# Differences of products of the extended Cesàro and composition operators from $F(p,q,s)$ space to $\mu$-Bloch type space on the unit ball

By LIANG ZHANG (Tianjin) and ZE-HUA ZHOU (Tianjin)

**Abstract.** This paper characterizes the boundedness of the differences of the products of extended Cesàro operators and composition operators from $F(p,q,s)$ space to $\mu$-Bloch type space on the unit ball of $\mathbb{C}^n$. At the same time, some asymptotically equivalent expressions of the essential norms of difference of products of generalized Cesàro operators and composition operators are presented. As some corollaries, the compactness of the difference operators are also characterized.

## 1. Introduction

Let $H(\mathbb{B}_n)$ be the space of all holomorphic functions on $\mathbb{B}_n$, the compact open topology on the space $H(\mathbb{B}_n)$ will be denoted by $co$, where $\mathbb{B}_n$ is the open unit ball of the complex $n$-dimensional Euclidean space $\mathbb{C}^n$. The collection of all the holomorphic self-maps of $\mathbb{B}_n$ will be denoted by $S(\mathbb{B}_n)$. Let $d\nu$ denote Lebesgue measure on $\mathbb{B}_n$ normalized so that $\nu(\mathbb{B}_n) = 1$ and $d\sigma$ the normalized Lebesgue measure on the boundary $\partial\mathbb{B}_n$ of $\mathbb{B}_n$.

For $f \in H(\mathbb{B}_n)$, let

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$$

be the radial derivative of $f$.

For $0 < \alpha < \infty$, the weighted Banach space $H_\alpha^\infty = H_\alpha^\infty(\mathbb{B}_n)$ consists of all $f \in H(\mathbb{B}_n)$ satisfying

$$\|f\|_{H_\alpha^\infty} = \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\alpha |f(z)| < \infty.$$

Let $0 < p$, $s < \infty$, $-n - 1 < q < \infty$, a function $f \in H(\mathbb{B}_n)$ is said to belong to $F(p, q, s)$ if

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} |\Re f(z)|^p (1 - |z|^2)^q g^s(z, a) d\nu(z) < \infty,$$

where the function $g(z, a)$ in the integral is a Green's function.

A positive continuous function $\mu$ on $[0, 1)$ is called normal if there exist constants $a$, $b$ $(0 < a < b)$, and $\delta \in (0, 1)$, such that

$$\frac{\mu(r)}{(1 - r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r \to 1^-} \frac{\mu(r)}{(1 - r)^a} = 0;$$

$$\frac{\mu(r)}{(1 - r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r \to 1^-} \frac{\mu(r)}{(1 - r)^b} = \infty.$$

If a function $\mu : \mathbb{B}_n \to [0, \infty)$ is normal, we will also assume that it is radial, that is, $\mu(z) = \mu(|z|)$, for each $z \in \mathbb{B}_n$.

For $\alpha \in (0, 1)$, $\Lambda_\alpha$ denotes the holomorphic $\alpha$-Lipschitz space which is the set of all $f \in H(\mathbb{B}_n)$ such that for some $C > 0$,

$$|f(z) - f(w)| \le C|z - w|^\alpha$$

for every $z, w \in \mathbb{B}_n$. It is well known (see, [23]) that $\Lambda_\alpha$ is endowed with a complete norm $\|.\|_{\Lambda_\alpha}$ given by

$$\|f\|_{\Lambda_\alpha} = |f(0)| + \sup_{z \neq w; z,w \in \mathbb{B}_n} \left\{ \frac{|f(z) - f(w)|}{|z - w|^\alpha} \right\}$$

If a function $\mu : \mathbb{B}_n \to [0, \infty)$ is normal, the $\mu$-Bloch type space $\mathcal{B}_\mu$ (see, e.g. [14]) is defined as the space of all $f \in H(\mathbb{B}_n)$ such that

$$\sup_{z \in \mathbb{B}_n} \mu(|z|)|\Re f(z)| < \infty.$$

As we all know that $\mathcal{B}_\mu$ is a Banach space under the norm defined by

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + \mu(|z|)|\Re f(z)|.$$

When $\mu(r) = (1 - r^2)^\alpha (\alpha > 0)$ and $\mu(r) = (1 - r^2)^{1-\alpha} (0 < \alpha < 1)$, the induced spaces $\mathcal{B}_\mu$ are the $\alpha$-Bloch space $\mathcal{B}^\alpha$ and $\alpha$-Lipschitz spaces $\Lambda_\alpha$, respectively.

Let $\varphi \in S(\mathbb{B}_n)$, the composition operator $C_\varphi$ induced by $\varphi$ is defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{B}_n),\ z \in \mathbb{B}_n.$$

This operator is well studied for many years, readers interested in this topic can refer to the books [4], [16], [22], which are excellent sources for the development of the theory of composition operators, and the recent papers [18], [21], [24], [25] and the references therein.

Assume that $g \in H(\mathbb{B}_n)$ with $g(0) = 0$ and $\varphi \in S(\mathbb{B}_n)$, then we introduce the integral operator on the unit ball

$$J_g f(z) = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}_n),\ z \in \mathbb{B}_n.$$

This operator is called generalized Cesàro operator. It has been well studied in many papers, see, e.g. [6], [7], [8] and so on.

It is natural to discuss the product of extended Cesàro operator and composition operator. For $g \in H(\mathbb{B}_n)$ with $g(0) = 0$ and $\varphi \in S(\mathbb{B}_n)$, the product can be expressed as

$$J_g C_\varphi f(z) = \int_0^1 f(\varphi(tz)) \Re g(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}_n),\ z \in \mathbb{B}_n.$$

The product operator has been also studied by some authors, see, e.g. [1], [2], [3] and the references therein.

