Publ. Math. Debrecen
45 / 3-4 (1994), 333–345

Curvature properties of certain pseudosymmetric manifolds

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1. Introduction

Let (M, g) be a connected n-dimensional, $n \geq 3$, semi-Riemannian manifold of class C^∞ with the metric g and the Levi-Civita connection ∇. Moreover, let S and κ be the Ricci tensor and the scalar curvature of (M, g), respectively. A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be Einstein manifold ([1], p. 3) if

$$S = \frac{\kappa}{n} g \tag{1}$$

holds on M. Certain Einstein manifolds can be realized as hypersurfaces immersed isometrically in a semi-Riemannian space of constant curvature $M^{n+1}(c)$ ([18], [20], [21], [22]). Such hypersurfaces have also a curvature property of pseudosymmetry type. Namely, in [15] (Proposition 3.2) it was proved that every Einstein hypersurface immersed isometrically in a semi-Riemannian space of constant curvature $M^{n+1}(c)$, $n \geq 4$, is a pseudosymmetric manifold. A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be pseudosymmetric [12] if at every point of M the following condition is satisfied:
$(*)$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.
The manifold (M, g) is pseudosymmetric if and only if

$$R \cdot R = L_R Q(g, R) \tag{2}$$

holds on the set $U_R = \{x \in M \mid Z(R) \neq 0 \text{ at } x\}$, where L_R is some function on U_R. The definitions of the tensors $R \cdot R$, $Q(g, R)$ and $Z(R)$ will be given in the next section. Pseudosymmetric manifolds are a generalization of the notion of locally symmetric manifolds ($\nabla R = 0$) as well as of the notion of semisymmetric manifolds ($R \cdot R = 0$, [23]). We refer [10] as a review paper on pseudosymmetric manifolds.

Another class of semi-Riemannian manifolds are manifolds with pseudosymmetric Weyl tensor. A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be manifold with pseudosymmetric Weyl tensor if at every point of M its Weyl conformal curvature tensor C satisfies the following condition ([16], [17]):

$(**)$ the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent.
We note that the condition $(**)$ is invariant under the conformal deformations of the metric tensor g. The manifold (M, g) is a manifold with pseudosymmetric Weyl tensor if and only if

$$C \cdot C = L_C Q(g, C) \tag{3}$$

holds on the set $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where L_C is some function on U_C. The definitions of the tensors $C \cdot C$ and $Q(g, C)$ will be given also in the next section. Note that $U_C \subset U_R$. If (M, g) is an Einstein manifold then $U_C = U_R$. The condition $(**)$ arose during the study of 4-dimensional warped products ([9]). It was shown in [9] (Theorem 2) that every warped product $M_1 \times_F M_2$, $\dim M_1 = \dim M_2 = 2$, fulfils $(**)$. Furthermore, it is easy to verify that if (M, g), $n \geq 4$, is an Einstein manifold then the relations (2) and (3), with $L_C = L_R - \frac{\kappa}{n(n-1)}$, are equivalent on U_R (cf. [3], Theorem 3.1). Examples of not Einsteinian semi-Riemannian manifolds realizing $(*)$ and $(**)$ are described in [3] (Theorem 4.1) and [11] (Theorem 4.1). In addition, these manifolds satisfying a certain another condition, weaker than (1). Namely, it is easy to check that at every point of the manifolds described in the above theorems also the following condition is fulfilled:
$(***)$ the tensors $S^2 - \frac{tr(S^2)}{n} g$ and $S - \frac{\kappa}{n} g$ are linearly dependent.
The tensor S^2 is defined by $S^2(X, Y) = S(\tilde{S}X, Y)$, where the tensor $\tilde{S}$, defined by $S(X, Y) = g(\tilde{S}X, Y)$, is the Ricci operator of (M, g) and X, Y are vector fields on M. However, the classes of manifolds satisfying $(*)$ or $(**)$ do not coincide. A suitable example is given in [3] (Example 5.2). Warped products realizing $(**)$ are considered in [16]. Hypersurfaces in Euclidean spaces realizing $(**)$ are studied in [5].

