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Diophantine quadruples in the sequence of shifted Tribonacci numbers

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Abstract. The Tribonacci sequence $\{T_n\}_{n\geq 0}$ has initial values $T_0 = 0$, $T_1{=}T_2{=}1$ and each term afterwards is the sum of the preceding three terms. In this paper, we study sequences $a_1, \dots, a_m$ of positive integers such that the product of any two different terms is a Tribonacci number. We prove that there is no such example with $m = 4$, give an example with $m = 3$, and leave as an open problem to find all examples for $m = 3$.

1. Introduction

Let $\mathbf{U} := \{U_n\}_{n\geq 0}$ be a sequence of integers. We say that a finite sequence $a_1, \dots, a_m$ of positive integers is *a subdiophantine sequence associated to* $\mathbf{U}$ if $a_i a_j$ is a member of $\{U_n\}_{n\geq 0}$ for all $1 \leq i < j \leq m$. We assume that $m \geq 3$ to avoid trivialities. The above definition is equivalent to $\{a_1, \dots, a_m\}$ being a Diophantine m-tuple with values in the sequence $\mathbf{U} - 1 := \{U_n - 1\}_{n\geq 0}$ in the sense of Fuchs, Luca and Szalay [7].

Some interesting problems appear when $\mathbf{U}$ is a linearly recurrent sequence. Consider the Fibonacci sequence $\mathbf{F} := \{F_n\}_{n\geq 0}$ given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. It is easy to see that there are no subdiophantine sequences of any size $m \geq 3$ associated to $\mathbf{F}$. Indeed, it is enough to show that there is no such sequence of size $m = 3$. Assume that there is one and so let $a < b < c$ be positive integers such that

$$ab = F_x, \qquad bc = F_y, \quad ac = F_z, \tag{1}$$

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for some positive integers x, y, z. Hence, $3 \leq x < z < y$. By CARMICHAEL's Primitive Divisor Theorem (see [3]), we know that if $n \geq 13$, then F_n has a *primitive prime factor*, that is a prime factor p which does not divide any F_m for $1 \leq m < n$. However, in (1), any prime factor p of F_y divides either b, or c, so, in particular, it divides either F_x or F_z. This shows that F_y has no primitive prime factors, therefore $y \leq 12$ and now a simple check reveals that (1) has no solutions.

In this paper, we look at the Tribonacci sequence $\mathbf{T} := \{T_n\}_{n\geq 0}$ given by $T_0 = 0$, $T_1 = T_2 = 1$, and

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \quad \text{for all } n \geq 0.$$

We study the existence of subdiophantine sequences associated to $\mathbf{T}$. Although the Tribonacci sequence does not enjoy the same nice arithmetic properties of the Fibonacci sequence, we prove nevertheless the following result.

Main Theorem. *There do not exist subdiophantine sequences associated to* $\mathbf{T}$ *with more than three terms, i.e., any set of* 4 *or more positive integers have at least two elements for which its product is not a number Tribonacci.*

Similar problems have been studied in [1], [4], [7], [8], [9], [11]. We conjecture that in fact there are only finitely many triples $\{a, b, c\}$ of distinct positive integers such that ab, bc, ac are all three in $\mathbf{T}$. We leave this as well as the calculation of all such examples as a project for the reader.

2. Preliminaries

2.1. The Tribonacci sequence. In the paper [5], DRESDEN and DU give a *Binet-like* formula for Tribonacci numbers:

$$T_n = c_\alpha \alpha^{n-1} + c_\beta \beta^{n-1} + c_\gamma \gamma^{n-1}, \tag{2}$$

where α is the real root of characteristic polynomial $\Psi(x) = x^3 - x^2 - x - 1$, associated $\mathbf{T}$, β, γ are its complex conjugated roots

$$\beta = \alpha^{-1/2} e^{i\theta} \quad \text{and} \quad \gamma = \alpha^{-1/2} e^{-i\theta} \text{ with } \theta \in (0, 2\pi). \tag{3}$$

and $c_z = (z-1)/(4z-6)$ for all $z \neq 3/2$.

In [5], it is also shown that the contribution of the roots complex β and γ, to the right-hand side of (2) is very small. More precisely, it is proved that the inequality

$$\left|T_n - c_\alpha \alpha^{n-1}\right| < \frac{1}{2} \quad \text{holds for all } n \geq 1. \tag{4}$$

Another well-known property of the Tribonacci numbers which is useful to us is the following (see [2]):

$$\alpha^{n-2} \le T_n \le \alpha^{n-1} \quad \text{for all } n \ge 1. \tag{5}$$

2.2. Linear forms in logarithms. Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers

$$f(X) := a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive. The *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d} \left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\eta^{(i)}|, 1\} \right).$$

Our main tool is a lower bound for a linear form in logarithms of algebraic numbers given by the following result of Matveev [**?**]:

Theorem 1 (Matveev's theorem)**.** *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\eta_1, \dots, \eta_t$ non-zero elements of $\mathbb{K}$, and $b_1, \dots, b_t$ rational integers. Put*

$$\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1 \quad \text{and} \quad B \ge \max\{|b_1|, \dots, |b_t|\}.$$

Let $A_i \ge \max\{Dh(\eta_i), |\log \eta_i|, 0.16\}$ be real numbers, for $i = 1, \dots, t$. Then, assuming that $\Lambda \ne 0$, we have

$$|\Lambda| > \exp(-3 \times 30^{t+4} \times (t+1)^{5.5} \times D^2 (1 + \log D)(1 + \log tB) A_1 \cdots A_t).$$

2.3. The Reduction Lemma. In the course of our calculations, we get some upper bounds on our variables which are very large, so we need to reduce them. With this aim, we use the following result which is a slight variation of a result due to Dujella and Pethő, which itself is a generalization of a result of Baker and Davenport (see [6] and [2]). For a real number x, we put $\|x\| = \min\{|x-n| : n \in \mathbb{Z}\}$ for the distance from x to the nearest integer.

