Publ. Math. Debrecen
42 / 3–4 (1993), 215–224

Geometry of space-time and generalised Lagrange gauge theory

By R. MIRON (Iaşi), R.K. TAVAKOL (London),
V. BALAN (Bucharest) and I. ROXBURGH (London)

Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. In §1 and §2 the authors present the Einstein and Maxwell equations for the generalised Lagrange space

$$GL^n = (M, g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x)),$$

and characterize the case of vanishing mixed curvature tensor field of the canonical linear d-connection. The Lagrangian gauge theory — in G.S. Asanov's sense [1] is developed in §3 for the tangent bundle endowed with (h,v)-metrics, obtaining the generalised Einstein - Yang Mills equations with respect to the metric gauge tensor fields and to the gauge field $\sigma(x,y)$ for three remarkable cases in which the metrics are derived from the fundamental tensor field $g_{ij}(x,y)$. Proofs are, in most cases, mechanical but rather tedious calculations. They are omitted.

Introduction

In a previous paper [6] it was shown that the EPS conditions can be satisfied in a more general frame than that of Finsler spaces, namely for convenient generalised Lagrange spaces. More precise, was examined the space $GL^n = (M, g_{ij}(x,y))$, where M is a n-dimensional differentiable manifold and

$$g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x), \tag{A1}$$

$\gamma_{ij}(x)$ being a metric tensor field on M, and $\sigma : TM \to \mathbb{R}$ a function of class C^∞ on $\widetilde{TM} = TM \setminus \{0\}$, continuous on the null section of the tangent bundle. The space was endowed with the non-linear connection

$$N^i{}_j(x,y) = \begin{Bmatrix} i \\ kj \end{Bmatrix} y^k, \tag{A2}$$

where $\left\{ {i \atop kj} \right\}$ are the Christoffel symbols for $\gamma_{ij}(x)$. Under these assumptions, GL^n represents a convenient relativistic model, since it has the same conformal and projective properties as the Riemannian space $V^n = (M, \gamma_{ij}(x))$.

§1. Einstein equations for GL^n

Developing the formalism presented in [3,4], we remind that the canonical h- and v-symmetrical d-connection of GL^n has the coefficients

$$\begin{aligned} L_{jk}{}^i &= \begin{Bmatrix} i \\ jk \end{Bmatrix} + \Lambda^i_{jk} \\ C_{bc}{}^a &= \delta^a_b \dot{\sigma}_c + \delta^a_c \dot{\sigma}_b - \gamma_{bc} \dot{\sigma}^a \end{aligned} \tag{1.1}$$

where we used the notations

$$\begin{aligned} \Lambda^i_{jk} &= \delta^i_j \sigma_k + \delta^i_k \sigma_j - \gamma_{jk} \sigma^i \\ \sigma_k &= \delta_k \sigma,\ \dot{\sigma}_a = \dot{\partial}_a \sigma,\ \sigma^k = \gamma^{ks} \sigma_s,\ \dot{\sigma}^a = \gamma^{ab} \dot{\sigma}_b \end{aligned} \tag{1.1'}$$

and

$$\delta_k = \partial_k - N^a_k(x,y)\dot{\partial}_a, \quad \dot{\partial}_b = \frac{\partial}{\partial y^b}; \quad \partial_k \equiv \frac{\partial}{\partial x^k}. \tag{1.1''}$$

Then the torsion d-tensor fields of the canonical linear d-connection $C\Gamma(N)$ of GL^n have the coefficients given by

$$\begin{cases} T_{jk}{}^i = 0, \quad S^a_{bc} = 0, \quad C_{ja}{}^i, \\ R^a_{kl} = r_d{}^a{}_{kl} y^d, \quad P^a_{kb} = -\Lambda^a_{bk}. \end{cases} \tag{1.2}$$

