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The dual spaces of martingale Hardy–Lorentz spaces

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Abstract. In this paper, the dual spaces of martingale Hardy–Lorentz spaces $H^s_{p,q}$ are identified by use of the technique of atomic decomposition. We show that the dual spaces of martingale Hardy–Lorentz spaces $H^s_{p,q}$ are $\Lambda_2(\alpha)$, where $0 < p < 1$, $0 < q \le 1$, $\alpha = \frac{1}{p} - 1$.

1. Introduction

Fefferman [1] has proved that the dual space of the Hardy space H_1 is BMO (the space of functions of bounded mean oscillation). In martingale setting, it was proved by Herz [3] that the dual space of martingale Hardy space H^s_1 is BMO_2, and considering a sequence of atomic σ-algebras he also proved in [4] that the dual spaces of $H^s_p (0 < p < 1)$ are $\Lambda_2(\alpha)$ $\big(\alpha = \frac{1}{p} - 1\big)$. Weisz[5] improved the result of Herz with the aid of atomic decomposition, he proved that the dual spaces of martingale Hardy spaces $H^s_p (0 < p \le 1)$ are $\Lambda_2(\alpha)$ $\big(\alpha = \frac{1}{p} - 1\big)$ for a sequence of arbitrary σ-algebras. On the main dual theorems in classical martingale H_p theory, the readers may refer to [2], [10] and [11].

The method of atomic decomposition has been successfully applied to study the dual spaces of martingale spaces. There is a lot of excellent work on this topic, see for example [6], [7], [15], [17]. The purpose of this paper is to apply the

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method of atomic decomposition to study the dual spaces of martingale Hardy–Lorentz spaces $H^s_{p,q}$ for $0 < p < 1$, $0 < q \leq 1$. The dual spaces of $H^s_{p,q}$ have been identified by Weisz in [10] for $1 < p, q < \infty$. However, the question of the dual spaces of $H^s_{p,q}$ is still not solved completely. In this paper, we show that the dual spaces of martingale Hardy–Lorentz spaces $H^s_{p,q}$ are $\Lambda_2(\alpha)$, where $0 < p < 1$, $0 < q \leq 1$, $\alpha = \frac{1}{p} - 1$.

The organization of this paper is divided into two further sections. Some basic knowledge, which we will use, is collected in the next section. Main result and its proof will be given in Section 3.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and f a measurable function defined on Ω. Denote its distribution function by

$$\lambda_f(t) = \mathbb{P}(x : |f(x)| > t), \quad t \geq 0,$$

and its decreasing rearrangement function f^* is defined as

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leq t\}, \quad t \geq 0.$$

The Lorentz space $L_{p,q}(\Omega) = L_{p,q}$, $0 < p < \infty$, $0 < q \leq \infty$, consists of those measurable functions f with finite quasinorm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} = \left(\int_0^\infty \left[t^{\frac{1}{p}} f^*(t)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}, \quad 0 < q < \infty,$$
$$\|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t), \quad q = \infty.$$

We recall three facts about Lorentz spaces. The first is that the quasinorm of Lorentz spaces has an equivalent definition, namely

$$\|f\|_{p,q} = \left(q \int_0^\infty \left[t\mathbb{P}(|f(x)| > t)^{\frac{1}{p}}\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}, \quad 0 < q < \infty,$$
$$\|f\|_{p,\infty} = \sup_{t>0} t\mathbb{P}(|f(x)| > t)^{\frac{1}{p}}, \quad q = \infty.$$

The second is that Lorentz spaces $L_{p,q}$ increase as the second exponent q increases, and decrease as the first exponent p increases (the second exponent q is not involved). Namely, $L_{p,q_1} \subset L_{p,q_2}$ for $0 < p < \infty$ and $0 < q_1 \leq q_2 \leq \infty$, $L_{r,s} \subset L_{p,q}$ for $0 < p < r \leq \infty$ and $0 < q, s \leq \infty$. Let $0 < p, q < \infty$ and $f \in L_{p,q}$, the third is that $L_{p,q}$ has absolutely continuous norm, and $\lim_{n\to\infty} \|f_n\|_{p,q} = 0$ if $|f_n| \leq |f|$ and $\lim_{n\to\infty} f_n = 0$ a.e. (see Theorem 2.3.4 and Proposition 2.3.3 in [16] or Proposition 3.5 in[9]).

