ and $\mathbf{D}=\mathbf{0}$, respectively.

Example 2.1. A Finsler metric F satisfying $F_{x^k}=FF_{y^k}$ is called a Funk metric. The standard Funk metric on the Euclidean unit ball is defined by

$$F(x,y):=\frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}{1-|x|^2}+\frac{\langle x,y\rangle}{1-|x|^2},\quad y\in T_xB^n(1)\simeq\mathbb{R}^n,$$

where $\langle\,,\rangle$ and $|.|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. It is easy to see that β is closed 1-form, hence F is a Douglas metric. Since $G^i=\frac{1}{2}Fy^i$, then we get $\Sigma_{ijkl}=F[C_{ijl|k}-C_{ijk|l}]$, i.e., F is not a stretch metric. Thus by Theorem 1.1, F is not Berwald metric.

3. Proof of Theorem 1.1

Let $F=\alpha+\beta$ be a Randers metric on a manifold M, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta(y)=b_i(x)y^i$ is a 1-form on M. Define $b_{i|j}$ by $b_{i|j}\theta^j:=db_i-b_j\theta_i{}^j$, where $\theta^i:=dx^i$ and $\theta_i{}^j:=\tilde{\Gamma}^j_{ik}dx^k$ denote the Levi-Civita connection forms of α. Let

$$r_{ij}:=\frac{1}{2}(b_{i|j}+b_{j|i}),\quad s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i}),\quad s^i{}_j:=a^{ih}s_{hj},\quad s_j:=b_is^i{}_j.$$

In this section, we will prove a generalized version of Theorem 1.1. Indeed, we prove the following.

Theorem 3.1. *Let $F=\alpha+\beta$ be a Randers metric on a manifold M. Suppose that F is a stretch metric and the following holds:*

$$\begin{aligned}&\big(cs_0-2s^ms_{m0}\big)\alpha^4+\big(2s_0^2+2s^m{}_0s_{m0}-4s^ms_{m0}\beta-cs_0\beta\big)\alpha^3+\big(6\beta s^m{}_0s_{m0}\\&\qquad-2s^ms_{m0}\beta^2+6\beta s_0^2\big)\alpha^2+6\beta^2s^m{}_0s_{m0}\alpha+2\beta^3s^m{}_0s_{m0}=0,\quad(2)\end{aligned}$$

where c is scalar function on M. Then F reduces to a Berwald metric.

Remark 3.1. According to Bácsó-Matsumoto's theorem, $F = \alpha + \beta$ is a Douglas metric if and only if β is a closed 1-form [4]. It is obvious that any closed 1-form β satisfies (2).

To prove the Theorem 3.1, we need the following.

Lemma 3.2. *Let $F = \alpha+\beta$ be a Randers metric. Then the stretch curvature of F is given by*

$$\Sigma_{ijkl} = \frac{2}{n+1}\big[h_{ik}J_{j|l} - h_{il}J_{j|k} + h_{jk}J_{i|l} - h_{jl}J_{i|k} + (J_{k|l} - J_{l|k})h_{ij}\big]. \tag{3}$$

Thus, if the mean Landsberg curvature is horizontally constant along geodesics, then F has vanishing stretch curvature.

Proof. By Lemma 2.1, F is C-reducible. Taking a horizontal derivation, C-reducibility implies that

$$L_{ijk} = \frac{1}{n+1}\big\{J_i h_{jk} + J_j h_{ik} + J_k h_{ij}\big\}. \tag{4}$$

By (4), we get

$$L_{ijk|l} = \frac{1}{n+1}\big\{J_{i|l} h_{jk} + J_{j|l} h_{ik} + J_{k|l} h_{ij}\big\}. \tag{5}$$

From (1) and (5), we obtain (3). $\Box$

Remark 3.2. Suppose that F be a stretch metric, i.e., $L_{ijk|l} = L_{ijl|k}$. Then we have $L_{ijk|l}y^l = 0$. This equation is equivalent to that for any linearly parallel vector fields u, v, w along a geodesic c, the following holds:

$$\frac{d}{dt}\big[\mathbf{L}_{\dot{c}}(u,v,w)\big] = 0.$$

The geometric meaning of this is that the rate of change of the Landsberg curvature is constant along any Finslerian geodesic.