and the curvature d-tensor fields have the expressions

$$\begin{cases} R_j{}^i{}_{kl} = r_j{}^i{}_{kl} + \delta^i_{(k} \sigma_{jl)} - \gamma^{is} \gamma_{j(k} \sigma_{sl)} + \gamma_{js} \dot{\sigma}^{(s} R^{i)}_{kl} \\ P_j{}^i{}_{kc} = \delta^i_k \overset{1}{\sigma}_{jc} - \delta^i_c \overset{2}{\sigma}_{jk} - \gamma^{is} \cdot \left[\gamma_{jk} \overset{1}{\sigma}_{sc} - \gamma_{jc} \overset{2}{\sigma}_{sk} \right] + \gamma^{is} \gamma_{ck} \sigma_{(s} \dot{\sigma}_{j)} \\ S^a_{bcd} = \delta^a_{(c} \dot{\sigma}_{bd)} - \gamma^{as} \gamma_{b(c} \dot{\sigma}_{sd)} \end{cases} \tag{1.3}$$

where $r_j{}^i{}_{kl}$ is the curvature tensor field of $\gamma_{ij}(x)$, $t_{(ij)} = t_{ij} - t_{ji}$, and we considered the following d-tensor fields

$$
(1.3')\qquad
\begin{aligned}
\sigma_{sl} &= \sigma_{s|l} + \sigma_s\sigma_l - \frac{1}{2}\gamma_{sl}\overset{H}{\sigma} = \sigma_{s\mathring{|}l} - \sigma_s\sigma_l + \frac{1}{2}\gamma_{sl}\overset{H}{\sigma}\\
\overset{1}{\sigma}_{sl} &= \sigma_s|_l + \sigma_s\dot{\sigma}_l - \frac{1}{2}\gamma_{sl}\overset{M}{\sigma} = \dot{\partial}_l\sigma_s - \dot{\sigma}_s\sigma_l + \frac{1}{2}\gamma_{sl}\overset{M}{\sigma}\\
\overset{2}{\sigma}_{sl} &= \dot{\sigma}_{s|l} + \dot{\sigma}_s\sigma_l - \frac{1}{2}\gamma_{sl}\overset{M}{\sigma} = \dot{\sigma}_{s\mathring{|}l} - \sigma_s\dot{\sigma}_l + \frac{1}{2}\gamma_{sl}\overset{M}{\sigma}\\
\dot{\sigma}_{sl} &= \dot{\sigma}_s|_l + \dot{\sigma}_s\dot{\sigma}_l - \frac{1}{2}\gamma_{sl}\overset{V}{\sigma} = \dot{\partial}_l\dot{\sigma}_s - \dot{\sigma}_s\dot{\sigma}_l + \frac{1}{2}\gamma_{sl}\overset{V}{\sigma}
\end{aligned}
$$

with $\overset{H}{\sigma} = \sigma_s\sigma^s$, $\overset{M}{\sigma} = \sigma_s\dot{\sigma}^s$, $\overset{V}{\sigma} = \dot{\sigma}_s\dot{\sigma}^s$.

We also used the covariant derivatives

$$
(1.3'')\quad \sigma_{s|l} = \delta_l\sigma_s - L^t_{sl}\sigma_t, \quad \sigma_s|_l = \dot{\partial}_l\sigma_s - C^t_{sl}\sigma_t, \quad \sigma_{s\mathring{|}l} = \partial_l\sigma_s - \begin{Bmatrix} t \\ sl \end{Bmatrix}\sigma_t.
$$

Then, by contracting the indices, one can derive the four Ricci tensor fields, expressed by

$$
(1.4)\qquad
\begin{aligned}
R_{ij} &= r_{ij} - \gamma_{ij}\dot{\sigma} + (2-n)\cdot\sigma_{ij} + \gamma_{is}\dot{\sigma}^{(s}r_t{}^{a)}{}_{ja}\\
\overset{1}{P}_{bk} &= (1-n)\overset{2}{\sigma}_{bk} + \overset{1}{\sigma}_{bk} - \gamma_{bk}\overset{1}{\sigma} + \sigma_{(k}\dot{\sigma}_{b)}\\
\overset{2}{P}_{bc} &= (1-n)\overset{1}{\sigma}_{bc} - \overset{2}{\sigma}_{bc} - \gamma_{bc}\overset{2}{\sigma} + \sigma_{(c}\dot{\sigma}_{b)}\\
S_{bc} &= (2-n)\dot{\sigma}_{bc} - \gamma_{bc}\dot{\sigma}
\end{aligned}
$$