In this paper we will investigate curvature properties of pseudosymmetric manifolds having pseudosymmetric Weyl tensor. Certain results concerned with this subject are present in [19]. Namely, by Theorems 1 and 2 and Proposition 4 of [19] (cf. [2], Theorem 4.2) we obtain some curvature properties of pseudosymmetric manifolds (M, g) satisfying the condition:

$$\omega(X)\tilde{C}(Y, Z) + \omega(Y)\tilde{C}(Z, X) + \omega(Z)\tilde{C}(X, Y) = 0, \tag{4}$$

where $\tilde{C}$ is the Weyl curvature operator of (M, g), ω is an 1-form on M and X, Y, Z are vector fields on M. We note that the tensor $C \cdot C$ vanishes at all point of a semi-Riemannian manifold fulfilling (4) at which the 1-form ω is nonzero ([13], Corollary 1).

Our main result (Theorem 3.1) states: If (M, g), $\dim M \geq 4$, is a semi-Riemannian manifold realizing $(*)$ and $(**)$ then at every point of M the condition $(***)$ is satisfied.

Furthermore, we will find also another curvature relations of manifolds realizing these both conditions. In the last section we shall investigate curvature properties of pseudosymmetry type of a certain four-dimensional Riemannian manifold described in [6]. This manifold is an example of a non Ricci-pseudosymmetric manifold with pseudosymmetric Weyl tensor.

2. Preliminaries

Let (M, g) be a connected n-dimensional, $n \geq 3$, semi-Riemannian manifold. We define on M the endomorphisms $\tilde{R}(X, Y)$, $X \wedge Y$ and $\tilde{C}(X, Y)$ by

$$\tilde{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y,$$

$$\tilde{C}(X,Y) = \tilde{R}(X,Y) - \frac{1}{n-2}\left((X \wedge \tilde{S}Y + \tilde{S}X \wedge Y) - \frac{\kappa}{n-1}X \wedge Y\right),$$

respectively, where $X, Y, Z \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M. Furthermore, we define the Riemann–Christoffel curvature tensor R, the Weyl conformal curvature tensor C, the tensor G and the concircular tensor $Z(R)$ of (M, g) by

$$R(X_1, X_2, X_3, X_4) = g(\tilde{R}(X_1, X_2)X_3, X_4),$$

$$C(X_1, X_2, X_3, X_4) = g(\tilde{C}(X_1, X_2)X_3, X_4),$$

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$$

and $Z(R) = R - \frac{\kappa}{n(n-1)}G$, respectively. For a $(0,k)$-tensor T on M, $k \geq 1$, we define the $(0, k+2)$-tensors $R \cdot T$ and $Q(g, T)$ by

$$\begin{aligned} R \cdot T(X_1, \ldots, X_k; X, Y) = &- T(\tilde{R}(X,Y)X_1, X_2, \ldots, X_k) \\ &- \cdots - T(X_1, \ldots, X_{k-1}, \tilde{R}(X,Y)X_k), \\ Q(g,T)(X_1, \ldots, X_k; X, Y) = &\, T((X \wedge Y)X_1, X_2, \ldots, X_k) \\ &+ \cdots + T(X_1, \ldots, X_{k-1}, (X \wedge Y)X_k), \end{aligned}$$

respectively. Similarly, we can define the $(0, k+2)$-tensor $C \cdot T$. Let now A and T be a symmetric $(0,2)$-tensor and a $(0,k)$-tensor on a semi-Riemannian manifold (M, g), $n \geq 3$, respectively. We define on M the