Lemma 1. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := \|\mu q\| - M\|\tau q\|$. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\tau - n + \mu < AB^{-s},$$

in positive integers m, n and s with

$$m \le M \quad \text{and} \quad s \ge \frac{\log(Aq/\epsilon)}{\log B}.$$

3. A first observation

Let $a < b < c < d$ a subdiophantine quadruple associated to $\mathbf{T}$. Then

$$ab = T_x, \quad bc = T_y, \quad cd = T_z, \quad ad = T_w, \tag{6}$$

for some positive integers x, y, z and w. We see easily that

$$3 \leq x < \min\{y, w\} \leq \max\{y, w\} < z. \tag{7}$$

The equalities in (6) lead us to

$$T_x T_z = T_y T_w, \tag{8}$$

a formula which we will use repeatedly. From inequalities (5), we have

$$\alpha^{x+z-4} \leq T_x T_z \leq \alpha^{x+z-2} \quad \text{and} \quad \alpha^{y+w-4} \leq T_y T_w \leq \alpha^{y+w-2}.$$

Hence, using (8), we deduce that

$$|(x+z) - (y+w)| \leq 2. \tag{9}$$

For the rest of this paper, we work with the Diophantine equation (8) by distinguishing two cases

$$x + z \neq y + w \quad \text{and} \quad x + z = y + w,$$

respectively.

4. The case $x + z \neq y + w$

By using formula (2) and inequality (4), we have that

$$T_n = c_\alpha \alpha^{n-1} + e(n), \quad \text{with } |e(n)| < 1/2. \tag{10}$$

Thus, by expanding both sides of equation (8) and performing some arithmetic, we get

$$\begin{aligned} c_\alpha^2 \alpha^{x+z-2} - c_\alpha^2 \alpha^{y+w-2} &= c_\alpha e(w) \alpha^{y-1} + c_\alpha e(y) \alpha^{w-1} + e(y)e(w) \\ &\quad - c_\alpha e(z) \alpha^{x-1} - c_\alpha e(x) \alpha^{z-1} - e(x)e(z). \end{aligned}$$

Dividing both sides of above equation by $c_\alpha^2\alpha^{x+z-2}$ and taking absolute values, we get

$$\left|1-\alpha^{-(x+z-y-w)}\right| < \frac{1}{\alpha^{x-1}}\left(\frac{1}{2c_\alpha}+\frac{1}{c_\alpha\alpha}+\frac{1}{2c_\alpha\alpha^2}+\frac{1}{2c_\alpha^2\alpha^4}\right) < \frac{2.6}{\alpha^{x-1}}, \quad (11)$$

where we have used (7). On the other hand, using inequality (9), we get

$$\min_{0<|x+z-y-w|\leq 2} |1-\alpha^{-(x+z-y-w)}| > 0.4563. \quad (12)$$

From (7), (11) and (12), we get $x=3$.

From equation (8), get $2T_z = T_\lambda T_\delta$, where

$$\lambda = \min\{y,w\} \leq \delta = \max\{y,w\}.$$

Replacing T_z, T_λ, T_δ according to (10) in the last equation above, we get

$$2c_\alpha\alpha^{z-1} - c_\alpha^2\alpha^{\lambda+\delta-2} = c_\alpha e(\delta)\alpha^{\lambda-1} + c_\alpha e(\lambda)\alpha^{\delta-1} + e(\lambda)e(\delta) - 2e(z).$$

Dividing both sides of above equation by $2c_\alpha\alpha^{z-1}$ and taking absolute values, we get

$$\left|1-2^{-1}c_\alpha\alpha^{\lambda+\delta-z-1}\right| < \frac{1/4}{\alpha^{z-\lambda}} + \frac{1/4}{\alpha^{z-\delta}} + \frac{5/(8c_\alpha\alpha^3)}{\alpha^{z-4}} < \frac{0.7}{\alpha^{z-\delta}}, \quad (13)$$

where we used the fact that $z-4 \geq z-\lambda \geq z-\delta$. However, by inequality (9) and the fact that $x=3$, we obtain that $|\lambda+\delta-z-1| \leq 4$. We check that

$$\min_{|\lambda+\delta-z-1|\leq 4} |1-2^{-1}c_\alpha\alpha^{\lambda+\delta-z-1}| > 0.046. \quad (14)$$

Thus, combining (7), (13), and (14), we conclude that $1 \leq z-\delta \leq 4$. Returning to inequality (9), we get that $4 \leq \lambda \leq 9$.

We go back again to the equality $2T_z = T_\lambda T_\delta$. Replacing T_z, T_δ according to (10), dividing both sides by $c_\alpha T_\lambda \alpha^{\delta-1}$ performing some algebra and taking value absolutes, we get

$$\left|2T_\lambda^{-1}\alpha^{z-\delta}-1\right| < \frac{1.3}{\alpha^{\delta-1}}. \quad (15)$$

By analyzing the minimum value of the left-hand side in (15), we get

$$\min_{\substack{4\leq\lambda\leq 9\\ 1\leq z-\delta\leq 4}} |2T_\lambda^{-1}\alpha^{z-\delta}-1| > 0.0334. \quad (16)$$

Hence, from inequalities (7), (15), and (16), we conclude that $5 \leq \delta \leq 7$. In particular, $5 \leq z \leq 11$.

Let us record what we have proved so far.

Lemma 2. *Let $3 \leq x < \min\{y,w\} \leq \max\{y,w\} < z$ be positive integers such that $T_xT_z = T_yT_w$ and $x+z \neq y+w$. Then*

$$x=3, \quad 4\leq y,\ w\leq 7, \quad \textit{and} \quad 5\leq z\leq 11.$$

5. The case $x+z=y+w$

By (2) and (3), we have that

$$T_n = d_\alpha \alpha^n + d_\beta \beta^n + d_\gamma \gamma^n,$$

where

$$\beta = \alpha^{-1/2} e^{i\theta}, \quad \gamma = \alpha^{-1/2} e^{-i\theta}, \quad \text{and} \quad d_z = (z-1)/z(4z-6).$$

Putting $d_\beta = \rho e^{i\omega}$ and $d_\gamma = \rho e^{-i\omega}$, with $\omega \in (0,\pi)$, we note that T_n can be rewritten as

$$T_n = d_\alpha \alpha^n \left(1 + \frac{2\rho/d_\alpha}{\alpha^{\frac{3}{2}n}} \cos(\omega + n\theta)\right). \tag{17}$$

After using the above identity in the Diophantine equation (8) and performing some calculations, we arrive at

$$\begin{aligned}\frac{\cos(\omega + x\theta)}{\alpha^{\frac{3}{2}x}} = &\frac{\cos(\omega+\lambda\theta)}{\alpha^{\frac{3}{2}\lambda}} + \frac{\cos(\omega+\delta\theta)}{\alpha^{\frac{3}{2}\delta}} - \frac{\cos(\omega+z\theta)}{\alpha^{\frac{3}{2}z}} \\ &+ \frac{(2\rho/d_\alpha)\cos(\omega+\lambda\theta)\cos(\omega+\delta\theta)}{\alpha^{\frac{3}{2}(x+z)}} \\ &- \frac{(2\rho/d_\alpha)\cos(\omega+x\theta)\cos(\omega+z\theta)}{\alpha^{\frac{3}{2}(\lambda+\delta)}}.\end{aligned} \tag{18}$$