If $L : X \to Y$ is a bounded linear operator, then the essential norm of the operator $L : X \to Y$, denoted by $\|L\|_{e,X \to Y}$, is defined as follows

$$\|L\|_{e,X \to Y} = \inf_{K \in \Omega} \{\|L + K\|_{X \to Y}\},$$

where $\Omega$ is the set of all compact operators from $X$ to $Y$. From this definition and since the set of all compact operators is a closed subset of the space of bounded operators, it follows that operator $L$ is compact if and only if $\|L\|_{e,X \to Y} = 0$.

Let $\varphi, \psi \in S(\mathbb{B}_n)$ and $g, h \in H(\mathbb{B}_n)$, $g(0) = 0, h(0) = 0$. Differences of products of extended Cesàro operators and composition operators on $H(\mathbb{B}_n)$ are defined as follows:

$$(J_g C_\varphi - J_h C_\psi) f(z) = \int_0^1 f(\varphi(tz)) \Re g(tz) \frac{dt}{t} - \int_0^1 f(\psi(tz)) \Re h(tz) \frac{dt}{t}.$$

Recently, there have been an increasing interest in studying the compact difference of composition operators acting on different spaces of holomorphic functions. Some related differences of the composition operators or weighted composition operators on weighted Banach spaces of analytic functions, Bloch-type space and weighted Bergman space on the unit disk can be found, for example, [10], on the polydisk, e.g., in [5], [11], and on the unit ball, e.g., in [9], [19], [20].

In 2011, HOSOKAWAN and OHNO [10] studied the boundedness and compactness of the differences of two weighted composition operators acting from the Bloch space $\mathcal{B}$ to the space $H^\infty$ of bounded analytic functions on the open unit disk. Such a study has a relationship to the topological structure problem of composition operators on $H^\infty$. Using this relation, they estimated the operator norms and the essential norms of the differences of two composition operators acting from $\mathcal{B}$ to $H^\infty$.

In this note, we limit our analysis to the differences of products of generalized Cesàro operators and composition operators and characterize the boundedness of the difference $J_g C_\varphi - J_h C_\psi$ acting from $F(p,q,s)$ spaces to the $\mu$-Bloch type space, and find some asymptotically equivalent expressions of the essential norms of the differences, which extend the results of [12], [13].

Throughout the paper, $C$ will denote a positive constant, the exact value of which will vary from one appearance to the next. We say $B$ is the upper bound or lower bound of $A$, if there is positive constant $C$ such that $A \leq CB$ or $A \geq \frac{1}{C}B$. The notation $A \asymp B$ means that there is a positive constant $C$ such that $B/C \leq A \leq CB$.

## 2. Auxiliary results

For any point $a \in \mathbb{B}_n \setminus \{0\}$, we define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle}, \quad z \in \mathbb{B}_n.$$

where $s_a = \sqrt{1 - |a|^2}$, $P_a$ is the orthogonal projection from $\mathbb{C}^n$ onto the one-dimensional subspace $[a]$ generated by $a$, and $Q_a = I - P_a$ is the projection onto the orthogonal complement of $[a]$, that is

$$P_a(z) = \frac{\langle z, a \rangle}{|a|^2} a, \quad Q_a(z) = z - P_a(z), \ z \in \mathbb{B}_n.$$

When $a = 0$, we simply define $\varphi_a(z) = -z$. It is well known that each $\varphi_a$ is a homeomorphism of the closed unit ball $\overline{\mathbb{B}_n}$ onto $\overline{\mathbb{B}_n}$. Then we define the

pseudohyperbolic metric on $\mathbb{B}_n$

$$\rho(a, z) = |\varphi_a(z)|.$$

We know that $\rho(a, z)$ is invariant under automorphisms (see, e.g. [23]).

For any two points $z$ and $w$ in $\mathbb{B}_n$, let $\gamma(t) = (r_1(t), \ldots, \gamma_n(t)) : [0, 1] \to \mathbb{B}_n$ be a smooth curve to connect $z$ and $w$. Define

$$l(\gamma) = \int_0^1 \sqrt{\langle B(\gamma(t))\gamma'(t), \gamma'(t)\rangle} dt.$$

The infimum of the set consisting of all $l(\gamma)$ is denoted by $\beta(z, w)$, where $\gamma$ is a smooth curve in $\mathbb{B}_n$ from $z$ and $w$. We call the $\beta$ the Bergman metric on $\mathbb{B}_n$. It is known that

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

Now let us state a couple of lemmas, which are used in the proof of the main results in the paper.

The following lemma is the crucial criterion for compactness, whose proof is an easy modification of that of Proposition 3.11 of [4].

**Lemma 2.1.** *Suppose that $\mu$ is normal on $[0, 1)$ and $\varphi, \psi \in S(\mathbb{B}_n)$, $g, h \in H(\mathbb{B}_n)$ and $g(0) = 0, h(0) = 0$. Then the operator $J_g C_\varphi - J_h C_\psi : F(p, q, s) \to \mathcal{B}_\mu$ is compact if and only if $J_g C_\varphi - J_h C_\psi : F(p, q, s) \to \mathcal{B}_\mu$ is bounded and for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $F(p, q, s)$ which converges to zero uniformly on compact subsets of $\mathbb{B}_n$ as $k \to \infty$, we have $\|(J_g C_\varphi - J_h C_\psi)f_k\|_{\mathcal{B}_\mu} \to 0$, as $k \to \infty$.*

**Lemma 2.2** (Lemma 6, [17])**.** *For each sequence $\{w_j\}_{j\in N}$ in $\mathbb{B}_n$ with $|w_j|\to 1$ as $j \to \infty$, there exists its subsequence $\{\eta_k\}_{k\in\mathbb{N}}$ and functions $\{f_{j_k}\}_{k\in\mathbb{N}}$ in $H^\infty(\mathbb{B}_n)$ such that*

$$\sum_{k=1}^\infty |f_{j_k}(z)| \le 1,$$

*for all $z \in \mathbb{B}_n$ and*

$$f_{j_k}(\eta_k) > 1 - \frac{1}{2^k}, k \in \mathbb{N}.$$

**Lemma 2.3.** *Suppose that $g \in H(\mathbb{B}_n)$, $g(0) = 0$ and $\varphi \in S(\mathbb{B}_n)$. Then for any $f \in F(p, q, s)$, we have*

$$\Re[J_g C_\varphi(f)](z) = f(\varphi(z))\Re g(z).$$

PROOF. The proof follows by standard arguments (see, e.g. [8]).  □

The following four lemmas can be found in [21].