First, we prove the following.

Lemma 3.3. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M. Then the mean Landsberg curvature of F satisfies*

$$\begin{aligned} J_{j|0} = {} & \frac{1}{\alpha+\beta}\big[\alpha r_{j0|0} - \alpha^2 s_{j|0} - \alpha^2 b^r{}_{|0} s_{rj} - \alpha^{-1}(r_{00|0} - \alpha s_{0|0} - \alpha b^r{}_{|0} s_{r0}) y_j\big] \\ & - \frac{\alpha}{2(\alpha+\beta)^2}\big[(r_{00|0} - 2\alpha s_{0|0} - 2\alpha b^r{}_{|0} s_{r0}) I_j + (r_{00} - 2\alpha s_0) J_j\big] + s_{j0|0}. \end{aligned} \tag{6}$$

PROOF. Denote by $l^i = \alpha^{-1}y^i$ the normalized supporting element. The fundamental tensor and angular metric of a Randers metric is written as

$$g_{ij} = \frac{F}{\alpha}a_{ij} + b_i b_j + \frac{1}{\alpha}(b_i y_j + b_j y_i) - \frac{\beta}{\alpha^3}y_i y_j, \tag{7}$$

$$h_{ij} = \frac{F}{\alpha}\left(a_{ij} - \frac{y_i y_j}{\alpha^2}\right). \tag{8}$$

The reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = \frac{\alpha}{F}\left[a^{ij} - \frac{1}{F}(y^i b^j + y^j b^i) + \frac{b^2\alpha + \beta}{\alpha F^2}y^i y^j\right], \tag{9}$$

where $b^2 = b_i b^i$ and $b^i = a^{ij}b_i$. Differentiating (7) with respect to y^k yields

$$C_{ijk} = \frac{1}{2\alpha}\big\{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\big\}, \tag{10}$$

where

$$I_i = b_i - \alpha^{-2}\beta y_i. \tag{11}$$

To finding the relation between the Cartan connection of α and F, we put the difference tensor $D^i{}_{jk} = \Gamma^i{}_{jk} - \gamma^i{}_{jk}$, where $\gamma^i{}_{jk}$ is the Christoffel symbols of α. It is computed in [13] by MATSUMOTO. Now, let ∇_k be the covariant differentiation by x^k with respect to associated Riemannian connection. Let

$$b_{ij} := \nabla_j b_i = \frac{\partial b_i}{\partial x^k} - b_r\gamma^r{}_{jk}, \quad r_{ijk} := \nabla_k r_{ij}, \quad s_{ijk} := \nabla_k s_{ij}. \tag{12}$$

We recall that the index 0 means contraction by y^i. For example $r_{0jk} = r_{ijk}y^i$. In [13], the following are obtained:

$$\begin{aligned} D^i &= 2\alpha s^i_0 + y^i\left(\frac{r_{00} - 2\alpha s_0}{F}\right), \\ D^i{}_j &= \alpha s^i{}_j + \frac{1}{\alpha}\big(s_{j0}y^i + s^i_0 y_j\big) + \left(\delta^i_j - \frac{y^i y_j}{\alpha^2}\right)\left(\frac{r_{00} - 2\alpha s_0}{2F}\right) \\ &\quad + \frac{y^i}{F}\left[r_{j0} - s_{j0} - \alpha s_j - \frac{1}{\alpha}\big(s_0 y_j + \beta s_{j0}\big) - \left(\frac{r_{00} - 2\alpha s_0}{2F}\right)I_j\right], \end{aligned} \tag{13}$$

where $s_j = b^i s_{ij}$ and $s^i{}_j = a^{ir}s_{rj}$. By contracting (13) with b_i, we have

$$b_i D^i{}_j = \frac{1}{F}\big[\alpha^2 s_j + s_0 y_j + \beta r_{j0}\big] + \frac{\alpha}{2F^2}\big[r_{00} - 2\alpha s_0\big]I_j \tag{14}$$