Multiplying by $\alpha^{\frac{3}{2}x}$ in both sides of (18) and taking absolute values, we get

$$\begin{aligned}|\cos(\omega+x\theta)| &< \frac{1}{\alpha^{\frac{3}{2}(\lambda-x)}} + \frac{1}{\alpha^{\frac{3}{2}(\delta-x)}} + \frac{1}{\alpha^{\frac{3}{2}(z-x)}} + \frac{4\rho/d_\alpha}{\alpha^{\frac{3}{2}z}} \\ &< \frac{1}{\alpha^{\frac{3}{2}(\lambda-x)}}\left(2 + \frac{1}{\alpha^{\frac{3}{2}}} + \frac{4\rho}{d_\alpha \alpha^6}\right) < \frac{2.5}{\alpha^{\frac{3}{2}(\lambda-x)}}.\end{aligned}$$

In the above estimates, we used inequalities (7), and $x+z=\lambda+\delta$. But

$$2\cos(\omega+x\theta) = 1 + e^{2i(\omega+x\theta)} = 1 - \left(-\frac{d_\beta}{d_\gamma}\right)\left(\frac{\beta}{\gamma}\right)^x.$$

Then

$$\left|1 - \left(-\frac{d_\beta}{d_\gamma}\right)\left(\frac{\beta}{\gamma}\right)^x\right| < \frac{2.5}{\alpha^{\frac{3}{2}(\lambda-x)}}. \tag{19}$$

In order to find an upper bound for $\lambda - x$, we use Theorem 1 with the parameters

$$t := 2, \quad \eta_1 := -\frac{d_\beta}{d_\gamma}, \quad \eta_2 := \frac{\beta}{\gamma}, \quad b_1 := 1, \quad b_2 := x.$$

Thus, $\Lambda_1 := 1 - (-d_\beta/d_\gamma)(\beta/\gamma)^x$, and from (19), we have that

$$|\Lambda_1| < \frac{2.5}{\alpha^{\frac{3}{2}(\lambda - x)}}. \tag{20}$$

The number field $\mathbb{K} = \mathbb{Q}(\alpha, \beta)$ contains η_1, η_2 and has degree $D = 6$ over $\mathbb{Q}$. A simple check shows that the minimal polynomials of η_1 and η_2 are

$$\prod_{\sigma \in G} \left(X + \sigma \left(\frac{d_\beta}{d_\gamma} \right) \right) = 11X^6 - 33X^5 + 64X^4 + 73X^3 + 64X^2 + 33X + 11, bb$$

and

$$\prod_{\sigma \in G} (X - \sigma(\beta/\gamma)) = X^6 + 4X^5 + 11bX^4 + 12X^3 + 11X^2 + 4X + 1, bb$$

respectively, where G is the Galois group $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$.

Furthermore, the conjugates of η_1 and η_2 satisfy

$$\left|\frac{d_\beta}{d_\gamma}\right| = \left|\frac{d_\gamma}{d_\beta}\right| = 1, \quad \left|\frac{d_\beta}{d_\alpha}\right| = \left|\frac{d_\gamma}{d_\alpha}\right| = 0.773\ldots, \quad \left|\frac{d_\alpha}{d_\beta}\right| = \left|\frac{d_\alpha}{d_\gamma}\right| = 1.293\ldots$$

and

$$\left|\frac{\beta}{\gamma}\right| = \left|\frac{\gamma}{\beta}\right| = 1, \quad \left|\frac{\beta}{\alpha}\right| = \left|\frac{\gamma}{\alpha}\right| = 0.4008\ldots, \quad \left|\frac{\alpha}{\beta}\right| = \left|\frac{\alpha}{\gamma}\right| = 2.494\ldots.$$

Hence, $h(\eta_1) = \frac{1}{6}\big(\log 11 + 2\log|\frac{d_\alpha}{d_\beta}|\big) < 0.5$ and $h(\eta_2) = \frac{1}{3}(\log|\alpha/\beta|) < 0.31$. So, we can take $A_1 := 3$ and $A_2 := 2$, given that $|\log \eta_1| < 2$ and $|\log \eta_2| < 2$. Finally, $\Lambda_1 \neq 0$ because β/γ is a algebraic integer while d_γ/d_β isn't. We put $B := x$.

By Theorem 1, we obtain

$$\begin{aligned} |\Lambda_1| &> \exp(-3 \cdot 30^6 \cdot 3^{5.5} \cdot 6^2 \cdot (1 + \log 6) \cdot (1 + \log(2x)) \cdot 3 \cdot 2) \\ &> \exp(-1.7 \cdot 10^{15} \log x). \end{aligned} \tag{21}$$

In above inequality, we used the inequality $1 + \log(2x) < 3\log x$ valid for all $x \geq 3$.

Therefore, combining (20) and (21), we conclude that

$$\lambda - x < 2 \cdot 10^{15} \log x. \tag{22}$$

Going back to equality (18), we group the dominant terms $\alpha^{-\frac{3}{2}(x-1)}$ and $\alpha^{-\frac{3}{2}(\lambda-1)}$ in one side and all the other terms in the other side, multiply the resulting equation by $2\alpha^{\frac{3}{2}x}$, and take absolute values, getting

$$\begin{aligned} \left| 2\cos(\omega + x\theta) - 2\cos(\omega + \lambda\theta)\alpha^{-\frac{3}{2}(\lambda - x)} \right| &< \frac{2}{\alpha^{\frac{3}{2}(\delta - x)}} + \frac{2}{\alpha^{\frac{3}{2}(z - x)}} + \frac{8\rho/d_\alpha}{\alpha^{\frac{3}{2}(\lambda + \delta)}} \\ &< \frac{1}{\alpha^{\frac{3}{2}(\delta - x)}} \left(2 + \frac{2}{\alpha^{\frac{3}{2}}} + \frac{8\rho}{d_\alpha \alpha^6} \right) < \frac{2.6}{\alpha^{\frac{3}{2}(\delta - x)}}. \end{aligned} \tag{23}$$