**Lemma 2.4.** *If* $f \in \mathcal{B}^\alpha$, *then*

$$|f(z)| \leq C \begin{cases} \|f\|_{\mathcal{B}}^\alpha, & 0 < \alpha < 1; \\ \|f\|_{\mathcal{B}}^\alpha \ln \dfrac{e}{1 - |z|^2}, & \alpha = 1; \\ \dfrac{\|f\|_{\mathcal{B}}^\alpha}{(1 - |z|^2)^{\alpha - 1}}, & \alpha > 1. \end{cases}$$

**Lemma 2.5.** *For* $0 < p, s < +\infty$, $-n - 1 < q < +\infty$, $q + s > -1$, *there exists* $C > 0$ *such that*

$$\sup_{a \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^p}{|1 - \langle z, w \rangle|^{n+1+q+p}} (1 - |z|^2)^q g^s(z, a) d\nu(z) \leq C$$

*for every* $\omega \in \mathbb{B}$.

**Lemma 2.6.** *There is a constant* $C > 0$ *so that for all* $t > -1$ *and* $z \in \mathbb{B}$, *we have*

$$\int_{\mathbb{B}_n} |\ln \frac{1}{1 - \langle z, w \rangle}|^2 \frac{(1 - |w|^2)^t}{|1 - \langle z, w \rangle|^{n+1+t}} d\nu(z) \leq C \left( \ln \frac{1}{1 - |z|^2} \right)^2.$$

**Lemma 2.7.** *Suppose that* $0 < p, s < \infty$, $-n - 1 < q < \infty$ *and* $q + s > -1$. *If* $f \in F(p, q, s)$, *then* $f \in \mathcal{B}^{(n+1+q)/p}$, *and* $\|f\|_{\mathcal{B}^{(n+1+q)/p}} \leq C \|f\|_{F(p,q,s)}$.

By Lemma 4, Lemma 7, and Lemma 3 in [17], we can easily obtain the following lemma.

**Lemma 2.8.** *Suppose that* $0 < p, s < \infty$, $-n - 1 < q < \infty$ *and* $q + s > -1$. *If* $f \in F(p, q, s)$, *then there is a positive constant* $C$ *independent of* $f$ *such that*

(i) *if* $n + 1 + q > p$, *then*

$$\left| (1 - |z|^2)^{\frac{n+1+q}{p} - 1} f(z) - (1 - |w|^2)^{\frac{n+1+q}{p} - 1} f(w) \right| \leq C \|f\|_{F(p,q,s)} \rho(z, w);$$

(ii) *if* $n + 1 + q = p$, *then*

$$\left| \frac{f(z)}{\ln \frac{e}{1 - |z|^2}} - \frac{f(w)}{\ln \frac{e}{1 - |w|^2}} \right| \leq C \|f\|_{F(p,q,s)} \rho(z, w);$$

(iii) *if* $n + 1 + q < p$, *then*

$$|f(z) - f(w)| \leq C \|f\|_{F(p,q,s)} \rho(z, w);$$

*for all* $z, w \in \mathbb{B}_n$.

**Lemma 2.9.** *Let $\{f_k\}_{k\in N}$ be a bounded sequence in $F(p, q, s)$ which converges to zero uniformly on compact subsets of the unit ball $\mathbb{B}_n$, where $(n+1+q)/p < 1$. Then $\lim_{k\to\infty} \sup_{z\in\mathbb{B}_n} |f_k(z)| = 0$.*

PROOF. Note that $F(p, q, s) \subseteq \mathcal{B}^{(n+1+q)/p}$ and $\|f\|_{\mathcal{B}^{(n+1+q)/p}} \leq C\|f\|_{F(p,q,s)}$. When $(n + 1 + q)/p < 1$, the proof of this lemma is similar to that of Lemma 3.6 of [14], hence the detail is omitted here. $\qquad\square$

## 3. The boundedness and essential norm of $J_g C_\varphi - J_h C_\psi : F(p, q, s) \to B_\mu$

In this section, we characterize the boundedness and the essential norm of $J_g C_\varphi - J_h C_\psi : F(p, q, s) \to B_\mu$ according to three cases, depending on two parameters $p, q$.

**3.1. The case for $(n + 1 + q)/p > 1$.** For the sake of convenience, we first introduce three quantities.

$$M_1 := \sup_{z\in\mathbb{B}_n} \frac{\mu(|z|)|\Re g(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1+q}{p}-1}} \rho(\varphi(z), \psi(z)) < \infty. \tag{1}$$

$$M_2 := \sup_{z\in\mathbb{B}_n} \frac{\mu(|z|)|\Re h(z)|}{(1 - |\psi(z)|^2)^{\frac{n+1+q}{p}-1}} \rho(\varphi(z), \psi(z)) < \infty. \tag{2}$$

$$M_3 := \sup_{z\in\mathbb{B}_n} \left| \frac{\mu(|z|)\Re g(z)}{(1 - |\varphi(z)|^2)^{\frac{n+1+q}{p}-1}} - \frac{\mu(|z|)\Re h(z)}{(1 - |\psi(z)|^2)^{\frac{n+1+q}{p}-1}} \right| < \infty. \tag{3}$$

**Theorem 3.1.** *Let $0 < p, s < \infty$, $-n - 1 < q < \infty$, $q + s > -1$ and $(n + 1 + q)/p > 1$. Assume that $\mu$ is normal on $[0, 1)$, $\varphi, \psi \in S(\mathbb{B}_n)$ and $g, h \in H(\mathbb{B}_n)$, $g(0) = 0, h(0) = 0$. Then the following statements are equivalent.*