Plugging $b_{i|j} = b_{ij} - b_r D_i{}^r{}_j$ in (14) yields

$$b_{i|0} = \frac{\alpha}{F} r_{j0} + s_{j0} - \frac{\alpha^2}{F} s_j - \frac{1}{F} s_0 y_j - \frac{\alpha}{2F^2}\big(r_{00} - 2\alpha s_0\big) I_j, \tag{15}$$

$$b_{0|0} = \frac{\alpha}{F}\big(r_{00} - 2\alpha s_0\big). \tag{16}$$

Since $F_{|0} = 0$, by (11), (15) and (16) we have

$$\begin{aligned} J_j &= b_{j|0} - \frac{1}{\alpha^2} b_{0|0} y_j \\ &= \frac{\alpha}{F}\big(r_{j0} - \alpha s_j\big) - \frac{1}{\alpha F}\big(r_{00} - \alpha s_0\big) y_j - \frac{\alpha}{2F^2}\big(r_{00} - 2\alpha s_0\big) I_j + s_{j0}. \end{aligned} \tag{17}$$

By taking a horizontal derivation of (17) along geodesics, we get (6). $\square$

Now, we are going to consider Randers metrics with vanishing stretch curvature.

Lemma 3.4. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M. Suppose that F is a stretch metric. Then*

$$r_{00} = c\alpha^2, \quad r_0 = c\beta, \tag{18}$$

where c is a function on M.

Proof. Contracting (6) with b^j we find

$$\begin{aligned} b^i J_{i|0} = {} & \frac{\alpha}{\alpha+\beta} r_{0|0} + s_{0|0} - \frac{\alpha^2}{\alpha+\beta} b^r{}_{|0} s_r - \frac{\beta}{\alpha(\alpha+\beta)}\big(r_{00|0} - \alpha s_{0|0} - \alpha b^r{}_{|0} s_{r0}\big) \\ & - \frac{\alpha}{2(\alpha+\beta)^2}\big[(r_{00|0} - 2\alpha s_{0|0} - 2\alpha b^r{}_{|0} s_{r0}) b^i I_i + (r_{00} - 2\alpha s_0) b^i J_i\big], \end{aligned} \tag{19}$$

where

$$r_{0|0} = b^j r_{j0|0},$$

$$r_{i0|0} = r_{i00} - r_{im} D_0{}^m{}_0 - r_{m0} D_i^m{}_0,$$

$$r_{00|0} = r_{000} - 2\alpha s_{m0} r_0^m - \frac{2r_{00}}{\alpha+\beta}(r_{00} - \alpha s_0),$$

$$\begin{aligned} b^i r_{i0|0} = {} & b^i r_{i00} - 2\alpha r_m s_0^m - \frac{r_0}{\alpha+\beta}(r_{00} - 2\alpha s_0) - \alpha s^m\ r_{m0} - \frac{1}{\alpha}(r_{00} s_0 + \beta r_{m0} s^m{}_0) \\ & - \frac{r_{00}}{2\alpha^2(\alpha+\beta)^2}[2\alpha(\alpha+\beta)(\alpha(r_0 - s_0) + 2\beta s_0) - (r_{00} - 2\alpha s_0)(b^2\alpha^2 - \beta^2)] \\ & - \frac{r_0}{2\alpha^2(\alpha+\beta)}(r_0\alpha^2 - \beta r_{00})(r_{00} - 2\alpha s_0), \end{aligned} \tag{20}$$

$$b^i s_{i0|0} = b^i s_{i00} - 3\alpha s_m s^m_0 - \frac{3s_0}{2(\alpha+\beta)}(r_{00} - 2\alpha s_0) - \frac{\beta}{\alpha} s^m_0 s_{m0}. \tag{21}$$