$(0, k+2)$-tensor $Q(A,T)$ by

$$Q(A,T)(X_1,\dots,X_k;X,Y) = T((X\wedge_A Y)X_1, X_2,\dots,X_k) + \dots + T(X_1,\dots,X_{k-1},(X\wedge_A Y)X_k),$$

where $X\wedge_A Y$ is the endomorphism defined by

$$(X\wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y.$$

In particular, we have $X\wedge_g Y = X\wedge Y$. Let (M,g) be a semi-Riemannian manifold covered by a system of charts $\{U;x^k\}$. We denote by

$$g_{ij},\ R_{hijk},\ S_{ij},\ S_i^j = g^{jk}S_{ij},\ G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}$$

and

$$C_{hijk} = R_{hijk} + \frac{\kappa}{(n-1)(n-2)}G_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}),$$

the local components of the metric tensor g, the Riemann-Christoffel curvature tensor R, the Ricci tensor S, the Ricci operator $\tilde{S}$, the tensor G and the Weyl tensor C, respectively. By $R\cdot T_{i_1\dots i_k\ell m}$ we will denote the local components of the tensor $R\cdot T$.

The semi-Riemannian manifold (M,g) is said to be Ricci-pseudosymmetric ([14], [7]) if at every point of M the following condition is satisfied:

$(****)$ the tensors $R\cdot S$ and $Q(g,S)$ are linearly dependent.
The manifold (M,g) is Ricci-pseudosymmetric if and only if

$$R\cdot S = L_S Q(g,S)$$

holds on the set $U_S = \{x\in M \mid S - \frac{\kappa}{n}g \neq 0\ at\ x\}$, where L_S is some fuction on U_S. It is clear that if at a point x of a manifold (M,g) $(*)$ is satisfied then also $(****)$ holds at x. The converse statement is not true ([14], [7]). For instance, the Cartan hypersurfaces of dimensions > 3 are not pseudosymmetric, Ricci-pseudosymmetric manifolds ([17], Theorem 1). These hypersurfaces satisfy also the condition $(***)$.

At the end of this section we present the following two lemmas.

Lenna 2.1. *The Weyl curvature tensor of every Ricci-pseudosymmetric semi-Riemannian manifold (M,g), $\dim M \geq 4$, fulfils the following equality*

$$C(\tilde{S}X,Y,Z,W) + C(\tilde{S}Z,Y,W,X) + C(\tilde{S}W,Y,X,Z) = 0\,, \tag{5}$$

where $X, Y, Z, W \in \Xi(M)$.

Proof. Evidently, (5) is satisfied at every point of $M - U_S$. Let now x be a point of U_S. Thus, by our assumptions, the following relation

$$\begin{aligned} &-R(\tilde{S}X_1, X_2, X_3, X_4) - R(X_1, \tilde{S}X_2, X_3, X_4) = \\ &= L_S(g(X_4, X_1)S(X_3, X_2) - g(X_3, X_1)S(X_4, X_2) \\ &+g(X_4, X_2)S(X_3, X_1) - g(X_3, X_2)S(X_4, X_1)) \end{aligned} \tag{6}$$

holds at x, where $X_1, \dots, X_4 \in T_x(M)$. Summing this cyclically in X_1, X_3, X_4 we get

$$R(\tilde{S}X_1, X_2, X_3, X_4) + R(\tilde{S}X_3, X_2, X_4, X_1) + R(\tilde{S}X_4, X_2, X_1, X_3) = 0.$$

But on the other hand, we can easily verify that the following identity

$$\begin{aligned} &C(\tilde{S}X, Y, Z, W) + C(\tilde{S}Z, Y, W, X) + C(\tilde{S}W, Y, X, Z) \\ &= R(\tilde{S}X, Y, Z, W) + R(\tilde{S}Z, Y, W, X) + R(\tilde{S}W, Y, X, Z) \end{aligned}$$

holds on any semi-Riemannian manifold. Using now the last two equations, we can conclude that (5) is fulfilled at x, which completes the proof.