and then the curvature scalar fields

$$
(1.5)\qquad
\begin{cases}
R = e^{-2\sigma}[r + 2(1-n)\overset{0}{\sigma} + 2r_{sc}y^s\dot{\sigma}^c]\\
\overset{1}{P} = -\overset{2}{P} = e^{-2\sigma}\cdot 2(1-n)\overset{1}{\sigma}, \quad S = e^{-2\sigma}\cdot 2(1-n)\dot{\sigma},
\end{cases}
$$

where

$$
\overset{0}{\sigma} = \gamma^{ij}\sigma_{ij}, \ \ \overset{1}{\sigma} = \gamma^{ij}\overset{1}{\sigma}_{ij}, \ \ \overset{2}{\sigma} = \gamma^{ij}\overset{2}{\sigma}_{ij}, \ \ \dot{\sigma} = \gamma^{ab}\dot{\sigma}_{ab}, \ \ r_{bk} = r^s_{bks}, \ \ r = \gamma^{bk}r_{bk}.
$$

Regarding the mixed curvature tensor field, the following result obtained by direct calculation holds true:

Theorem. *The following assertions are valid:*

a) $P^a_{bkc} - P^a_{bck} = 0$ *iff* $\overset{1}{\sigma}_{bc} + \overset{1}{\sigma}_{cb} = 0$

b) $P^a_{bkc} + P^a_{bck} = 0$ *iff* $\overset{1}{\sigma}_{bc} - \overset{1}{\sigma}_{cb} = \frac{2}{n}\sigma_{(b}\dot{\sigma}_{c)} = 0.$

c) $P_b{}^a{}_{kc} = 0$ *implies* $\overset{H}{\sigma}\overset{V}{\sigma} = \overset{M}{\sigma}{}^2$ *and* $\overset{1}{\sigma} = \overset{2}{\sigma} = 0$

d) $\overset{1}{P}_{bk} = \overset{2}{P}_{bk} = 0$ *iff* $\overset{1}{\sigma}_{bk} = \frac{1}{n}\sigma_{(b}\dot{\sigma}_{k)}$

e) $\overset{1}{\sigma}_{jk} = \overset{2}{\sigma}_{kj}$.

Remark. Using a), b), and e) it becomes obvious that $P_b{}^a{}_{kc} = 0$ implies $\overset{1}{\sigma}_{bk} = \overset{2}{\sigma}_{bk} = 0$.

The Einstein equations of the space GL^n have the generic form

$$\begin{cases} R_{ij} - \frac{1}{2}Rg_{ij} = \kappa \overset{H}{T}_{ij} \\ S_{ab} - \frac{1}{2}Sg_{ab} = \kappa \overset{V}{T}_{ab} \end{cases} \tag{1.6}$$

where $\overset{H}{T}_{ij}$ and $\overset{V}{T}_{ab}$ are called the h- and the v-components of the energy-momentum tensor field of the space GL^n respectively, $g_{ab} = \delta^i_a \delta^j_b \cdot g_{ij}$ and κ is the gravific constant. Using the expressions of the Ricci d-tensor fields (1.4) and of the curvature scalars (1.5) we obtain that the Einstein equations of the space GL^n admit the equivalent form

$$\begin{cases} r_{ij} - \frac{1}{2}r\gamma_{ij} + t_{ij} = \kappa \overset{H}{T}_{ij} \\ (2-n)(\dot{\sigma}_{ab} - \dot{\sigma}\gamma_{ab}) = \kappa \overset{V}{T}_{ab} \end{cases} \tag{1.6'}$$

where

$$t_{ij} = (n-2)(\gamma_{ij}\overset{0}{\sigma} - \sigma_{ij}) + \gamma_{ij} r_{st} y^s \dot{\sigma}^t + \gamma_{is}\dot{\sigma}^{(s} r^{a)}_{tja} y^t$$

We remark that in the first equation of the system (1.6') the term t_{ij} is complementary to the classical Einstein equations of the Riemannian space $V^n = (M, \gamma_{ij}(x))$.