(i) *$J_g C_\varphi - J_h C_\psi : F(p, q, s) \to \mathcal{B}_\mu$ is bounded.*

(ii) *The conditions (1) and (3) hold.*

(iii) *The conditions (2) and (3) hold.*

PROOF. First, we prove the implication (i) $\implies$ (ii). Assume that $J_g C_\varphi - J_h C_\psi : F(p, q, s) \to \mathcal{B}_\mu$ is bounded. Fix $w \in \mathbb{B}_n$, define the function

$$P_w(z) = \frac{1 - |\varphi(w)|^2}{(1 - \langle z, \varphi(w)\rangle)^{\frac{n+1+q}{p}}}$$

for $z \in \mathbb{B}_n$, then,

$$\frac{\partial P_w(z)}{\partial z_k} = \frac{n+1+q}{p} \cdot \frac{\varphi_k(w)(1-|\varphi(w)|^2)}{(1-\langle z, \varphi(w)\rangle)^{\frac{n+1+q}{p}+1}}, \quad k = 1, 2, \ldots n.$$

So, by Lemma 2.5, it is easy to check that $P_w \in F(p,q,s)$ and $\sup_{w \in \mathbb{B}_n} \|P_w\|_{F(p,q,s)} \leq C_1$ for a positive constant $C_1$.

If $\varphi(w) \neq \psi(w)$, we consider the text function $f_w$ defined by

$$f_w(z) = P_w(z) \frac{\langle \varphi_{\psi(w)}(z), \varphi_{\psi(w)}(\varphi(w))\rangle}{|\varphi_{\psi(w)}(\varphi w))|} \tag{4}$$

for $z \in \mathbb{B}_n$. Thus $f_w \in F(p,q,s)$ and $\sup_{w \in \mathbb{B}_n} \|f_w\|_{F(p,q,s)} \leq C_1$ for a constant $C_1$. It is clear that

$$f_w(\varphi(w)) = \frac{\rho(\varphi(w), \psi(w))}{(1-|\varphi(w)|^2)^{\frac{n+1+q}{p}-1}}, \quad f_w(\psi(w)) = 0. \tag{5}$$

By the boundedness of $J_g C_\varphi - J_h C_\psi : F(p,q,s) \to \mathcal{B}_\mu$, and using (5) and Lemma 2.3, we have

$$\begin{aligned}
\infty > \|(J_g C_\varphi - J_h C_\psi)f_w\|_{\mathcal{B}_\mu} &= \sup_{z \in \mathbb{B}_n} \mu(|z|)|\Re((J_g C_\varphi - J_h C_\psi)f_w)(z)| \\
&= \sup_{z \in \mathbb{B}_n} \mu(|z|)|f_w(\varphi(z))\Re g(z) - f_w(\psi(z))\Re h(z)| \\
&\geq \frac{\mu(|w|)|\Re g(w)|\rho(\varphi(w), \psi(w))}{(1-|\varphi(w)|^2)^{\frac{n+1+q}{p}-1}}
\end{aligned} \tag{6}$$

for any $w \in \mathbb{B}_n$ with $\varphi(w) \neq \psi(w)$.

Note that $\rho(\varphi(w), \psi(w)) = 0$ if $\varphi(w) = \psi(w)$. Thus (1) follows immediately from (6), since $w$ is an arbitrary element of $\mathbb{B}_n$.

Next we prove (3). For given $w \in \mathbb{B}_n$, we consider the test function

$$Q_w(z) = \frac{1-|\psi(w)|^2}{(1-\langle z, \psi(w)\rangle)^{\frac{n+1+q}{p}}}.$$

Similarly, by Lemma 2.5 and a direct computation, we can obtain that $Q_w \in F(p,q,s)$ with $\|Q_w\|_{F(p,q,s)} \leq C_2$ for a constant $C_2$. It follows from Lemma 2.3 that

$$\infty > \|(J_g C_\varphi - J_h C_\psi)Q_w\|_{\mathcal{B}_\mu} \geq \mu(|w|)|\Re((J_g C_\varphi - J_h C_\psi)Q_w)(w)|$$

$$= \mu(|w|)|Q_w(\varphi(w))\Re g(w) - Q_w(\psi(w))\Re h(w)|$$

$$= \left| \frac{\Re g(w)\mu(|w|)(1 - |\psi(w)|^2)}{(1 - \langle \varphi(w), \psi(w) \rangle)^{\frac{n+1+q}{p}}} - \frac{\mu(|w|)\Re h(w)}{(1 - |\psi(w)|^2)^{\frac{n+1+q}{p}-1}} \right|$$

$$= |I(w) + J(w)|, \qquad (7)$$

where

$$I(w) = \frac{\mu(|w|)|\Re g(w)|}{(1 - |\varphi(w)|^2)^{\frac{n+1+q}{p}-1}} - \frac{\mu(|w|)|\Re h(w)|}{(1 - |\psi(w)|^2)^{\frac{n+1+q}{p}-1}}.$$

And,

$$J(w) = \frac{\Re g(w)\mu(|w|)(1 - |\psi(w)|^2)}{(1 - \langle \varphi(w), \psi(w) \rangle)^{\frac{n+1+q}{p}}} - \frac{\mu(|w|)|\Re g(w)|}{(1 - |\varphi(w)|^2)^{\frac{n+1+q}{p}-1}}$$

$$= \frac{\mu(|w|)\Re g(w)}{(1 - |\varphi(w)|^2)^{\frac{n+1+q}{p}-1}} \Big[ (1 - |\varphi(w)|^2)^{\frac{n+q+1}{p}-1} Q_w(\varphi(w))$$

$$- (1 - |\psi(w)|^2)^{\frac{n+1+q}{p}-1} Q_w(\psi(w)) \Big].$$

By (1) and Lemma 2.8, we conclude that

$$|J(w)| \leq C \frac{\mu(|w|)|\Re g(w)|}{(1 - |\varphi(w)|^2)^{\frac{n+1+q}{p}-1}} \|Q_w\|_{F(p,q,s)} \rho(\varphi(w), \psi(w))$$

$$\leq C \frac{\mu(|w|)|\Re g(w)|}{(1 - |\varphi(w)|^2)^{\frac{n+1+q}{p}-1}} \rho(\varphi(w), \psi(w)) < \infty.$$

Thus we obtain that $|J(w)| < \infty$ for all $w \in \mathbb{B}_n$. This combines with (7) we obtain $|I(w)| < \infty$ for all $w \in \mathbb{B}_n$. Thus the desired result (3) follows.