We remark that

i) $2\cos(\omega + x\theta) - 2\cos(\omega + \lambda\theta)\alpha^{-\frac{3}{2}(\lambda - x)} \neq 0$. Indeed, otherwise

$$\frac{2\cos(\omega + x\theta)}{\alpha^{\frac{3}{2}x}} = \frac{2\cos(\omega + \lambda\theta)}{\alpha^{\frac{3}{2}\lambda}}.$$

Multiplying by ρ and then adding d_α to each side, we obtain by (17) that $T_x/\alpha^x = T_\lambda/\alpha^\lambda$, or equivalently $\alpha^{-(\lambda - x)} = T_x/T_\lambda$. However, as α^{-1} is a unit (an algebraic integer whose reciprocal is also an algebraic integer), we have that $T_x/T_\lambda = 1$, or $x = \lambda$, which is not possible.

ii) $2\cos(\omega + x\theta) - 2\cos(\omega + \lambda\theta)\alpha^{-\frac{3}{2}(\lambda - x)}$ is equal to

$$\begin{aligned} & 2\mathrm{Re}\left[e^{i(\omega + x\theta)}\left(1 - \alpha^{-\frac{3}{2}(\lambda - x)}e^{i(\lambda - x)\theta}\right)\right] \\ = {} & e^{-i(\omega + x\theta)}\left(1 - (\alpha^{-\frac{3}{2}}e^{-i\theta})^{\lambda - x}\right)\left[1 - e^{2i\omega}(e^{2i\theta})^x\frac{(\alpha^{-\frac{3}{2}}e^{i\theta})^{\lambda - x} - 1}{(\alpha^{-\frac{3}{2}}e^{-i\theta})^{\lambda - x} - 1}\right]. \end{aligned}$$

But, given that

$$\alpha^{-\frac{3}{2}}e^{i\theta} = \beta/\alpha, \qquad \alpha^{-\frac{3}{2}}e^{-i\theta} = \gamma/\alpha, \qquad e^{2i\omega} = d_\beta/d_\gamma, \qquad e^{2i\theta} = \beta/\gamma, \tag{24}$$

we conclude from (23) and (5) that

$$\left|1 - \left(\frac{\gamma}{\alpha}\right)^{\lambda - x}\right|\left|1 - \left(\frac{d_\beta}{d_\gamma}\right)\left(\frac{\beta}{\gamma}\right)^x\frac{\left(\frac{\beta}{\alpha}\right)^{\lambda - x} - 1}{\left(\frac{\gamma}{\alpha}\right)^{\lambda - x} - 1}\right| < \frac{2.6}{\alpha^{\frac{3}{2}(\delta - x)}}. \tag{25}$$

As in the previous application of the linear forms in logarithms, we consider the number field $\mathbb{K} = \mathbb{Q}(\alpha, \beta)$ and $D = 6$. Further, we compute that $h(\gamma/\alpha) = h(\beta/\gamma) < 0.31$. Hence, we obtain from Theorem 1 that

$$\begin{aligned} \left|1 - \left(\frac{\gamma}{\alpha}\right)^{\lambda - x}\right| & > \exp(-3 \cdot 30^5 \cdot 2^{5.5} \cdot 6^2 \cdot (1 + \log 6) \cdot (1 + \log(\lambda - x)) \cdot 2) \\ & > \exp(-1.4 \cdot 10^{12}\log(2(\lambda - x)) > \exp(-6 \cdot 10^{14}\log\log x). \end{aligned} \tag{26}$$

We have used in above inequality the fact that $1 + \log(\lambda - x) < 2\log(2(\lambda - x))$, for all $\lambda - x \geq 1$ and, by (22), that $\log(2(\lambda - x)) < 4 \cdot 10^2 \log\log x$ for all $x \geq 3$. On the other hand,

$$h\left(\frac{(\beta/\alpha)^{\lambda - x} - 1}{(\gamma/\alpha)^{\lambda - x} - 1}\right) \leq 2h((\beta/\alpha)^{\lambda - x} - 1).$$

Given that β/α is an algebraic integer and

$$|\sigma(\beta/\alpha)^{\lambda-x} - 1| \leq 2.5^{\lambda-x} + 1, \quad \text{for all } \sigma \in G,$$

we conclude that $h((\beta/\alpha)^{\lambda-x}-1) \leq \log(2.5^{\lambda-x}+1) < (\lambda-x)\log 3$. Furthermore,

$$\begin{aligned}\left|\log\left(\frac{(\beta/\alpha)^{\lambda-x}-1}{(\gamma/\alpha)^{\lambda-x}-1}\right)\right| &\leq |\log(1-(\beta/\alpha)^{\lambda-x})| + |\log(1-(\gamma/\alpha)^{\lambda-x})| \\ &\leq 2\sum_{m=1}^{\infty}\frac{1}{m}\left|\frac{\beta}{\alpha}\right|^{(\lambda-x)m} < 2\sum_{m=1}^{\infty}\left(\frac{1}{2^{(\lambda-x)}}\right)^m = \frac{2}{2^{\lambda-x}-1}.\end{aligned}$$

Below we shall apply Theorem 1 with the data

$$t := 3, \quad \eta_1 := -\frac{d_\beta}{d_\gamma}, \quad \eta_2 := \frac{\beta}{\gamma}, \quad \eta_3 := \frac{(\beta/\alpha)^{\lambda-x}-1}{(\gamma/\alpha)^{\lambda-x}-1},$$
$$b_1 := 1, \quad b_2 := x, \quad b_3 := 1.$$

We put $B := x$. Referring to the previous calculations, we take $A_1 := 3$, $A_2 := 2$ and $A_3 := 12(\lambda-x)\log 3 \geq \max\{Dh(\eta_3), |\log(\eta_3)|\}$. The conclusion of Theorem 1 leads us to the following inequality:

$$\begin{aligned}&\left|1-(d_\beta/d_\gamma)(\beta/\gamma)^x\left((\beta/\alpha)^{\lambda-x}-1\right)\left((\gamma/\alpha)^{\lambda-x}-1\right)^{-1}\right| \\ &\quad > \exp(-3\cdot 30^7\cdot 4^{5.5}\cdot 6^2\cdot(1+\log 6)(1+\log(3x)))\cdot 3\cdot 2\cdot 12(\lambda-x)\log 3) \\ &\quad > \exp(-6.5\cdot 10^{33}\log^2 x). \qquad (27)\end{aligned}$$

In above inequality, we used that the inequality $1+\log(3x) < 3\log x$ holds for all $x \geq 3$, as well as inequality (22). Combining the inequalities (25), (26) and (27), we get

$$\exp(-6.5\cdot 10^{33}\log^2 x - 6\cdot 10^{14}\log\log x) < \frac{2.6}{\alpha^{\frac{3}{2}(\delta-x)}},$$

which leads us to

$$\delta - x < 7.2\cdot 10^{33}\log^2 x + 6.6\cdot 10^{14}\log\log x < 8\cdot 10^{33}\log^2 x. \qquad (28)$$

Given that $z - x = (\delta - x) + (\lambda - x)$, we obtain

$$z - x < 7.2\cdot 10^{33}\log^2 x + 6.6\cdot 10^{14}\log\log x + 2\cdot 10^{15}\log x < 8\cdot 10^{33}\log^2 x. \qquad (29)$$

Let us record what we have proved so far.