§2. Maxwell equations for GL^n

Let the h- and v-deflection tensor fields be given respectively by

$$D^i{}_j = y^i{}_{|j}, \quad d^i{}_j = y^i|_j$$

and the corresponding ones having the indices lowered

$$D_{ij} = g_{is}D^s{}_j, \quad d_{ij} = g_{is}d^s{}_j.$$

Then we define the h- and v-electromagnetic tensor fields by

$$F_{ij} = \frac{1}{2}(D_{ij} - D_{ji}), \quad f_{ij} = \frac{1}{2}(d_{ij} - d_{ji}) \tag{2.1}$$

respectively, and infer the Maxwell equations

$$\text{(2.2)} \qquad \begin{cases} \underset{ijk}{\sigma} F_{ij|k} = \underset{ijk}{\sigma} e^{2\sigma} {r_0}^s{}_{kj} y_i \dot{\sigma}_s \\ \underset{ijk}{\sigma} F_{ij}|_k + \underset{ijk}{\sigma} f_{ij|k} = 0 \\ \underset{ijk}{\sigma} f_{ij}|_k = 0 \end{cases}$$

where $\underset{ijk}{\sigma}$ denotes cyclic summation in the indices i, j, k and

$$ {r_0}^s{}_{kj} = {r_t}^s{}_{kj} y^t, \quad y_i = \gamma_{is} y^s. $$

Indeed, (2.2) takes place since D_{ij} and d_{ij} satisfy the relations

$$\begin{cases} D_{ij|k} - D_{ik|j} = R_{0ijk} - d_{is} R^s_{jk} \\ D_{ij}|_k - d_{ik|j} = P_{0ijk} - D_{is} C^s_{jk} - d_{is} P^s_{jk} \\ d_{ij}|_k - d_{ik}|_j = S_{0ijk} \end{cases}$$

and the Bianchi identities lead to (2.2). Also, the same equations come up clearly if we consider the expressions

$$F_{ij} = e^{2\sigma}(y_i \sigma_j - y_j \sigma_i), \quad f_{ij} = e^{2\sigma}(y_i \cdot \sigma_j - \gamma_j \sigma_i)$$

using direct computation.

In the following we examine the Maxwell equations for some remarkable particular examples of spaces GL^n:

1. If $\sigma(x,y) = \frac{1}{2}\gamma_{rs} y^r y^s$, then the Maxwell equations are trivially satisfied, since as $\sigma_k = 0$ and $\dot{\sigma}_k = y_k$, we infer that

$$F_{ij} = f_{ij} = 0.$$

2. If GL^n is locally Minkowskian, i.e. at any point $(x,y) \in TM$ there exists a domain of a local map in which

$$g_{ij}(x,y) = e^{2\sigma(y)} \gamma_{ij} \quad \text{with} \quad \partial_k \gamma_{ij} = 0,$$

then $F_{ij} = 0$ and $f_{ij} = e^{2\sigma(y)} y_{(i} \dot{\sigma}_{j)}$, and the Maxwell equations

$$\underset{ijk}{\sigma} f_{ij}|_k = 0$$

take place.

3. For a dispersive medium [7] $(M, V^i(x), n(x, V(x)))$ where $V^i(x)$ is the speed of the particle and $n(x, V(x))$ is the refraction index, we can consider

$$\sigma(x,y) = \alpha \cdot \left(1 - \frac{1}{n^2(x,y)}\right), \quad \alpha \in \mathbb{R}, \alpha > 0.$$

Remark that for $y^i = V^i(x)$ the Maxwell equations have to be computed directly, being not implied by (2.2).

§3. Generalized Einstein - Yang Mills equations

Let TM be endowed with a (h,v)-metric G_* [3] given by

$$G_* = g_{ij} dx^i \otimes dx^j + h_{ab} \delta y^a \otimes \delta y^b \tag{3.1}$$

where $\{dx^i, \delta y^a \mid i, a = \overline{1,n}1,n\}$ is the adapted basis for the cotangent bundle, dual to (1.1"), and $g_{ij}(x,y)$, $h_{ab}(x,y)$ are symmetric d-tensor fields of rank n.