Lemma 2.2. *Let (M, g), $dim M \geq 4$, be a pseudosymmetric semi-Riemannian manifold with pseudosymmetric Weyl tensor. Then the relation*

$$\begin{aligned} &((n-2)(L_R - L_C) + \frac{\kappa}{n-1})Q(g, C)(X_1, \dots, X_4; X, Y) \\ &= Q(S, C)(X_1, \dots, X_4; X, Y) \\ &+g(X_1, Y)C(\tilde{S}X, X_2, X_3, X_4) - g(X_1, X)C(\tilde{S}Y, X_2, X_3, X_4) \\ &-g(X_2, Y)C(\tilde{S}X, X_1, X_3, X_4) + g(X_2, X)C(\tilde{S}Y, X_1, X_3, X_4) \\ &+g(X_3, Y)C(\tilde{S}X, X_4, X_1, X_2) - g(X_3, X)C(\tilde{S}Y, X_4, X_1, X_2) \\ &-g(X_4, Y)C(\tilde{S}X, X_3, X_1, X_2) + g(X_4, X)C(\tilde{S}Y, X_3, X_1, X_2) \end{aligned} \tag{7}$$

holds on U_C, where $X, Y, X_1, \dots, X_4$ are vector fields on U_C.

Proof. First of all we note that from (2) it follows that the relation

$$R \cdot C = L_R Q(g, C) \tag{8}$$

holds on U_C. Moreover, it is easy to verify that the following identity is satisfied on U_C

$$C \cdot C(X_1, \dots, X_4; X, Y) = R \cdot C(X_1, \dots, X_4; X, Y)$$

$$
\begin{aligned}
&+\frac{\kappa}{(n-1)(n-2)} Q(g,C)(X_1,\dots,X_4;X,Y)\\
&-\frac{1}{n-2} Q(S,C)(X_1,\dots,X_4;X,Y)\\
&-\frac{1}{n-2}(g(X_1,Y)C(\tilde{S}X,X_2,X_3,X_4)-g(X_1,X)C(\tilde{S}Y,X_2,X_3,X_4)\\
&-g(X_2,Y)C(\tilde{S}X,X_1,X_3,X_4)+g(X_2,X)C(\tilde{S}Y,X_1,X_3,X_4)\\
&+g(X_3,Y)C(\tilde{S}X,X_4,X_1,X_2)-g(X_3,X)C(\tilde{S}Y,X_4,X_1,X_2)\\
&-g(X_4,Y)C(\tilde{S}X,X_3,X_1,X_2)+g(X_4,X)C(\tilde{S}Y,X_3,X_1,X_2)).
\end{aligned}
$$

Now, applying in this (3) and (8) we get (7), completing the proof.

3. Manifolds satisfying $(*)$ and $(**)$

Theorem 3.1. *Let (M,g), $dimM \geq 4$, be a pseudosymmetric manifold with pseudosymmetric Weyl tensor. Then the conditions $(***)$ and*

$$
Q\left(S-(\frac{\kappa}{n-1}-L_R+L_C)g, C-\frac{\mu}{n(n-2)}G\right)=0, \quad \mu\in\mathbb{R}, \tag{9}
$$

hold at every point of U_C.

Proof. Let x be a point of U_C. We can present (7) in the following form

$$
\begin{aligned}
&\left((n-2)(L_R-L_C)+\frac{\kappa}{n-1}\right) Q(g,C)_{hijk\ell m} = Q(S,C)_{hijk\ell m}\\
&+g_{hm}S^p_\ell C_{pijk}-g_{h\ell}S^p_m C_{pijk}-g_{im}S^p_\ell C_{phjk}+g_{i\ell}S^p_m C_{phjk}\\
&+g_{jm}S^p_\ell C_{pkhi}-g_{j\ell}S^p_m C_{pkhi}-g_{km}S^p_\ell C_{pjhi}+g_{k\ell}S^p_m C_{pjhi}.
\end{aligned} \tag{10}
$$