(ii) $\implies$ (iii). Assume that (1) and (3) hold, we need only to show that (2) holds. Note that the pseudohyperbolic metric $\rho$ is less than 1. Then we have

$$\frac{\mu(|z|)|\Re h(z)|}{(1 - |\psi(z)|^2)^{\frac{n+1+q}{p}-1}} \rho(\varphi(z), \psi(z)) \leq \frac{\mu(|z|)|\Re g(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1+q}{p}-1}} \rho(\varphi(z), \psi(z))$$

$$+ \left| \frac{\mu(|z|)|\Re g(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1+q}{p}-1}} - \frac{\mu(|z|)|\Re h(z)|}{(1 - |\psi(z)|^2)^{\frac{n+1+q}{p}}} \right| \rho(\varphi(z), \psi(z)). \qquad (8)$$

From which, and employing (1) and (3), we can get (2).

(iii) $\implies$ (i). Assume that (2) and (3) hold. By Lemma 2.3, Lemma 2.7 and Lemma 2.8, for any $f \in F(p, q, s)$, we have

$$\mu(|z|)|\Re((J_g C_\varphi - J_h C_\psi)f)(z)| = \mu(|z|)|f(\varphi(z))\Re g(z) - f(\psi(z))\Re h(z)|$$

$$\leq C\|f\|_{F(p,q,s)}\left|\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q}{p}-1}}-\frac{\mu(|z|)|\Re h(z)|}{(1-|\psi(z)|^2)^{\frac{n+1+q}{p}-1}}\right|$$

$$+C\|f\|_{F(p,q,s)}\frac{\mu(|z|)|\Re h(z)|}{(1-|\psi(z)|^2)^{\frac{n+1+q}{p}-1}}\rho(\varphi(z),\psi(z))\leq C\|f\|_{F(p,q,s)}<\infty.$$

from which it follows that $J_gC_\varphi - J_hC_\psi : F(p,q,s) \to \mathcal{B}_\mu$ is bounded. The whole proof is complete.                                                                              $\square$

Let $h = 0$ in the above theorem, we get a characterization for the boundedness of the product of extended Cesàro operator and composition operator.

**Corollary 3.1.** *Let* $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$ *and* $(n+1+q)/p > 1$. *Assume that* $\mu$ *is normal on* $[0,1)$, $\varphi \in S(\mathbb{B}_n)$ *and* $g \in H(\mathbb{B}_n)$ *with* $g(0) = 0$. *Then* $J_gC_\varphi : F(p,q,s) \to \mathcal{B}_\mu$ *is bounded if and only if*

$$\sup_{z\in\mathbb{B}_n}\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q}{p}-1}}<\infty.$$

Next, we characterize the essential norm of $J_gC_\varphi - J_hC_\psi : F(p,q,s) \to B_\mu$. Denote

$$M_4 := \limsup_{|\varphi(z)|\to 1}\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q}{p}-1}}\rho(\varphi(z),\psi(z)). \tag{9}$$

$$M_5 := \limsup_{|\psi(z)|\to 1}\frac{\mu(|z|)|\Re h(z)|}{(1-|\psi(z)|^2)^{\frac{n+1+q}{p}-1}}\rho(\varphi(z),\psi(z)). \tag{10}$$

$$M_6 := \limsup_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1}\left|\frac{\mu(|z|)\Re g(z)}{(1-|\varphi(z)|^2)^{\frac{n+1+q}{p}-1}}-\frac{\mu(|z|)\Re h(z)}{(1-|\psi(z)|^2)^{\frac{n+1+q}{p}-1}}\right|. \tag{11}$$

**Theorem 3.2.** *Let* $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$ *and* $\frac{n+1+q}{p} > 1$. *Assume that* $\mu$ *is normal on* $[0,1)$ *and* $\varphi, \psi \in S(\mathbb{B}_n)$, $g, h \in H(\mathbb{B}_n)$ *and* $g(0) = 0, h(0) = 0$, $\max\{\|\varphi\|_\infty, \|\psi\|_\infty\} = 1$. *If* $J_gC_\varphi, J_hC_\psi : F(p,q,s) \to \mathcal{B}_\mu$ *are bounded, then the essential norm* $\|J_gC_\varphi - J_hC_\psi\|_{e,F(p,q,s)\to\mathcal{B}_\mu}$ *is equivalent to maximum* $M = \max\{M_4, M_5, M_6\}$.

PROOF. First we show that there is positive constant $C$ such that

$$\|J_gC_\varphi - J_hC_\psi\|_{e,F(p,q,s)\to\mathcal{B}_\mu} \leq CM.$$

That is, the maximum $M = \max\{M_4, M_5, M_6\}$ is a upper bounded for the essential norm.

Consider the operators on $H(\mathbb{B}_n)$ defined by

$$T_k(f)(z) = f\left(\frac{k}{k+1}z\right), \quad k \in \mathbb{N}.$$

It is easy to see that they are continuous on the *co* topology and that $T_k(f) \to f$ on any compact subsets of $\mathbb{B}_n$ as $k \to \infty$.