Lemma 3. *Let $3 \leq x < \min\{y, w\} \leq \max\{y, w\} < z$ be positive integers such that $T_x T_z = T_y T_w$ and $x + z = y + w$. If $h = \lambda - x$, $k = \delta - x$ and $l = z - x$, then the inequalities*

$$l < 8 \cdot 10^{33} \log^2 x, \qquad k < 8 \cdot 10^{33} \log^2 x, \qquad h < 2 \cdot 10^{15} \log x \tag{30}$$

hold.

Once again we return to equation (18) and this time we rewrite it as follows

$$\begin{aligned} &\frac{\cos(\omega + x\theta)}{\alpha^{\frac{3}{2}x}} + \frac{\cos(\omega + z\theta)}{\alpha^{\frac{3}{2}z}} - \frac{\cos(\omega + \delta\theta)}{\alpha^{\frac{3}{2}\delta}} - \frac{\cos(\omega + \lambda\theta)}{\alpha^{\frac{3}{2}\lambda}} \\ &= \frac{(2\rho/d_\alpha)\cos(\omega + \lambda\theta)\cos(\omega + \delta\theta)}{\alpha^{\frac{3}{2}(x+z)}} - \frac{(2\rho/d_\alpha)\cos(\omega + x\theta)\cos(\omega + z\theta)}{\alpha^{\frac{3}{2}(\lambda+\delta)}}. \end{aligned}$$

As before, we multiply both sides above by $2\alpha^{\frac{3}{2}x}$ and the take absolute values, to get

$$\begin{aligned} \big| 2\cos(\omega + x\theta) &+ 2\cos(\omega + z\theta)\alpha^{-\frac{3}{2}(z-x)} - 2\cos(\omega + \delta\theta)\alpha^{-\frac{3}{2}(\delta-x)} \\ &- 2\cos(\omega + \lambda\theta)\alpha^{-\frac{3}{2}(\lambda-x)}\big| < \frac{1}{\alpha^{\frac{3}{2}x}}. \end{aligned} \tag{31}$$

We let A stand for the term inside the absolute value on the left-hand side of the above inequality. With the aim to use once more time a linear forms in logarithms, we show that A is not zero and then rewrite A in a way that allows us to use Theorem 1. To see that $A \neq 0$, assume otherwise. We get

$$\frac{2\cos(\omega + x\theta)}{\alpha^{\frac{3}{2}x}} + \frac{2\cos(\omega + z\theta)}{\alpha^{\frac{3}{2}z}} = \frac{2\cos(\omega + \lambda\theta)}{\alpha^{\frac{3}{2}\lambda}} + \frac{2\cos(\omega + \delta\theta)}{\alpha^{\frac{3}{2}\delta}}.$$

We multiply by ρ and add δ_α in both sides. We recognize from (17) that the resulting expression is equivalent to

$$\frac{T_x}{\alpha^x} + \frac{T_z}{\alpha^z} = \frac{T_\lambda}{\alpha^\lambda} + \frac{T_\delta}{\alpha^\delta}.$$

Further, by equation (8), and the fact that $x + z = \lambda + \delta$, it follows that also $(T_x/\alpha^x)(T_z/\alpha^z) = (T_\lambda/\alpha^\lambda)(T_\delta/\alpha^\delta)$. So, we have that the sets $\{T_x/\alpha^x, T_z/\alpha^z\}$ and $\{T_\lambda/\alpha^\lambda, T_\delta/\alpha^\delta\}$ give the roots of the same quadratic equation. Thus,

$$T_x/\alpha^x = T_\lambda/\alpha^\lambda, \quad \text{or } T_x/\alpha^x = T_\delta/\alpha^\delta,$$

and in any case we get a contradiction, as we noted earlier. On the other hand, A can be rewriten as

$$e^{i(\omega+x\theta)}\left(1 + \alpha^{-\frac{3}{2}(z-x)}e^{i(z-x)\theta} - \alpha^{-\frac{3}{2}(\delta-x)}e^{i(\delta-x)\theta} - \alpha^{-\frac{3}{2}(\lambda-x)}e^{i(\lambda-x)\theta}\right)$$

$$+e^{-i(\omega+x\theta)}\left(1+\alpha^{-\frac{3}{2}(z-x)}e^{-i(z-x)\theta}-\alpha^{-\frac{3}{2}(\delta-x)}e^{-i(\delta-x)\theta}-\alpha^{-\frac{3}{2}(\lambda-x)}e^{-i(\lambda-x)\theta}\right)$$

and putting

$$B=1+\alpha^{-\frac{3}{2}(z-x)}e^{i(z-x)\theta}-\alpha^{-\frac{3}{2}(\delta-x)}e^{i(\delta-x)\theta}-\alpha^{-\frac{3}{2}(\lambda-x)}e^{i(\lambda-x)\theta},$$

we get

$$A=e^{-i(\omega+x\theta)}\overline{B}\left(1+e^{2i(\omega+x\theta)}\frac{B}{\overline{B}}\right), \tag{32}$$

where $\overline{B}$ denotes the complex conjugate of B. Moreover,

$$\overline{B}=\left(1-\left(e^{-i\theta}\alpha^{-\frac{3}{2}}\right)^{\delta-x}\right)\left[1-\left(e^{-i\theta}\alpha^{-\frac{3}{2}}\right)^{\lambda-x}\frac{\left(e^{-i\theta}\alpha^{-\frac{3}{2}}\right)^{z-\lambda}-1}{\left(e^{-i\theta}\alpha^{-\frac{3}{2}}\right)^{\delta-x}-1}\right]. \tag{33}$$

Hence, from equations (31), (32), (33) and the identities in (24), we conclude that

$$\left|1-\left(-\frac{d_\beta}{d_\gamma}\right)\left(\frac{\beta}{\gamma}\right)^x\frac{B}{\overline{B}}\right|\left|1-\left(\frac{\gamma}{\alpha}\right)^{\delta-x}\right|\times\left|1-\left(\frac{\gamma}{\alpha}\right)^{\lambda-x}\frac{\left(\frac{\gamma}{\alpha}\right)^{z-\lambda}-1}{\left(\frac{\gamma}{\alpha}\right)^{\delta-x}-1}\right|<\frac{3}{\alpha^{\frac{3}{2}x}}. \tag{34}$$

Here, we use linear forms in one, two and three logarithms to find a lower bound on each of the above absolute values.