Contracting this with g^{hm} and using the identities

$$
\begin{aligned}
g^{hm}Q(g,C)_{hijk\ell m} &= (n-1)C_{\ell ijk},\\
g^{hm}Q(S,C)_{hijk\ell m} &= \kappa C_{\ell ijk}+S^p_i C_{p\ell jk}+S^p_j C_{pi\ell k}+S^p_k C_{pij\ell}-S^p_\ell C_{pijk},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&(n-1)(n-2)(L_R-L_C)C_{\ell ijk} = S^p_i C_{p\ell jk}+S^p_j C_{pi\ell k}\\
&+S^p_k C_{pij\ell}+(n-2)S^p_\ell C_{pijk}+g_{j\ell}A_{ki}-g_{k\ell}A_{ij},
\end{aligned} \tag{11}
$$

where

$$A_{ij} = S^{pq}C_{ipqj} = S^{pq}R_{ipqj} + \frac{2}{n-2}S^2_{ij} - \frac{n\kappa}{(n-1)(n-2)}S_{ij} + \frac{1}{n-2}\left(\frac{\kappa^2}{n-1} - S^{pq}S_{pq}\right)g_{ij}. \tag{12}$$

But (11), in view of Lemma 2.1, reduces to

$$(n-1)(n-2)(L_R - L_C)C_{\ell ijk} = S^p_i C_{p\ell jk} + (n-1)S^p_\ell C_{pijk} + g_{j\ell}A_{ki} - g_{k\ell}A_{ij}. \tag{13}$$

Symmetrizing this in ℓ, i we find

$$S^p_i C_{p\ell jk} + S^p_\ell C_{pijk} = \frac{1}{n}(g_{k\ell}A_{ij} + g_{ki}A_{j\ell} - g_{j\ell}A_{ik} - g_{ij}A_{k\ell}). \tag{14}$$

Thus (13) turns into

$$(n-1)(n-2)(L_R - L_C)C_{\ell ijk} = (n-2)S^p_\ell C_{pijk} + g_{j\ell}A_{ki} - g_{k\ell}A_{ij} + \frac{1}{n}(g_{k\ell}A_{ij} + g_{ki}A_{j\ell} - g_{j\ell}A_{ik} - g_{ij}A_{k\ell}). \tag{15}$$

It is easy to check that the identity

$$-S^p_i C_{pjk\ell} - S^p_j C_{pik\ell} = \frac{1}{n-2}(g_{jk}S^2_{i\ell} - g_{j\ell}S^2_{ik} + g_{ik}S^2_{j\ell} - g_{i\ell}S^2_{jk}) + R \cdot S_{ijk\ell} - \frac{\kappa}{(n-1)(n-2)}(g_{jk}S_{i\ell} - g_{j\ell}S_{ik} + g_{ik}S_{j\ell} - g_{i\ell}S_{jk}),$$

by making use of the relation $R \cdot S = L_R Q(g, S)$, which follows immediately from (2), turns into

$$-S^p_i C_{pjk\ell} - S^p_j C_{pik\ell} = g_{j\ell}B_{ik} - g_{jk}B_{i\ell} + g_{i\ell}B_{jk} - g_{ik}B_{j\ell}, \tag{16}$$

$$B_{jk} = L_R S_{jk} + \frac{\kappa}{(n-1)(n-2)}S_{jk} - \frac{1}{n-2}S^2_{jk}. \tag{17}$$

Moreover, contracting the equality

$$-S^p_i R_{pj\ell m} - S^p_j R_{pi\ell m} = L_R(g_{im}S_{j\ell} + g_{jm}S_{i\ell} - g_{i\ell}S_{jm} - g_{j\ell}S_{im})$$

with $g^{\ell m}$ we obtain

$$S^{pq}R_{jpq\ell} = S^2_{j\ell} - nL_R\left(S_{j\ell} - \frac{\kappa}{n}g_{j\ell}\right).$$