Substituting (21) in (19) yields

$$\alpha A + B = 0, \tag{22}$$

where

$$\begin{aligned}
A := {} & 2s_m s^m \alpha^6 + \big[8\beta s_m s^m - 2s^m{}_0 s_m - 2r^m{}_0 s_m\big]\alpha^5 \\
& + \big[2b^2 s_0^2 - 6\beta s^m_0 s_m - 4\beta r^m_0 s_m + 2\beta^2 s_m s^m + 2r_0 s_0\big]\alpha^4 \\
& + \big[2\beta r_0 s_0 - r_0 r_{00} - 6\beta^2 s^m_0 s_m - 2\beta^2 r^m_0 s_m - 2b^2 r_{00} s_0 - r_{00} s_0\big]\alpha^3 \\
& + \big[\frac{1}{2} b^2 r_{00}^2 - 4\beta r_{00} s_0 - 2\beta^2 s_0^2 - \beta r_{00} r_0 - 2\beta^3 s_m s^m_0\big]\alpha^2 \\
& + \big[\beta r_{00}^2 - \beta^2 r_{00} s_0\big]\alpha + \frac{1}{2}\beta^2 r_{00}^2,
\end{aligned} \tag{23}$$

$$\begin{aligned}
B := {} & \big[-6b^2 s_m s^m_0 - 4r_m s^m_0 - 2r_{m0} s^m - 2b^2 s^m s_{m0} - 6s^m_0 s_m\big]\alpha^6 \\
& + \big[8b^2 s_0^2 - 2\beta s^m s_{m0} + 6r_0 s_0 + 2b^2 r^m_0 s_{m0} + 2b^2 s^m_0 s_{m0} + 6s_0^2 \\
& - 12\beta r_m s^m_0 + 2b^2 b^i s_{i00} + 2b^2 r_{m0} s^m_0 - 30\beta s_m s^m_0 - 12b^2\beta s_m s^m_0 \\
& + 2b^i r_{i00} + 2b^i s_{i00} - 2b^2\beta s^m s_{m0} - 6\beta r_{m0} s^m\big]\alpha^5 \\
& + \big[2\beta r^m_0 s_{m0} - 6\beta^2 r_{m0} s^m + 6\beta b^i r_{i00} + r_{00} s_0 - 5r_0 r_{00} - 6b^2\beta^2 s_m s^m_0 \\
& - 12\beta^2 r_m s^m_0 + 6\beta r_{m0} s^m_0 + \beta 6b^2 s_0^2 + 2b^2\beta r^m_0 s_{m0} + 26\beta s_0^2 - 48\beta^2 s_m s^m_0 \\
& - 2\beta^2 s^m s_{m0} + 12\beta r_0 s_0 + 10\beta b^i s_{i00} + 4b^2\beta r_{m0} s^m_0 - b^2 r_{000} - 8b^2 r_{00} s_0 \\
& + 2b^2\beta s^m_0 s_{m0} + 4b^2\beta b^i s_{i00}\big]\alpha^4 + \big[6\beta^2 b^i r_{i00} - 30\beta^3 s_m s^m_0 - 7b^2\beta r_{00} s_0 \\
& - 4\beta^3 r_m s^m_0 + 2\beta^2 r^m_0 s_{m0} - 13\beta r_{00} s_0 + 24\beta^2 s_0^2 + 3b^2 r_{00}^2 + 16\beta^2 b^i s_{i00} \\
& - 10\beta r_0 r_{00} - 2\beta^3 r_{m0} s^m + 2b^2\beta^2 r_{m0} s^m_0 - 2b^2\beta r_{000} - 2\beta r_{000} + 6\beta^2 r_0 s_0 \\
& + 2b^2\beta^2 b^i s_{i00} - 6\beta^2 s^m_0 s_{m0} - 2b^2\beta^2 s^m_0 s_{m0} + 16\beta^2 r_{m0} s^m_0\big]\alpha^3 \\
& + \big[6\beta^3 s_0^2 - 20\beta^2 r_{00} s_0 - 5\beta^2 r_0 r_{00} - 6\beta^4 s^m_0 s_m - 2b^2\beta^3 s^m_0 s_{m0} \\
& + 10\beta^3 b^i s_{i00} - 5\beta^2 r_{000} + 3b^2\beta r_{00}^2 + 2\beta^3 b^i r_{i00} + 5\beta r_{00}^2 + 14\beta^3 r_{m0} s^m_0 \\
& - b^2\beta^2 r_{000} - 14\beta^3 s^m_0 s_{m0}\big]\alpha^2 + \big[2\beta^4 b^i s_{i00} - 4\beta^3 r_{000} + 4\beta^4 r_{m0} s^m_0 \\
& - 10\beta^4 s^m_0 s_{m0} + 7\beta^2 r_{00}^2 - 7\beta^3 r_{00} s_0\big]\alpha - 2\beta^5 s^m_0 s_{m0} - \beta^4 r_{000} + 2\beta^3 r_{00}^2.
\end{aligned} \tag{24}$$