Let the coordinate transform on TM be given by

$$\begin{cases} x^i = x^i(x^j), & \det(\partial x^i / \partial \bar{x}^j) \neq 0, \\ y^a = \frac{\partial x^a}{\partial \bar{x}^b} \bar{y}^b, & i,j,a,b = \overline{1,n}. \end{cases} \tag{3.2}$$

A generalized gauge transformation on TM is a diffeomorphism of TM compatible with the tangent bundle structure, given locally by

$$\begin{cases} x^i = X^i(\tilde{x}^j), & \det(\partial X^i / \partial \tilde{x}^j) \neq 0, \\ y^a = Y^a_b(\tilde{x}) \tilde{y}^b, & \det\left(Y^a_b(\tilde{x})\right) \neq 0. \end{cases} \tag{3.3}$$

A gauge d-tensor field is a d-tensor field which obeys tensor-type transformation rules with respect to (3.3); e.g. the gauge d-tensor field $\{w^{i\,a}_{j\,b}(x,y)\}$ transforms relative to (3.2) and (3.3) according to

$$\begin{cases} \bar{w}^{i\,a}_{j\,b}(\bar{x},\bar{y}) \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^c}{\partial \bar{x}^a} = w^{k\,c}_{l\,d}(x,y) \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^d}{\partial \bar{x}^b}, \\ \tilde{w}^{ia}_{jb}(\tilde{x},\tilde{y}) \frac{\partial X^k}{\partial \tilde{x}^i} \frac{\partial Y^c}{\partial \tilde{y}^a} = w^{k\,c}_{l\,d}(x,y) \frac{\partial X^l}{\partial \tilde{x}^j} \frac{\partial Y^d}{\partial \tilde{y}^b}. \end{cases} \tag{3.4}$$

Let TM be endowed with the nonlinear connection $N = \{N^a_i(x,y)\}$. If δ_k given by (1.1) yields d-covector fields when acting on gauge scalar fields (i.e. functions $f \in F(TM)$ which obey $\bar{f}(\bar{x},\bar{y}) = f(x,y), \tilde{f}(\tilde{x},\tilde{y}) = f(x,y)$), then N is called a generalized gauge non-linear connection. Further, if $_{|k}$ and $|_c$ are the h- and v-covariant derivations associated with a linear d-connection D on TM (i.e. a connection that preserves by parallelism the distributions N and VTM [3]), having the coefficients

$$D\Gamma(N) = \left\{ L^i_{jk}(x,y), \quad L^a_{bk}(x,y), \quad C^i_{ja}(x,y), \quad C^a_{bc}(x,y) \right\}$$

given by the relations

$$\begin{cases} D_{\delta_j} \delta_i &= L^k_{ij} \delta_k, \quad D_{\dot{\partial}_a} \delta_j = C^i_{ja} \delta_i \\ D_{\delta_k} \dot{\partial}_b &= L^a_{bk} \dot{\partial}_a, \quad D_{\dot{\partial}_c} \dot{\partial}_b = C^a_{bc} \dot{\partial}_a \end{cases} \tag{3.5}$$

then we say that D is a gauge linear d-connection iff $_{|k}$ and $|_c$ preserve the gauge tensorial character.

In the following we suppose that N is given by $(A2)$, that g_{ij} and h_{ab} in (3.1) are gauge d-tensor fields, that γ_{ij} and σ in $(A1)$ are gauge fields (tensor, respectively scalar), and we fix the coefficients

$$C^i_{ja} = 0, \quad L^a_{bk} = \dot{\partial}_b N^a_k = \begin{Bmatrix} a \\ bk \end{Bmatrix}.$$

Then we can infer the following

Proposition. *If* $g_{ij|k} = 0$ *and* $h_{ab}|_c = 0$, $L^i_{jk} = L^i_{kj}$ *and* $C^a_{bc} = C^a_{cb}$, *then*

$$\begin{cases} L^i_{jk} = \frac{1}{2} g^{is}(\delta_j g_{sk} + \delta_k g_{sj} - \delta_s g_{jk}) \\ C^a_{bc} = \frac{1}{2} h^{as}(\dot{\partial}_b h_{sc} + \dot{\partial}_c h_{sb} - \dot{\partial}_s h_{bc}). \end{cases} \tag{3.6}$$

We remark that if g_{ij} is given by $(A1)$ and $h_{ab} = \delta^i_a \delta^j_b \cdot g_{ij}$, then L^i_{jk} and C^a_{bc} are those given by (1.1).