As in the previous applications of Theorem 1, we have $\mathbb{K}=\mathbb{Q}(\alpha,\beta)$, $D=6$, $h(\gamma/\alpha)=h(\beta/\gamma)<0.31$, $h(-d_\beta/d_\gamma)<0.5$ and

$$h\left(\frac{(\gamma/\alpha)^{z-\lambda}-1}{(\gamma/\alpha)^{\delta-x}-1}\right)\le 2h\left((\gamma/\alpha)^{z-\lambda}-1\right)<2\log 3(z-x).$$

Also, given that

$$B=(\beta/\alpha)^{z-x}-(\beta/\alpha)^{\delta-x}-(\beta/\alpha)^{\lambda-x}+1\in\mathcal{O}_{\mathbb{K}},\quad h(B)=h(\overline{B})$$

and $|\sigma(B)|<3\cdot 2.5^{z-x}+1<3^{z-x}$ for all $\sigma\in G$ and $z-x\ge 2$, we get

$$h(B/\overline{B})\le 2h(B)\le 2\log 3(z-x).$$

Furthermore,

$$\begin{aligned}\left|\log(B/\overline{B})\right| &\le 2\left|\log\left(1-\left((\beta/\alpha)^{\lambda-x}+(\beta/\alpha)^{\delta-x}-(\beta/\alpha)^{z-x}\right)\right)\right|\\ &<2\sum_{m=1}^{\infty}\left(\left|\frac{\beta}{\alpha}\right|^{\lambda-x}\cdot(2+(|\beta|/\alpha))\right)^m=\frac{2(2+(|\beta|/\alpha))}{(\alpha/|\beta|)^{\lambda-x}-(2+(|\beta|/\alpha))}.\end{aligned}$$

In last inequality above, we used the fact that

$$\begin{aligned}\left|(\beta/\alpha)^{\lambda-x}+(\beta/\alpha)^{\delta-x}-(\beta/\alpha)^{z-x}\right| &\le (|\beta|/\alpha)^{\lambda-x}\left(1+(|\beta|/\alpha)^{\delta-\lambda}+(|\beta|/\alpha)^{z-\lambda}\right)\\ &\le (|\beta|/\alpha)^{\lambda-x}(2+(|\beta|/\alpha))<1.\end{aligned}$$

Applying Theorem 1 three times with the above information, we can conclude that

$$\left|1-\left(-\frac{d_\beta}{d_\gamma}\right)\left(\frac{\beta}{\gamma}\right)^x\frac{B}{\overline{B}}\right| > \exp\left(-2.6\cdot 10^{52}\log^3 x\right), \tag{35}$$

$$\left|1-\left(\frac{\gamma}{\alpha}\right)^{\delta-x}\right| > \exp\left(-5.6\cdot 10^{14}\log\log x\right) \tag{36}$$

and

$$\left|1-\left(\frac{\gamma}{\alpha}\right)^{\lambda-x}\frac{\left(\frac{\gamma}{\alpha}\right)^{z-\lambda}-1}{\left(\frac{\gamma}{\alpha}\right)^{\delta-x}-1}\right| > \exp\left(-6.8\cdot 10^{51}\log^2 x\log\log x\right). \tag{37}$$

Thus, the inequalities (34), (35), (36) and (37) lead us to inequality

$$x<1.8\cdot 10^{52}\log^3 x+4.6\cdot 10^{51}\log^2 x\log\log x+3.8\cdot 10^{14}\log\log x,$$

from which we have $x<4.5\cdot 10^{58}$. We record what we have just proved.

Lemma 4. *Let $3\le x<\min\{y,w\}\le\max\{y,w\}<z$ be positive integers such that $T_xT_z=T_yT_w$ and $x+z=y+w$. If $h=\lambda-x$, $k=\delta-x$ and $l=z-x$, then the inequalities*

$$x<4.5\cdot 10^{58},\quad k<l<1.5\cdot 10^{38},\quad h<2.8\cdot 10^{17} \tag{38}$$

hold.

6. Reducing h,k,l and x when $x+z=y+w$

We use Lemma 1 to reduce the bounds given in the inequalities (38) to cases that can be treated computationally.

6.1. Reduction of h. From inequality (19) is clear that

$$\left|\sin\left(\omega+x\theta-\frac{\pi}{2}\right)\right| = |\cos(\omega+x\theta)| < 2.5\cdot\alpha^{-\frac{3}{2}h}.$$

Putting $m := \lfloor(\omega + x\theta - \frac{\pi}{2})/\pi\rceil$, where $\lfloor y\rceil$ is the nearest integer to the real number y, we obtain that $-\pi/2 \le \omega + x\theta - \frac{\pi}{2} - m\pi \le \pi/2$. Hence,

$$\begin{aligned} 2.5 \cdot \alpha^{-\frac{3}{2}h} > \left|\sin\left(\omega + x\theta - \frac{\pi}{2}\right)\right| &= \left|\sin\left(\omega + x\theta - \frac{\pi}{2} - m\pi\right)\right| \\ &\ge \left|\frac{2\omega}{\pi} + \left(\frac{2\theta}{\pi}\right)x - 2m - 1\right|, \end{aligned} \tag{39}$$

where we have used the inequality

$$|\sin y| = \sin|y| \ge \frac{2}{\pi}|y| \quad \text{for all } -\pi/2 \le y \le \pi/2.$$

Thus, we conclude from inequality (39) that

$$\left|\left(\frac{\theta}{\pi}\right)x - m + \left(\frac{\omega}{\pi} - \frac{1}{2}\right)\right| < 1.3 \cdot \alpha^{-\frac{3}{2}h}. \tag{40}$$

We take

$$\Gamma_1 := (\theta/\pi)x - m + (\omega/\pi - 1/2),$$

which is nonzero.