Substituting this in (12) and using (17) we get

$$A_{ij} = -nB_{ij} + tr(B)g_{ij}. \tag{18}$$

Next, transvecting (15) with S^{ij} we find

$$\left((n-2)(L_R - L_C) + \frac{\kappa}{n(n-1)}\right) A_{\ell k} = E_{\ell k} - \frac{1}{n} tr(E) g_{\ell k}, \tag{19}$$

where $E_{\ell k} = S^p_\ell A_{pk}$. On the other hand, transvecting (13) with S^k_m and symmetrizing the resulting equality in j, m we obtain

$$\begin{aligned}&(n-1)(n-2)(L_R - L_C)(S^p_m C_{pji\ell} + S^p_j R_{pmi\ell}) = S^k_i(S^p_m C_{pj\ell k} + S^p_j C_{pm\ell k})\\ &+(n-1)S^k_\ell(S^p_m C_{pjik} + S^p_j C_{pmik}) + g_{\ell m}E_{ij} + g_{j\ell}E_{im} - S_{j\ell}A_{im} - S_{\ell m}A_{ij}.\end{aligned}$$

This, by an application of (14), yields

$$\begin{aligned}&\frac{(n-1)(n-2)}{n}(L_R - L_C)(g_{\ell m}A_{ij} + g_{j\ell}A_{im} - g_{ij}A_{\ell m} - g_{im}A_{j\ell})\\ &= -\frac{1}{n}(g_{\ell m}E_{ij} + g_{j\ell}E_{im} - S_{ij}A_{\ell m} - S_{im}A_{j\ell}) + \frac{n-1}{n}(S_{\ell m}A_{ij}\\ &+S_{j\ell}A_{im} - g_{ij}E_{\ell m} - g_{im}E_{j\ell}) + g_{\ell m}E_{ij} + g_{j\ell}E_{im} - A_{ij}S_{\ell m} - A_{im}S_{j\ell}\end{aligned}$$

which, after antisymmetrization in ℓ, i, turns into

$$(n-1)(n-2)(L_R - L_C)Q(g, A) = -Q(S, A) + (n-1)Q(g, E).$$

Now, in virtue of (19), we have

$$Q(S - \frac{\kappa}{n}g, A) = 0. \tag{20}$$

It is clear that if the tensor $S - \frac{\kappa}{n}g$ vanishes at x then also $(***)$ and

$$A = \mu(S - \frac{\kappa}{n}g), \quad \mu \in \mathbb{R}, \tag{21}$$

are satisfied at x. Further, in view of Lemma 2.4(i) [15], we can conclude from (20) that if the tensor $S - \frac{\kappa}{n}g$ is nonzero at x then also (21) is fulfilled at x. Now (18), by (17) and (21), leads to $(***)$. Finally, we prove that (9) holds at x. In fact, the relation (15), in virtue of (21), turns into

$$\begin{aligned}S^p_\ell C_{pijk} &= (n-1)(L_R - L_C)C_{\ell ijk} - \frac{1}{n^2}\mu\kappa G_{\ell ijk} + \frac{\mu}{n-2}(g_{\ell k}S_{ij} - g_{j\ell}S_{ik})\\ &- \frac{\mu}{n(n-2)}(g_{ik}S_{j\ell} + g_{\ell k}S_{ij} - g_{j\ell}S_{ik} - g_{ik}S_{\ell k}).\end{aligned} \tag{22}$$

Applying this in (10) we get

$$Q\left(\left(\frac{\kappa}{n-1} - L_R + L_C\right)g, C\right) = Q\left(S, C - \frac{\mu}{n(n-2)}G\right),$$

which turns immediately into (9), completing the proof.

Combining Theorem 3.1 with Lemma 1.2 of [4] we get immediately the following corollary.

Corollary 3.1. *Let (M, g), $\dim M \geq 4$, be a pseudosymmetric manifold with pseudosymmetric Weyl tensor. Then the tensors $S^2 - \frac{tr(S^2)}{n} g$ and $S - \frac{\kappa}{n} g$ are linearly dependent at every point of M.*

Theorem 3.2. *Let (M, g), $\dim M \geq 4$, be a pseudosymmetric manifold with pseudosymmetric Weyl tensor. Moreover, let V be a vector at $x \in U_C$ such that the scalar $\varrho = a(V)$ is nonzero, where a is a covector defined by*

$$a(X) = S(X, V) - \left(\frac{\kappa}{n-1} - L_R + L_C \right) g(X, V),$$

$X \in T_x(M)$ and let F be a $(0, 2)$-tensor defined by

$$F(X, Y) = C(V, X, Y, V) - \frac{\mu}{n(n-2)} G(V, X, Y, V), \; X, Y \in T_x M,$$

and μ is the real number given in (9).