Let $(\mathcal{F}_n)_{n\geq 0}$ be a non-decreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_{n\geq 0} \mathcal{F}_n)$. The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. For a martingale $f = (f_n)_{n\geq 0}$ relative to $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_n)_{n\geq 0})$, denote $df_i = f_i - f_{i-1}$($i \geq 0$, with convention $df_0 = 0$) and

$$M_n f = \sup_{0\leq i\leq n} |f_i|, \quad Mf = \sup_{0\leq i} |f_i|,$$

$$s_n(f) = \left(\sum_{i=0}^{n} \mathbb{E}_{i-1} |df_i|^2 \right)^{\frac{1}{2}}, \quad s(f) = \left(\sum_{i=0}^{\infty} \mathbb{E}_{i-1} |df_i|^2 \right)^{\frac{1}{2}}.$$

For $0 < p < \infty, 0 < q \leq \infty$, martingale Hardy–Lorentz spaces $H^s_{p,q}$ are defined by

$$H^s_{p,q} = \{ f = (f_n)_{n\geq 0} : \|f\|_{H^s_{p,q}} = \|s(f)\|_{p,q} < \infty \}.$$

Let $1 \leq q < \infty$, $\alpha > 0$ and $\mathcal{T}$ be the set of stopping times relative to $(\mathcal{F}_n)_{n\geq 0}$, the Lipschitz space $\Lambda_q(\alpha)$ denotes the space of functions $f \in L_q$ for which

$$\|f\|_{\Lambda_q(\alpha)} = \sup_{\nu\in\mathcal{T}} \mathbb{P}(\nu < \infty)^{-\frac{1}{q}-\alpha} \|f - f^\nu\|_q < \infty.$$

We recall that $\Lambda_q(0) = \mathrm{BMO}_q$ (see [10]).

Throughout this paper, we denote the set of integers and the set of non-negative integers by $\mathbf{Z}$ and $\mathbf{N}$, respectively. We use C to denote positive constants and may denote different constants at different occurrences.

3. Main result and its proof

In order to prove the main result, we first establish weak atomic decomposition for martingale Hardy–Lorentz spaces $H^s_{p,q}$, which is also a complement to Theorem 1 in [13]. An atomic decomposition theorem for martingale Hardy–Lorentz spaces $H^s_{p,q}$ can be found in [14].

Definition 3.1 ([12], [5]). *A measurable function a is called a weak atom of the first category (or $w-1$-atom, briefly) if there exists a stopping time ν (ν is called the stopping time associated with a) such that*

(i) $a_n = \mathbb{E}_n a = 0$ *if* $\nu \geq n$,

(ii) $\|s(a)\|_\infty < \infty$.

Lemma 3.2. *Let* $0 < p,\ q < \infty$. *If* $f = (f_n)_{n\geq 0} \in H^s_{p,q}$, *then there exist a sequence* $(a^k)_{k\in\mathbf{Z}}$ *of* $w-1$*-atoms and the corresponding stopping times* $(\nu_k)_{k\in\mathbf{Z}}$ *such that*

(i) $f_n = \sum_{k\in\mathbf{Z}} \mathbb{E}_n a^k$, $\forall n \in \mathbf{N}$;

(ii) $s(a^k) \leq A\cdot 2^k$, $\forall k \in \mathbf{Z}$ *for some constant* $A > 0$, $\{2^k\mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k\in\mathbf{Z}} \in l^q$, *and*

$$\|\{2^k\mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k\in\mathbf{Z}}\|_{l^q} \leq C\|f\|_{H^s_{p,q}}. \tag{3.1}$$

Moreover, the sum $\sum_{k\in\mathbf{Z}} a^k$ *converges to* f *in* $H^s_{p,q}$.