It can be shown by direct calculation that the torsion and curvature d-tensor fields of $D\Gamma(N)$ are gauge d-tensor fields, and we have [3,4]

$$T^i_{jk} = 0, \quad S^a_{bc} = 0, \quad P^a_{kb} = 0, \quad C^i_{ja} = 0, \quad R^a_{kl} = r^a_{skl} y^s \tag{3.7}$$

and

$$\begin{gathered} R^a_{bkl} = r^a_{bkl} + C^a_{bd} R^d_{kl}, \quad P^i_{jkc} = \dot{\partial}_c L^i_{jk} \\ P^a_{bkc} = -C^a_{bc|k}, \quad S^i_{jbc} = 0, \end{gathered} \tag{3.8}$$

with R^i_{jkl} and S^a_{bcd} given by (1.3) for L^i_{jk} and C^a_{bc} given by (1.1). So that $D\Gamma(N)$ admits one torsion and five curvature non-trivial gauge d-tensor fields. It is obvious that the mixed Lagrangian

$$L = l_1 R + l_5 S + \hat{L} \tag{3.9}$$

is a gauge scalar field. Here

$$\begin{gathered} \hat{L} = l_0 L_0 + l_2 \hat{R} + l_3 \overset{h}{P} + l_4 \overset{v}{P}, \quad L_0 = R^a_{kl} R^{kl}_a, \quad \overset{v}{R} = R^a_{bkl} R^{bkl}_a \\ \overset{h}{P} = P^i_{jkc} P^{jkc}_i, \quad \overset{v}{P} = P^a_{bkc} P^{bkc}_a, \quad l_0, l_1, l_2, l_3, l_4, l_5 \in \mathbb{R} \end{gathered} \tag{3.10}$$

and R, S are given by (1.5), the raising/lowering of the indices being performed using $g_{ij}(x,y)$ and $h_{ab}(x,y)$ for the corresponding index-types. We remark that the Lagrangian (3.9) depends functionally on the gauge fields $g_{ij}(x,y)$, $h_{ab}(x,y)$ and their derivatives, by means of (3.8) and (3.6). So that applying the gauge variational principle

$$\delta \int \mathcal{L} dx^n dy^n = 0$$

for the Lagrangian density

$$\mathcal{L} = LG, \quad G = |\det(g_{ij})|^{1/2} \cdot |\det(h_{ab})|^{1/2}, \tag{3.11}$$

we infer the generalized Einstein-Yang Mills equations for $\mathcal{L}$, obtained by vanishing of the Euler-Lagrange derivatives

$$\frac{\delta\mathcal{L}}{\delta\phi} \equiv \partial_k\left(\frac{\partial\mathcal{L}}{\partial(\partial_k\phi)}\right) + \dot{\partial}_a\left(\frac{\partial\mathcal{L}}{\partial(\dot{\partial}_a\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0, \quad \phi \in \{g_{ij}(x,y), h_{ab}(x,y)\}.$$

By direct calculation, we obtain the following equivalent expressions for these equations:

$$\begin{aligned} R_{ij} - \frac{1}{2}Rg_{ij} = \frac{1}{l_1}g_{il}g_{jm}\{ \frac{\partial\hat{L}}{\partial g_{lm}} + g^{st}\frac{\partial R_{st}}{\partial g_{lm}} - \\ -\frac{1}{G}\left[\partial_k\left(G\frac{\partial L}{\partial(\partial_k g_{lm})}\right) + \dot{\partial}_a\left(G\frac{\partial L}{\partial(\dot{\partial}_a g_{lm})}\right)\right]\} + \frac{1}{2l_1}g_{ij}(\hat{L} + l_5 S), \end{aligned} \tag{3.12}$$

(3.13)

$$\begin{aligned} S_{ab} - \frac{1}{2}Sh_{ab} = \frac{1}{l_5}h_{ae}h_{bf} \cdot \{ \frac{\partial\hat{L}}{\partial h_{ef}} + h^{uv}\frac{\partial S_{uv}}{\partial h_{ef}} - \\ -\frac{1}{G}\left[\partial_k\left(G\frac{\partial L}{\partial(\partial_k h_{ef})}\right) + \dot{\partial}_d\left(G\frac{\partial L}{\partial(\dot{\partial}_d h_{ef})}\right)\right]\} + \frac{1}{2l_5}h_{ab}(\hat{L} + l_1 R). \end{aligned}$$