(i) If the tensor $S - (\frac{\kappa}{n-1} - L_R + L_C) g - \frac{1}{\varrho} a \otimes a$ vanishes then the relations:

$$C(X, Y, Z, W) = \tag{23}$$
$$= \frac{1}{\varrho^2} (F((X \wedge_{a \otimes a} Y)Z, W) - F((X \wedge_{a \otimes a} Y)W, Z)),$$

$$a(X)\tilde{C}(Y, Z) + a(Y)\tilde{C}(Z, X) + a(Z)\tilde{C}(X, Y) = 0, \tag{24}$$

$$C \cdot C = 0, \; L_C = 0, \tag{25}$$

$$L_R = \frac{\kappa}{n(n-1)}, \tag{26}$$

$$S = \frac{\kappa}{n} g + \frac{1}{\varrho} a \otimes a, \tag{27}$$

hold at x, where $X, Y, Z, W \in T_x(M)$.

(ii) If the tensor $S - (\frac{\kappa}{n-1} - L_R + L_C) g - \frac{1}{\varrho} a \otimes a$ is nonzero then the equality

$$\varrho C(X, Y, Z, W) =$$
$$\left(\frac{\mu \varrho}{(n-1)(n-2)} + \lambda \left(\frac{\kappa}{n-1} - L_R + L_C \right)^2 \right) G(X, Y, Z, W)$$

$$+\lambda S((X \wedge_S Y)Z, W) - \lambda \left(\frac{\kappa}{n-1} - L_R + L_C \right) (g(X,W)S(Y,Z)$$
$$+g(Y,Z)S(X,W) - g(X,Z)S(Y,W) - g(Y,W)S(X,Z))$$

holds at x, where $X, Y, Z, W \in T_x(M)$.

Proof. We will use notations of Proposition 4.1 of [2].

(i) Applying Proposition 4.1(i) of [2] we get

$$\varrho^2 \left(C - \frac{\mu}{n(n-2)} G \right)_{\ell ijk} = a_\ell a_k F_{ij} + a_i a_j F_{\ell k} - a_\ell a_j F_{ik} - a_i a_k F_{\ell j}. \tag{28}$$

(i1) We assume that μ vanishes. Then (28) reduces to (23). Thus (24) is satisfied. Moreover, from (23), in view of Theorem 1 of [13], it follows that (25) is fulfilled. Next, applying Theorem 4.2 of [2] we get (26). Now we see that the Ricci tensor S takes the required form (27).

(i2) Suppose that μ is nonzero. Then (28), after contraction with g^{ij}, yields

$$a^p a_p F_{\ell k} = -\frac{n-1}{n(n-2)} \mu \varrho^2 g_{\ell k} + a_\ell b_k - a_k a_\ell, \tag{29}$$

where $b_k = a^p F_{\ell k}$. We note that if $a^p a_p$ vanishes then (29) reduces to

$$g_{\ell k} = \tau(a_\ell b_k + a_k b_\ell), \quad \tau \in \mathbb{R} - \{0\},$$

a contradiction. Thus $a^p a_p$ must be nonzero. Multiplying now (28) by $a^p a_p$ and using (29) we get

$$a^p a_p \left(C - \frac{\lambda}{n(n-2)} G \right)_{\ell ijk} =$$
$$= -\frac{n-1}{n(n-2)} \mu (a_\ell a_k g_{ij} + a_i a_j g_{\ell k} - a_i a_k g_{\ell j} - a_\ell a_j g_{ik}).$$

Contracting this with $g^{\ell k}$ we get

$$g_{ij} = \tilde{\tau} a_i a_j, \quad \tilde{\tau} \in \mathbb{R} - \{0\},$$

a contradiction.