If $\Gamma_1 > 0$, then, by (40), we get

$$0 < \left(\frac{\theta}{\pi}\right)x - m + \left(\frac{\omega}{\pi} - \frac{1}{2}\right) < 1.3 \cdot \alpha^{-\frac{3}{2}h}. \tag{41}$$

We put

$$\tau := \frac{\theta}{\pi}, \quad \mu := \frac{\omega}{\pi} - \frac{1}{2}, \quad A := 1.3, \quad B := \alpha^{\frac{3}{2}}.$$

Inequality (41) can be rewritten as

$$0 < \tau x - m + \mu < AB^{-h}. \tag{42}$$

The fact that $\mathbf{T}$ is non-degenerate ensures that γ is an irrational number. Lastly, we take $M := 4.5 \cdot 10^{58}$ which is an upper bound on x by inequalities in (38), and apply Lemma 1 to inequality (42). With the help of Mathematica, we found that the maximum value of $\lfloor \log(Aq/\epsilon)/\log B \rfloor$ is 19, which is an upper bound on h, according to Lemma 1. This was when $\Gamma_1 > 0$. But if $\Gamma_1 < 0$, then, from (40), we obtain

$$0 < \left(\frac{\pi}{\theta}\right)m - x + \left(\frac{\pi}{2\theta} - \frac{\omega}{\theta}\right) < \alpha^{-\frac{3}{2}h}. \tag{43}$$

In this other case, we have that

$$0 < \tau m - x + \mu < AB^{-x}, \tag{44}$$

where
$$\tau := \frac{\pi}{\theta}, \quad \mu := \frac{\pi}{2\theta} - \frac{\omega}{\theta}, \quad A := 1, \quad B := \alpha^{3/2}.$$

Finally, we take $M := 6.3 \cdot 10^{58}$ which is an upper bound on m because $m = \lfloor (\omega + x\theta - \frac{\pi}{2})/\pi \rceil < 1.4x$, and apply again Lemma 1 to inequality (44). With the help of Mathematica, we found that the maximum value of $\lfloor \log(Aq/\epsilon)/\log B \rfloor$ is 18, which is an upper bound on h, according to Lemma 1.

So, in summary, we have $1 \le h \le 19$.

6.2. Reduction of k and l. Here, we assume that $k \ge 2$. Note first that

$$\min_{1 \le h \le 19} \left| 1 - \left(\frac{\gamma}{\alpha}\right)^h \right| > 0.9376. \tag{45}$$

Thus, from inequality (25), we conclude that

$$\left| \left(\frac{d_\beta}{d_\gamma}\right) \left(\frac{\beta}{\gamma}\right)^x \frac{\left(\frac{\beta}{\alpha}\right)^h - 1}{\left(\frac{\gamma}{\alpha}\right)^h - 1} - 1 \right| < \frac{2.8}{\alpha^{\frac{3}{2}k}}. \tag{46}$$

We put
$$\Lambda_2 := (d_\beta/d_\gamma)\,(\beta/\gamma)^x\,((\beta/\alpha)^h - 1)((\gamma/\alpha)^h - 1)^{-1} - 1.$$

Thus, given that $k \ge 2$, we then get $|\Lambda_2| < 1/2$.

Taking
$$\log w = \log|w| + i \arg w \quad \text{with} \quad -\pi < \arg w \le \pi,$$

for the logarithm of a complex number w, we get

$$\log(1+w) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{w^n}{n} \quad \text{for } w \in \mathbb{C} \text{ with } |w| < 1.$$

From here, one easily shows that

$$|\log(1+w)| \le 2|w| \quad \text{if } |w| \le 1/2.$$

Hence, with $w = \Lambda_2$, and recalling that the complex logarithm is additive only modulo $2\pi i$ and that $\log(\beta/\gamma) = 2\theta i$, we obtain from inequality (46)

$$\left| \log\left(\left(\frac{d_\beta}{d_\gamma}\right) \frac{(\beta/\alpha)^h - 1}{(\gamma/\alpha)^h - 1} \right) + 2\theta x i + 2\pi t i \right| < \frac{5.6}{\alpha^{\frac{3}{2}k}} \tag{47}$$

for some $t \in \mathbb{Z}$. Using $|\log(1+\Lambda_2)| \le 2|\Lambda_2| < 1$ and inequalities (38) and (47), we have that

$$2\pi|t| \le 1 + 2\theta x + \max_{1\le h\le 19}\left|\log\left(\left(\frac{d_\beta}{d_\gamma}\right)\frac{(\beta/\alpha)^h - 1}{(\gamma/\alpha)^h - 1}(\gamma/\alpha)^h - 1\right)\right| < 9\cdot 10^{58},$$

which leads to $|t| < 1.5\cdot 10^{58}$. Let

$$\zeta(h) := \operatorname{Re}\left[-i\log\left(\left(\frac{d_\beta}{d_\gamma}\right)\frac{(\beta/\alpha)^h - 1}{(\gamma/\alpha)^h - 1}\right)\right].$$

We see from inequality (47) that

$$|2\theta x + 2\pi t + \zeta(h)| < \frac{5.6}{\alpha^{\frac{3}{2}k}}.$$

Furthermore, as $-2.01812 < \zeta(h) < -1.21087$ holds for all $1 \le h \le 19$ and $2\theta \in (\pi/2, \pi)$, we get that t must be a negative integer. Hence, by replacing t with $-t$, we can assume no less generality that

$$|2\theta x - 2\pi t + \zeta(h)| < \frac{5.6}{\alpha^{\frac{3}{2}k}}, \tag{48}$$

where x and t are positive integers $< 4.5\cdot 10^{48}$.

For each $h \in [1, 19]$, we used the reduction method of Lemma 1. Putting $\Gamma_2 := 2\theta x - 2\pi t + \zeta(h)$, we have $\Gamma_2 \ne 0$ since $\Lambda_2 \ne 0$. We describe in parallel the cases $\Gamma_2 > 0$ and $\Gamma_2 < 0$.