Remark that, for the case $h_{ab} = \delta^i_a\delta^j_b \cdot g_{ij}$, the equations (3.12, 3.13) contain explicitly the Einstein h- and v-gauge tensor fields considered in (1.6)

$$\begin{cases} E_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} \\ E_{ab} = S_{ab} - \frac{1}{2}Sh_{ab} \end{cases} \tag{3.14}$$

and that the right-hand terms of these equations stand for the energy-momentum h- and v-tensor fields, respectively.

In order to obtain solutions $\{g_{ij}(x,y), h_{ab}(x,y)\}$ for (3.12) and (3.13) having the predefined form (A1), a necessary condition will be (as L will depend on $\sigma(x,y)$ and its derivatives) the vanishing of the corresponding Euler-Lagrange derivative

$$\frac{\delta L}{\delta\sigma} \equiv \frac{1}{G}\left\{\partial_k\left[\frac{\partial\mathcal{L}}{\partial(\partial_k\sigma)}\right] + \dot{\partial}_a\left(\frac{\partial\mathcal{L}}{\partial\dot{\sigma}_a}\right) - \frac{\partial\mathcal{L}}{\partial\sigma}\right\} = 0. \tag{3.15}$$

We shall describe the generalised Einstein-Yang Mills equation (3.15) for the gauge Lagrangian L given in (3.9) for three special cases.

1°. If $g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x)$ and $h_{ab} = \delta^i_a\delta^j_b\ g_{ij}$, then we obtain the case of the almost Hermitian model $H^{2n} = (TM, G_*, F)$ [3] given by the

N-lift of the generalized Lagrange metric $(A1)$ to TM, and the almost complex structure F on TM given in the local adapted frame (1.1") by

$$F(\delta_i) = -\dot{\partial}_i, \quad F(\dot{\partial}_i) = \delta_i.$$

In this case, we remark that L^i_{jk} and C^a_{bc} are given by (1.1), R and S are described in (1.5), and (3.15) becomes

$$\frac{\delta L}{\delta \sigma} = \frac{1}{G}\left[\partial_k\left(l_1 H \frac{\partial \overset{0}{\sigma}}{\partial \beta_k} - 2l_4 G \frac{\partial (C^a_{bc|l})}{\partial \beta_k} P^{blc}_a\right) + \dot{\partial}_a\left(G\frac{\partial \bar{L}}{\partial \dot{\sigma}_a}\right)\right] + + 2(1-n)\bar{L} + 2(\bar{L} - l_5 S) = 0 \tag{3.16}$$

where $H = Ge^{-2\sigma}$, $\beta_k = \partial_k \sigma$, and

$$\bar{L} = l_2 \overset{v}{R} + l_3 \overset{h}{P} + l_4 \overset{v}{P} + l_5 S. \tag{3.17}$$

2°. For $g_{ij} = \gamma_{ij}(x)$, $h_{ab} = e^{2\sigma(x,y)}\gamma_{ab}(x)$, we have $L^i_{jk} = \left\{ {i \atop jk} \right\}$, $R = r$, C^a_{bc} given by (1.1-2), and

$$\frac{\delta L}{\delta \sigma} = \frac{1}{G}\left[-2l_4 \partial_k\left(G P^{blc}_a \frac{\partial (C^a_{bc|l})}{\partial \beta_k}\right) + \dot{\partial}_a\left(G\frac{\partial \bar{L}}{\partial \dot{\sigma}_a}\right)\right] + + 2(l_4 \overset{v}{P} + l_5 S - l_0 L_0) - nL = 0. \tag{3.18}$$