On the other hand, since $M_p(f,r) = (\int_{\partial\mathbb{B}_n} |f(r\zeta)|^p d\sigma(\zeta))^p$ are nondecreasing in $r$, by the polar coordinates formula

$$\int_{B_n} f(z)d\nu(z) = 2n\int_0^1 r^{2n-1}dr \int_{\partial\mathbb{B}_n} f(r\zeta)d\sigma(\zeta)$$

and the definition of $F(p,q,s)$, we get

$$\|T_k(f)\|_{F(p,q,s)} \leq \|f\|_{F(p,q,s)}, \quad k \in \mathbb{N},$$

which implies that $\sup_{k\in\mathbb{N}} \|T_k\|_{F(p,q,s)\to F(p,q,s)} \leq 1$. Moreover, it is easy to know that $T_k(k \in \mathbb{N})$ are compact on $F(p,q,s)$. By the boundedness of $J_gC_\varphi, J_hC_\psi : F(p,q,s) \to \mathcal{B}_\mu$, we know that $(J_gC_\varphi - J_hC_\psi)T_k$ are also compact operators.

Let $r \in (0,1)$ be fixed and $f \in F(p,q,s)$ such that $\|f\|_{F(p,q,s)} \leq 1$. Let

$$g_k := (I - T_k)f, \quad k \in \mathbb{N}.$$

We can easily obtain $g_k \in F(p,q,s)$, $k \in \mathbb{N}$ and $\sup_{k\in\mathbb{N}} \|g_k\|_{F(p,q,s)} \leq 2$. Therefore,

$$\|J_gC_\varphi - J_hC_\psi\|_{e,F(p,q,s)\to\mathcal{B}_\mu} \leq \|(J_gC_\varphi - J_hC_\psi)I - J_gC_\varphi - J_hC_\psi)T_k\|_{F(p,q,s)\to\mathcal{B}_\mu}$$

$$= \sup_{\|f\|_{F(p,q,s)}\leq 1} \|(J_gC_\varphi - J_hC_\psi)(I - T_k)f\|_{\mathcal{B}_\mu} = \sup_{\|f\|_{F(p,q,s)}\leq 1} \|(J_gC_\varphi - J_hC_\psi)g_k\|_{\mathcal{B}_\mu}$$

$$\leq \sup_{\|f\|_{F(p,q,s)}\leq 1} \sup_{|\varphi(z)|>r} \mu(|z|)|g_k(\varphi(z))\Re g(z) - g_k(\psi(z))\Re h(z)|$$

$$+ \sup_{\|f\|_{F(p,q,s)}\leq 1} \sup_{|\psi(z)|>r} \mu(|z|)|g_k(\varphi(z))\Re g(z) - g_k(\psi(z))\Re h(z)|$$

$$+ \sup_{\substack{\|f\|_{F(p,q,s)}\leq 1 \\ \min\{|\varphi(z)|,|\psi(z)|\}\leq r}} \mu(|z|)|g_k(\varphi(z))\Re g(z) - g_k(\psi(z))\Re h(z)|$$

$$= I_{k,1}(r) + I_{k,2}(r) + I_{k,3}(r). \tag{12}$$

First we estimate $I_{k,1}(r)$. By Lemma 2.7 and Lemma 2.8 and $\sup_{k\in\mathbb{N}} \|g_k\|_{F(p,q,s)} \leq 2$, we have

$$\mu(|z|)|g_k(\varphi(z))\Re g(z) - g_k(\psi(z))\Re h(z)| \leq 2C\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q}{p}-1}}\rho(\varphi(z),\psi(z))$$

$$+2\left|\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q}{p}-1}}-\frac{\mu(|z|)|\Re h(z)|}{(1-|\psi(z)|^2)^{\frac{n+1+q}{p}-1}}\right|. \tag{13}$$

A similar estimate is obtained for $I_{k,2}(r)$.

It is clear that for every $f \in H(\mathbb{B}_n)$, $\lim_{k\to\infty}(I-T_k)f(z) = 0$ and the space $H(\mathbb{B}_n)$ endowed with compact open topology $co$ is a Fréchet space. Hence, by Banach–Steinhaus theorem, $(I-T_k)f$ converges to zero uniformly on compacts of $(H(\mathbb{B}_n), co)$. Since the unit ball of $F(p,q,s)$ is a compact subset of $(H(\mathbb{B}_n), co)$ it follows that

$$\lim_{k\to\infty}\sup_{\|f\|_{F(p,q,s)}\leq 1}\sup_{|\zeta|\leq r}|(I-T_k)(f)(\zeta)| = 0. \tag{14}$$

From the boundedness of the operators $J_g C_\varphi, J_h C_\psi : F(p,q,s) \to \mathcal{B}_\mu$, we can easily obtain the boundedness of $J_g C_\varphi - J_h C_\psi : F(p,q,s) \to \mathcal{B}_\mu$. Then by Theorem 3.1, we get (3).

On the other hand we apply (14) to the sequence $\{g_k\}_{k\in\mathbb{N}}$, we have

$$\lim_{k\to\infty}\sup_{\|f\|_{F(p,q,s)}\leq 1}\sup_{|\zeta|\leq r}|g_k(\zeta)| = 0. \tag{15}$$

Hence for each $r \in (0,1)$ and $|\psi(z)| \leq r$, using (15) and (3), we have

$$\limsup_{k\to\infty} I_{k,1}(r) \leq 2C \sup_{|\varphi(z)|>r}\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q}{p}-1}}\rho(\varphi(z),\psi(z)).$$

If $|\psi(z)| > r$, it follows from (13) that

$$\limsup_{k\to\infty} I_{k,1}(r) \leq 2C \sup_{|\varphi(z)|>r}\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q}{p}-1}}\rho(\varphi(z),\psi(z))$$

$$+\limsup_{\min\{|\varphi(z)|,|\psi(z)|\}>r} 2\left|\frac{\mu(|z|)|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q}{p}-1}}-\frac{\mu(|z|)|\Re h(z)|}{(1-|\psi(z)|^2)^{\frac{n+1+q}{p}-1}}\right|$$

Letting $r \to 1$ in the above inequality, we get an estimate for $\limsup_{r\to 1}\limsup_{k\to\infty} I_{k,1}(r) \leq 2CM_4 + 2M_5 \leq CM$. Similar estimate is obtained for $\limsup_{r\to 1}\limsup_{k\to\infty} I_{k,2}(r)$.