$\Gamma_2 > 0$	$\Gamma_2 < 0$
$0 < \left(\frac{\theta}{\pi}\right)x - t + \frac{\zeta(h)}{2\pi} < 0.9\alpha^{-\frac{3}{2}k}$	$0 < \left(\frac{\pi}{\theta}\right)t - x - \frac{\zeta(h)}{2\theta} < 2.91\alpha^{-\frac{3}{2}k}$
$\tau := \frac{\theta}{\pi},\ \mu_h := \frac{\zeta(h)}{2\pi}$	$\tau := \frac{\pi}{\theta},\ \mu_h := -\frac{\zeta(h)}{2\theta}$
$A := 0.9,\ B := \alpha^{\frac{3}{2}}$	$A := 2.91,\ B := \alpha^{\frac{3}{2}}$.

Conditions on x and t in equation (48), allow us to take $M := 4.5\cdot 10^{48}$ in both cases. Further, τ is a irrational number, because otherwise $\beta^m = \gamma^m$ for some positive integer m (by (24)). But this equality is not possible since if it were, than conjugating the above relation with the automorphism $\sigma : \alpha \to \gamma$, $\beta \to \beta$, $\gamma \to \alpha$, we obtain $\beta^m = \alpha^m$. However, in the above equation the absolute value of the left-hand side is < 1, while the absolute value of the right-hand side is > 1, which gives us the contradiction.

A new implementation of Lemma 1 in Mathematica tells us that the maximum value of $\lfloor \log(Aq/\epsilon)/\log B \rfloor$ is 6 and 7, respectively. So, we conclude that $2 \le k \le 7$. On the other hand, as $h \le k$ and

$$l = z - x = (\lambda - x) + (\delta - x) = h + k,$$

we get that $1 \le h \le 7$ and $2 \le l \le 14$.

Finally, we note that if $k = 1$, then $h = 1$ and $l = 2$. To summarize the last two subsections, we present the following inequalities:

$$1 \le h \le k < l, \quad k \le 7, \quad l \le 14. \tag{49}$$

6.3. Reduction of x. With the purpose of reducing the bound to x, we go back to inequality (34). By inequalities (45) and

$$\min_{\substack{k\le 7,\ l\le 14 \\ 1\le h\le k<l}} \left| 1 - \left(\frac{\gamma}{\alpha}\right)^h \frac{\left(\frac{\gamma}{\alpha}\right)^{l-h} - 1}{\left(\frac{\gamma}{\alpha}\right)^k - 1} \right| > 0.91,$$

we can conclude that

$$\left| \left(-\frac{d_\beta}{d_\gamma}\right) \left(\frac{\beta}{\gamma}\right)^x \left(\frac{B}{\overline{B}}\right) - 1 \right| < \frac{3.6}{\alpha^{\frac{3}{2}x}}, \tag{50}$$

where

$$B = (\beta/\alpha)^l - (\beta/\alpha)^k - (\beta/\alpha)^h + 1.$$

Let

$$\Lambda_3 := (-d_\beta/d_\gamma)\,(\beta/\gamma)^x\,(B/\overline{B}) - 1.$$

By inequality (50), it is clear that $|\Lambda_3| < 1/2$. From the arguments used at equations (46) and (47), we obtain that

$$\left| \log\left(\left(-\frac{d_\beta}{d_\gamma}\right) \left(\frac{B}{\overline{B}}\right) \right) + 2\theta x i + 2\pi t i \right| < \frac{7.2}{\alpha^{\frac{3}{2}x}},$$

for some $t \in \mathbb{Z}$. Even more,

$$\max_{\substack{k\le 7,\ l\le 14 \\ 1\le h\le k<l}} \left| \log\left(\left(-\frac{d_\beta}{d_\gamma}\right) \left(\frac{B}{\overline{B}}\right) \right) \right| < 1.41,$$

so $|t| < 1.5 \cdot 10^{58}$. Finally, we take

$$\zeta(l,k,h) := \mathrm{Re}\left[-i \log\left(\left(-\frac{d_\beta}{d_\gamma}\right)\left(\frac{B}{\overline{B}}\right)\right)\right].$$

Using an argument analogous to the one used at inequality (48), we get

$$\left|\left(\frac{\theta}{\pi}\right)x - t + \frac{\zeta(l,k,h)}{2\pi}\right| < \frac{1.2}{\alpha^{\frac{3}{2}x}},$$

for positive integers $x,\ t$ smaller than $4.5 \cdot 10^{48}$.

Now, putting

$$\Gamma_3 := (\theta/\pi)x - t + (\zeta(l,k,h)/2\pi),$$

it is clear that $\Gamma_3 \neq 0$.The cases $\Gamma_3 > 0$ and $\Gamma_3 < 0$ can be treated analogously using Lemma 1.Making the appropriate choices of upper bound M, convergence p/q, number ϵ, etc., we get that $x \leq 13$.

We conclude with the following result, which summarizes both (49) and the above bound on x.

Lemma 5. *Let $3 \leq x < \min\{y,w\} \leq \max\{y,w\} < z$ be positive integers such that $T_xT_z = T_yT_w$ and $x+z = y+w$. Then*

$$3 \leq x < 13, \qquad 4 \leq y, \qquad w \leq 20, \qquad 5 \leq z \leq 27$$

hold.

7. The proof of Main Theorem

Case $x+z \neq y+w$. We list the values of T_x, T_y, T_z, T_w, with x, y, z, w in the range given by Lemma 2, which leads us to the conclusion that equation (8) no has solutions. So, there is no quadruple of positive integers that satisfies (6) in this case.

Case $x+z = y+w$. A quick inspection with the information given by Lemma 5, shows that the only solutions to (8) are

x	9	9	12	9	12
y	12	12	13	13	13
w	12	13	15	13	16
z	15	16	16	17	17

However, equation (6) has no solutions in any of the cases either. Thus, the main theorem is proved.

8. A open problem

Although we have proved that there are no subdiophantine sequences with four or more terms associated to $\mathbf{T}$, we know nothing about the subdiophantine triples.

By studying the Diophantine equation

$$aT_n = bT_m, \tag{51}$$

in positive integers a, b, n, m with $a < b$ relatively prime in a small range, we obtained by computer search that for $a, b \in \{9, 56, 103\}$ the equation (51) always has solutions. Interestingly $\{9, 56, 103\}$ is a subdiophantine triple associated to $\mathbf{T}$ because

$$9 \times 56 = T_{13}, \quad 9 \times 103 = T_{14}, \quad 56 \times 103 = T_{17}.$$

We propose to the reader to prove first that there are only finitely many subdiophantine triples for $\mathbf{T}$ and to determine all of them.

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