By (22), we have $A = 0$ and $B = 0$. Since $A = 0$, it follows that α^2 must be a factor of r_{00} and then we get (18). □

Proof of Theorem 3.1. Substituting (18) in (23) yields

$$\big[4s_m s^m + c^2\beta^2\big]\alpha^4 + \big[8\beta s_m s^m - 6cs_0 - 4s^m{}_0 s_m - 4b^2 cs_0\big]\alpha^3$$

$$+\left[4b^2 s_0 - 12 s_m s^m_{\ 0}\beta - c^2\beta^2 - 12\beta^2 c s_0 + 4 s_m s^m \beta^2\right]\alpha^2$$
$$-\left[12\beta^2 s_m s^m_{\ 0} + 2c\beta^2 s_0\right]\alpha - 4 s_m s^m_{\ 0}\beta^3 - 4\beta^2 s_0^2 = 0. \quad (25)$$

By (17) and (18), we get

$$\begin{aligned} g^{ij}J_iJ_j = \frac{1}{4F^4}\big[&(4s_m s^m + b^2c^2)\alpha^6 + (8s_m s^m\beta - 4cs_0 - 4s^m s_{m0} \\ &- 4b^2cs_0 - 4s^m_{\ 0}s_m)\alpha^5 + (4s_0^2 - 12s_m s^m_{\ 0}\beta - 12cs_0\beta \\ &+ 4s_m s^m\beta^2 + 4s^m_{\ 0}s_{m0} - 12s^m s_{m0}\beta - c^2\beta^2 + 4b^2s_0^2)\alpha^4 \\ &+ (16\beta s^m_{\ 0}s_{m0} - 4cs_0\beta^2 - 12s^m s_{m0}\beta^2 + 16\beta s_0^2 - 12s_m s^m_{\ 0}\beta^2)\alpha^3 \\ &+ (8\beta^2 s_0^2 - 4\beta^3 s^m s_{m0} - 4s_m s^m_{\ 0}\beta^3 + 24\beta^2 s^m_{\ 0}s_{m0})\alpha^2 \\ &+ 16\beta^3 s^m_{\ 0}s_{m0}\alpha + 4\beta^4 s^m_{\ 0}s_{m0}\big]. \end{aligned} \quad (26)$$

By (25) and (26) it results that $J_i = 0$ if the following holds:

$$\begin{aligned} (cs_0 - 2s^m s_{m0})\alpha^4 + (2s_0^2 + 2s^m_{\ 0}s_{m0} - 4s^m s_{m0}\beta - cs_0\beta)\alpha^3 + (6\beta s^m_{\ 0}s_{m0} \\ - 2s^m s_{m0}\beta^2 + 6\beta s_0^2)\alpha^2 + 6\beta^2 s^m_{\ 0}s_{m0}\alpha + 2\beta^3 s^m_{\ 0}s_{m0} = 0. \end{aligned} \quad (27)$$

Therefore, by assumption, F is a weakly Landsberg metric. Thus, by the C-reducibility, it follows that F reduces to a Landsberg metric, and hence F is a Berwald metric. □

A Finsler spaces is said to be R-quadratic if its Riemann curvature R_y is quadratic in $y \in T_xM$ [5]. By definition, every Berwald metric is R-quadratic. It is proved that every R-quadratic metric is a stretch metric (see [15], [17]). Then by Theorem 1.1, we get the following.

Corollary 3.1. *Let $F = \alpha + \beta$ be a Douglas metric on a manifold M. Then F is R-quadratic if and only if it is a Berwald metric.*

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