Proof. Assume that $f = (f_n)_{n\geq 0} \in H^s_{p,q}$. Define stopping times for all $k \in \mathbf{Z}$ as following:

$$\nu_k = \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\} \quad (\inf\emptyset = \infty).$$

Obviously, $\nu_k \uparrow \infty$ $(k \to \infty)$. Let $f^{\nu_k} = (f_{n\wedge\nu_k})_{n\geq 0}$ be the stopping martingale. Then

$$\begin{aligned}\sum_{k\in\mathbf{Z}}(f_n^{\nu_{k+1}} - f_n^{\nu_k}) &= \sum_{k\in\mathbf{Z}}\left(\sum_{m=0}^{n}\chi(m\leq\nu_{k+1})df_m - \sum_{m=0}^{n}\chi(m\leq\nu_k)df_m\right)\\ &= \sum_{m=0}^{n}\left(\sum_{k\in\mathbf{Z}}\chi(\nu_k < m \leq \nu_{k+1})df_m\right) = f_n,\end{aligned} \tag{3.2}$$

where $\chi(A)$ denotes the characteristic function of the set A. Now let $a_n^k = f_n^{\nu_{k+1}} - f_n^{\nu_k}$ $(k \in \mathbf{Z},\ n \in \mathbf{N})$. It is clear that for any fixed $k \in \mathbf{Z}$, $a^k = (a_n^k)_{n\geq 0}$ is a martingale. Since $s(f^{\nu_k}) = s_{\nu_k}(f) \leq 2^k$, we have

$$\begin{aligned}s(a^k) &= \left(\sum_{i=0}^{\infty}\mathbb{E}_{i-1}|da_i^k|^2\right)^{\frac{1}{2}}\left(\sum_{i=0}^{\infty}\mathbb{E}_{i-1}|df_i^{\nu_{k+1}} - df_i^{\nu_k}|^2\right)^{\frac{1}{2}}\\ &\leq s(f^{\nu_{k+1}}) + s(f^{\nu_k}) \leq 3\cdot 2^k.\end{aligned} \tag{3.3}$$

Thus $\|Ma\|_2 \leq C\|s(a^k)\|_2 \leq C\cdot 3\cdot 2^k$ and $(a_n^k)_{n\geq 0}$ is L_2-bounded. So $(a_n^k)_{n\geq 0}$ converges almost everywhere. Denote the limit still by a^k. Then $\mathbb{E}_n a^k = a_n^k$ $(\forall n \geq 0)$. It is clear that $a_n^k = 0$ if $n \leq \nu_k$, by (3.3) we have $\|s(a^k)\|_\infty < \infty$, so a^k is a w-1-atom. It follows from (3.2) that

$$f_n = \sum_{k\in\mathbf{Z}}(f_n^{\nu_{k+1}} - f_n^{\nu_k}) = \sum_{k\in\mathbf{Z}} a_n^k = \sum_{k\in\mathbf{Z}}\mathbb{E}_n a^k.$$

Hence we get (i).

Since $\{\nu_k < \infty\} = \{s(f) > 2^k\}$, for any $k \in \mathbf{Z}$ we have

$$\sum_{k\in\mathbf{Z}} 2^{kq}\mathbb{P}(\nu_k < \infty)^{\frac{q}{p}} = \sum_{k\in\mathbf{Z}} 2^{kq}\mathbb{P}(s(f) > 2^k)^{\frac{q}{p}} \le C\bigg(\sum_{k\in\mathbf{Z}} \int_{2^{k-1}}^{2^k} y^{q-1}dy\mathbb{P}(s(f) > 2^k\bigg)^{\frac{q}{p}}$$
$$\le C\bigg(\sum_{k\in\mathbf{Z}} \int_{2^{k-1}}^{2^k} y^{q-1}\mathbb{P}(s(f) > y\bigg)^{\frac{q}{p}} dy$$
$$\le C\int_0^\infty y^{q-1}\mathbb{P}(s(f) > y)^{\frac{q}{p}} dy \le C\|f\|^q_{H^s_{p,q}},$$

from which we get (3.1).

Let $f \in H^s_{p,q}$, since

$$f - \sum_{k=l}^{m} a^k = (f - f^{\nu_{m+1}}) + f^{\nu_l}, s(f - f^{\nu_{m+1}}) \le s(f), s(f^{\nu_l}) \le s(f),$$

and the a.e. limits of $s(f - f^{\nu_{m+1}})$ and $s(f^{\nu_l})$ are equal to 0 as $m \to \infty, l \to -\infty$, we know that the sum $\sum_{k\in\mathbf{Z}} a^k$ converges to f in $H^s_{p,q}$. The proof is complete. $\square$

Theorem 3.3. *The dual of $H^s_{p,q}$ is $\Lambda_2(\alpha)$, where $0 < p < 1$, $0 < q \le 1$,$\alpha = \frac{1}{p} - 1$.*