Next we estimate $\lim_{k\to\infty} I_{k,3}$. We take the test function $f(z)=1 \in F(p,q,s)$, the boundedness of $J_g C_\varphi, J_h C_\psi : F(p,q,s) \to \mathcal{B}_\mu$ implies that $g, h \in \mathcal{B}_\mu$. Combing this with (15), it is easy to get $\lim_{k\to\infty} I_{k,3}(r) = 0$. The desired upper estimate follows.

Next we show that the maximum $M = \max\{M_4, M_5, M_6\}$ is a lower bound for the essential norm. Choose a sequence $\{z_k\}_{k\in\mathbb{N}}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ and

$$M_4 = \lim_{k\to\infty} \frac{\mu(|z_k|)|\Re g(z_k)|}{(1 - |\varphi(z_k)|^2)^{\frac{n+1+q}{p}-1}} \rho(\varphi(z_k), \psi(z_k)).$$

If such a sequence does not exist, then the estimate vacuously holds. Since $|\varphi(z_k)| \to 1$ as $k \to \infty$, by Lemma 2.2, we can find an sequence $f_k \in H^\infty(\mathbb{B}_n)$, $k \in \mathbb{N}$, such that

$$\sum_{k=1}^{\infty} |f_k(z)| \le 1, \tag{16}$$

for all $z \in \mathbb{B}_n$, and

$$f_k(\varphi(z_k)) > 1 - \frac{1}{2^k}, k \in \mathbb{N}. \tag{17}$$

Define the test functions similar to that of Theorem 3.1

$$G_k(z) = f_k(z)P_k(z) \cdot \frac{\langle \varphi_{\psi(z_k)}(z), \varphi_{\psi(z_k)}(\varphi(z_k)) \rangle}{|\varphi_{\psi(z_k)}(\varphi(z_k))|}, \tag{18}$$

where

$$P_k(z) = \frac{1 - |\varphi(z_k)|^2}{(1 - \langle z, \varphi(z_k) \rangle)^{\frac{n+1+q}{p}}}$$

and $C_1 > 0$ is a constant.

Similar to that of Theorem 3.1, we have that $\sup_{k\in\mathbb{N}} \|G_k\|_{F(p,q,s)} \le C_1$.

Note that

$$G_k(\varphi(z_k)) = f_k(\varphi(z_k)) \frac{\rho(\varphi(z_k), \psi(z_k))}{(1 - |\varphi(z_k)|^2)^{\frac{n+1+q}{p}-1}}, \quad G_k(\psi(z_k)) = 0 \tag{19}$$

and, $G_k \to 0$ uniformly on the compact subsets of $\mathbb{B}_n$ as $k \to \infty$. Then for each compact operators $K : F(p,q,s) \to \mathcal{B}_\mu$, we have $\lim_{k\to\infty} \|KG_k\|_{\mathcal{B}_\mu} = 0$. Therefore

$$\begin{aligned}
C_1 \|J_g C_\varphi - J_h C_\psi - K\|_{F(p,q,s)\to\mathcal{B}_\mu} &\ge \limsup_{k\to\infty} \|(J_g C_\varphi - J_h C_\psi - K)G_k\|_{\mathcal{B}_\mu} \\
&\ge \limsup_{k\to\infty} \|(J_g C_\varphi - J_h C_\psi)G_k\|_{\mathcal{B}_\mu} - \limsup_{k\to\infty} \|KG_k\|_{\mathcal{B}_\mu} \\
&\ge \limsup_{k\to\infty} \mu(|z_k|)|G_k(\varphi(z_k))\Re g(z_k) - G_k(\psi(z_k))\Re h(z_k)| \\
&= \limsup_{k\to\infty} \mu(|z_k|)\left| f_k(\varphi(z_k)) \frac{\rho(\varphi(z_k), \psi(z_k))\Re g(z_k)}{(1 - |\varphi(z_k)|^2)^{\frac{n+1+q}{p}-1}} \right|
\end{aligned}$$

$$\geq \limsup_{k\to\infty} \left(1 - \frac{1}{2^k}\right) \frac{\mu(|z_k|)|\Re g(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+1+q}{p}-1}} \rho(\varphi(z_k), \psi(z_k)) = M_4. \qquad (20)$$

Therefore, $M_4$ is a lower bound for the essential norm. Similarly, $M_5$ is also a lower bound.

Next we prove that $M_6$ is a lower bound for the essential norm. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence $\{z_k\}_{k\in\mathbb{N}}$ with $\min\{|\varphi(z_k)|, |\psi(z_k)|\} \to 1$ as $k \to \infty$ and

$$M_6 = \limsup_{k\to\infty} \left| \frac{\mu(|z_k|)\Re g(z_k)}{(1-|\varphi(z_k)|^2)^{\frac{n+1+q}{p}-1}} - \frac{\mu(|z_k|)\Re h(z_k)}{(1-|\psi(z_k)|^2)^{\frac{n+1+q}{p}-1}} \right|.$$

Setting $l := \lim_{k\to\infty} \rho(\varphi(z_k), \psi(z_k))$ (if necessary, we choose a subsequence), then $l \geq 0$.

If $l > 0$, when $\min\{|\varphi(z_k)|, |\psi(z_k)|\} \to 1$ as $k \to \infty$, we have $M_6 \leq (M_4 + M_5)/l$. Since we have proved that both $M_4$ and $M_5$ are low bounds for the essential norm, it follows that $M_6$ is also a lower bound. Now we can assume that $l = 0$ when $\min\{|\varphi(z_k)|, |\psi(z_k)|\} \to 1$ as $k \to \infty$.