3°. For $g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x)$, $h_{ab} = \gamma_{ab}(x)$, we have L^i_{jk} given by (1.1-1), $P = S = 0$, and

$$\begin{aligned} \frac{\delta L}{\delta \sigma} = & \frac{1}{G}\Big\{ l_1 \partial_k\left(H\frac{\partial \overset{o}{\sigma}}{\partial \beta_k}\right) + \\ & + 2\Big[l_2 \frac{\partial}{\partial y^a}\left(G R^{bkl}_f R^d_{kl}(\delta^f_b \delta^a_d + \delta^f_d \delta^a_b - \gamma_{bd}\gamma^{af})\right) \\ & + l_3 \frac{\partial}{\partial y^a}\left(G P^{jkc}_i(\gamma_{kj}\gamma^{is}N^a_s - \delta^i_j N^a_k - \delta^i_k N^a_j)\right)\Big]\Big\} + \\ & + 2(2l_0 L_0 + l_1 R + 2l_2 \overset{v}{R} + l_3 \overset{h}{P}) - n\bar{L} = 0. \end{aligned} \tag{3.19}$$

An open problem is the one of the one of determining valid solutions $\sigma(x,y)$ for the equation (3.15), for suitable constants $l_\circ, \ldots, l_5$ in the Lagrangian field (3.9).

Conclusions. The space GL^n, as a model for the geometry of space-time [6] can be examined from the point of view of determining its Einstein and Maxwell equations, using for the first ones two approaches: the theory of d-object fields for generalised Lagrange spaces and the generalized gauge theory for vector bundles endowed with (h, v)-metrics. Explicit forms for these equations are obtained, as a preliminary step for determining their solutions. The Einstein-Yang Mills equation with respect to the scalar gauge field $\sigma(x, y)$ is also derived for three cases of (h, v)-metrics related to the fundamental metric of GL^n.

References

[1] G.S. Asanov, S.F. Ponomarenko, Finsler Bundles over Space-Time. Associated Gauge Fields and Connections (Nauka, ed.), *Chishinew*, 1989. (in Russian)

[2] G.S. Asanov, P.C. Stavrions, Finslerian Derivations of Geodesics over Tangent Bundle, vol. 30., *Rep. on Math. Phys.*, 1991.

[3] R. Miron, M. Anastasiei, Vector Bundles. Lagrange Spaces. Applications to the Theory of Relativity (Academy, ed.), *Bucharest*, 1987. (in Romanian)

[4] R. Miron, V. Balan and R.K. Tavakol, Einstein and Maxwell Equations for the Space $GL^n = (M, e^{2\sigma(x,y)}\gamma_{ij}(x))$, *Proc. of the National Sem. on Finsler, Lagrange and Hamilton Spaces*, *Brasov*, 1992, (*preprint*).

[5] R. Miron, T. Kawaguchi, Relativistic Geometrical Optics, *Int. Jour. of Theor. Phys.* **30.** no. 11. (1991), 1521–1543.

[6] R. Miron, R.K. Tavakol, Geometry of Space-Time and Generalised Lagrange Spaces, *Rep. on Math. Phys.* (*to appear*).

[7] J.L. Synge, Relativity-the General Theory, *Amsterdam, North Holland*, 1960.

[8] S. Watanabe, S. Ikeda, F. Ikeda, On a Metrical Finsler Connection of A Generalised Metric $g_{ij} = e^{2\sigma(x,y)}\gamma_{ij}(x)$, vol. 40., *Tensor N.S.*, 1983.

[9] K. Yano, S. Ishihara, Tangent and Cotangent Bundles, *M. Dekker. Inc.*, *New York*, 1973.

R. MIRON
UNIVERSITY "AL.I.CUZA"–IAŞI,
FACULTY OF MATHEMATICS,
6600, IAŞI, ROMANIA

R.K. TAVAKOL
UNIV. OF LONDON, SCHOOL OF MATH. SCI.
QUEEN MARY & WESTFIELD COLLEGE
MILE END RD., E1.4NS, LONDON, U.K.

V. BALAN
POLYTECHNIC INST. BUCHAREST
DEPARTMENT OF MATHEMATICS I
SPLAIUL INDEPENDENTEI 313
BUCHAREST, ROMANIA

I. ROXBURGH
UNIV. OF LONDON, ASTRONOMY UNIT,
SCHOOL OF MATH. SCI., QMW COLLEGE,
MILE END RD., E1.4NS, LONDON, U.K.

(Received September 15, 1992)