Proof. By Lemma 3.2 and the inequality

$$\|f\|_{H^s_{p,q}} \le \|s(f)\|_2 = \|f\|_2, \tag{3.4}$$

we know that L_2 is dense in $H^s_{p,q}$. Set

$$l_\phi(f) = \mathbb{E}(f\phi) \quad (f \in L_2),$$

where $\phi \in \Lambda_2(\alpha)$ is arbitrary. If $f \in L_2$, one can show that

$$\sum_{k\in\mathbf{Z}} a^k = \sum_{k\in\mathbf{Z}} (f^{\nu_{k+1}} - f^{\nu_k}) = f$$

holds a.e. and also in L_2 norm as in Lemma 3.2. So we have

$$l_\phi(f) = \sum_{k\in\mathbf{Z}} \mathbb{E}(a^k\phi).$$

By (i) of the definition of th weak atom a^k

$$\mathbb{E}(a^k\phi) = \mathbb{E}[a^k(\phi - \phi^{\nu_k})].$$

It follows that

$$\begin{aligned}|l_\phi(f)| &\le \sum_{k\in\mathbf{Z}} \|a^k\|_2\|\phi-\phi^{\nu_k}\|_2 \le C\sum_{k\in\mathbf{Z}} 2^k\mathbb{P}(\nu_k<\infty)^{\frac12}\|\phi-\phi^{\nu_k}\|_2\\ &\le C\sum_{k\in\mathbf{Z}} 2^k\mathbb{P}(\nu_k<\infty)^{\frac1p}\mathbb{P}(\nu_k<\infty)^{\frac12-\frac1p}\|\phi-\phi^{\nu_k}\|_2\\ &\le C\sum_{k\in\mathbf{Z}} 2^k\mathbb{P}(\nu_k<\infty)^{\frac1p}\|\phi\|_{\Lambda_2(\alpha)}\end{aligned}$$

Since $0<q\le 1$, we have

$$|l_\phi(f)|^q \le C\sum_{k\in\mathbf{Z}} 2^{kq}\mathbb{P}(\nu_k<\infty)^{\frac qp}\|\phi\|^q_{\Lambda_2(\alpha)}.$$

Consequently, we obtain from Lemma 3.2 that

$$|l_\phi(f)| \le C\|f\|_{H^s_{p,q}}\|\phi\|_{\Lambda_2(\alpha)}.$$

Conversely, let $l\in(H^s_{p,q})^*$ be a bounded linear functional on $H^s_{p,q}$. According to (3.4), there exists $\phi\in L_2$ such that

$$l_\phi(f)=\mathbb{E}(f\phi)\quad (f\in L_2).$$

Let ν be an arbitrary stopping time and set

$$g=\frac{\phi-\phi^\nu}{\|\phi-\phi^\nu\|_2\mathbb{P}(\nu_k<\infty)^{\frac1p-\frac12}}.$$

Obviously, $s(g)=s(g)\chi(\nu<\infty)$, then by Hölder inequality we have

$$\begin{aligned}\|g\|_{H^s_{p,q}} &= \left(\int_0^{\mathbb{P}(\nu<\infty)}(t^{\frac1p}s(g)^*(t))^q\frac{dt}{t}\right)^{\frac1q}\\ &\le \left(\int_0^{\mathbb{P}(\nu<\infty)}(s(g)^*(t))^2dt\right)^{\frac12}\left(\int_0^{\mathbb{P}(\nu<\infty)}t^{(\frac qp-1)\cdot\frac{2}{2-q}}dt\right)^{\frac{2-q}{2q}}\\ &\le C_{p,q}\|g\|_{H^s_2}\mathbb{P}(\nu<\infty)^{\frac1p-\frac12}\\ &= C_{p,q}\|g\|_2\mathbb{P}(\nu<\infty)^{\frac1p-\frac12}=C_{p,q},\end{aligned}$$

where $C_{p,q}=(\frac{2p-pq}{2q-pq})^{\frac{2-q}{2q}}$. Thus

$$C_{p,q}\|l\|\ge|l(g)|=\mathbb{E}[g\cdot(\phi-\phi^\nu)]=\mathbb{P}(\nu<\infty)^{\frac12-\frac1p}\|\phi-\phi^\nu\|_2,$$

which indicates that $\|\phi\|_{\Lambda_2(\alpha)}\le C_{p,q}\|l\|$. The proof is complete. □

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