(ii) The second assertion is an immediate consequence of Proposition 4.1(ii) of [2]. Our theorem is thus proved.

4. Examples

In this section we present some examples of four-dimensional Riemannian manifolds with pseudosymmetric Weyl tensor.

Let (M, g) be the four-dimensional manifold defined in [6] (Lemme 1.1). As it was proved in [6] (see Lemme 1.1 and Remarqué 1.5), (M, g) is a non conformally flat and non semisymmetric, Weyl-semisymmetric manifold, i.e. the tensors C and $R \cdot R$ are nonzero and the condition $R \cdot C = 0$ holds on M. Furthermore, using formulas for the local components of the Ricci tensor and the Riemann-Christoffel tensor of the manifold (M, g) obtained in [6] (Lemme 1.1), we can easily verify that (M, g) is a non Ricci-pseudosymmetric manifold satisfying at every point the condition $(***)$.

Let V be a connected subset of the set $U = \{x \in M : \ u(x) \neq 0 \ \}$, where u is the function defined in [6] (Lemme 1.1). By formula (10) of [6] we have $U = U_C$. Let on V be given the conformal deformation $g \rightarrow \bar{g}_k = \frac{1}{(u+k)^2} g$ of the metric g, where k is a constant such that $k \geq 0$ when $u > 0$ on V or $k \leq 0$ when $u < 0$ on V.

First we consider the deformation $g \rightarrow \bar{g}_0 = \frac{1}{u^2} g$. The manifold $(V, \bar{g}_0)$ is an Einstein manifold ([6], Lemme 1.1(viii)). Moreover, as it was stated in [8] (Example 3) the relation

$$\bar{R} \cdot \bar{R} = -\frac{1}{12}(u^3 - pq) Q(\bar{g}_0, \bar{R})$$

holds on $(V, \bar{g}_0)$, where $\bar{R}$ is the Riemann-Christoffel curvature tensor of $\bar{g}_0$ and p, q are some constants. Now, in view of Theorem 3.1 of [3], the equality

$$\bar{C} \cdot \bar{C} = -\frac{1}{12}(u^3 - pq + \bar{\kappa}) Q(\bar{g}_0, \bar{C})$$

holds on V, where $\bar{C}$ is the Weyl conformal curvature tensor of $\bar{g}_0$ and the constant $\bar{\kappa}$ is the scalar curvature of $\bar{g}_0$. From this we get easily

$$C \cdot C = -\frac{u^3 - pq + \bar{\kappa}}{12u^2} Q(g, C).$$

Thus we see that the condition $(**)$ is fulfilled on (M, g), i.e. (M, g) is a manifold with pseudosymmetric Weyl tensor.

Now we consider on V the conformal deformation $g \rightarrow \bar{g}_k = \frac{1}{(u+k)^2} g$, where k is a positive or negative constant. Since (V, g) is a manifold with pseudosymmetric Weyl tensor, then $(V, \bar{g}_k)$, $k \neq 0$, is also a manifold with pseudosymmetric Weyl tensor. Moreover, as it was stated in [8] (Example 2) this manifold is a non Ricci-pseudosymmetric manifold.

Finally, we state that (M, g) is a Riemannian manifold which cannot be realized as a hypersurface immersed isometrically in a 5-dimensional Euclidean space. This is an immediate consequence of Corollary 3.1 of [15] and the fact that the tensor $R \cdot R - Q(S, R)$ is a nonzero tensor on M. More generally, (M, g) cannot be realized as a hypersurface immersed isometrically in a 5-dimensional space of constant curvature. This statement follows immediately from Proposition 3.1 of [15] and the fact that the tensors $R \cdot R - Q(S,R)$ and $Q(g,C)$ are not linearly dependent at every point of M.

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(Received August 30, 1993)