Let $\{f_k\}_{k\in\mathbb{N}}$ satisfy (16) and (17), and $\{P_k\}_{k\in\mathbb{N}}$ is defined in (18). Then we choose the function $H_k(z) = f_k(z)P_k(z)$, $k \in \mathbb{N}$.

We can know that $\sup_{k\in\mathbb{N}} \|H_k\|_{F(p,q,s)} \leq C_1$. Moreover, it is obvious that $H_k \to 0$ uniformly on the compact subsets of $\mathbb{B}_n$ as $k \to \infty$. By Lemma 2.1, for each compact operator $K : F(p,q,s) \to \mathcal{B}_\mu$, we have $\lim_{k\to\infty} \|KH_k\|_{\mathcal{B}_\mu} = 0$. Therefore, we have

$$C_1 \|J_g C_\varphi - J_h C_\psi - K\|_{F(p,q,s)\to\mathcal{B}_\mu}$$
$$\geq \limsup_{k\to\infty} \|(J_g C_\varphi - J_h C_\psi)H_k\|_{\mathcal{B}_\mu} - \limsup_{k\to\infty} \|KH_k\|_{\mathcal{B}_\mu}$$
$$\geq \limsup_{k\to\infty} \mu(|z_k|)|H_k(\varphi(z_k))\Re g(z_k) - H_k(\psi(z_k))\Re h(z_k)|$$
$$\geq \limsup_{k\to\infty} \left| \frac{\mu(|z_k|)\Re g(z_k)}{(1-|\varphi(z_k)|^2)^{\frac{n+1+q}{p}-1}} - \frac{\mu(|z_k|)\Re h(z_k)}{(1-|\psi(z_k)|^2)^{\frac{n+1+q}{p}-1}} \right| \left(1 - \frac{1}{2^k}\right)$$
$$- C \limsup_{k\to\infty} \frac{\mu(|z_k|)|\Re h(z_k)|}{(1-|\psi(z_k)|^2)^{\frac{n+1+q}{p}-1}} \rho(\varphi(z_k), \psi(z_k))$$
$$= \limsup_{k\to\infty} \left| \frac{\mu(|z_k|)\Re g(z_k)}{(1-|\varphi(z_k)|^2)^{\frac{n+1+q}{p}-1}} - \frac{\mu(|z_k|)\Re h(z_k)}{(1-|\psi(z_k)|^2)^{\frac{n+1+q}{p}-1}} \right|. \qquad (21)$$

So we know that the expression $M_6$ is also a lower bound for the essential norm. The whole proof is complete. $\qquad\square$

From the above theorem, we can easily obtain the following corollary about the compactness.

**Corollary 3.2.** *Let $0 < p, s < \infty$, $-n - 1 < q < \infty$, $q + s > -1$ and $(n + 1 + q)/p > 1$. Suppose that $\mu$ is normal on $[0, 1)$, $\varphi \in S(\mathbb{B}_n)$ and $g \in H(\mathbb{B}_n)$. If $J_g C_\varphi : F(p, q, s) \to \mathcal{B}_\mu$ is bounded, then $J_g C_\varphi : F(p, q, s) \to \mathcal{B}_\mu$ is compact if and only if*

$$\limsup_{|\varphi(z)| \to 1} \frac{\mu(|z|)|\Re g(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1+q}{p} - 1}} = 0.$$

**3.2. The case for $(n + 1 + q)/p = 1$.** For the sake of convenience, we also introduce three quantities.

$$M_7 := \sup_{z \in \mathbb{B}_n} \mu(|z|)|\Re g(z)| \left| \ln \frac{e}{1 - |\varphi(z)|^2} \right| \rho(\varphi(z), \psi(z)) < \infty. \tag{22}$$

$$M_8 := \sup_{z \in \mathbb{B}_n} \mu(|z|)|\Re h(z)| \left| \ln \frac{e}{1 - |\psi(z)|^2} \right| \rho(\varphi(z), \psi(z)) < \infty. \tag{23}$$

$$M_9 := \sup_{z \in \mathbb{B}_n} \left| \mu(|z|)\Re g(z) \ln \frac{e}{1 - |\varphi(z)|^2} - \mu(|z|)\Re h(z) \ln \frac{e}{1 - |\psi(z)|^2} \right| < \infty. \tag{24}$$

**Theorem 3.3.** *Let $0 < p, s < \infty$, $-n - 1 < q < \infty$, $q + s > -1$ and $(n + 1 + q)/p = 1$. Suppose that $\mu$ is normal on $[0, 1)$, $\varphi, \psi \in S(\mathbb{B}_n)$, $g, h \in H(\mathbb{B}_n)$ and $g(0) = 0, h(0) = 0$. Then the following statements are equivalent.*

(i) *$J_g C_\varphi - J_h C_\psi : F(p, q, s) \to \mathcal{B}_\mu$ is bounded.*

(ii) *The conditions (22) and (24) hold.*

(iii) *The conditions (23) and (24) hold.*

PROOF. First, we prove the implication (i) $\implies$ (ii). Assume that $J_g C_\varphi - J_h C_\psi : F(p, q, s) \to \mathcal{B}_\mu$ is bounded. Fix $w \in \mathbb{B}_n$, define the function

$$P_w(z) = \ln \frac{e}{1 - \langle z, \varphi(w) \rangle}$$

for $z \in \mathbb{B}_n$.

By Lemma 2.6, we know that $P_w \in F(p, q, s)$ and $\sup_{w \in \mathbb{B}_n} \|P_w\|_{F(p,q,s)} \leq C$. If $\varphi(w) \neq \psi(w)$, we consider the text function $f_w$ defined by

$$f_w(z) = P_w(z) \frac{\langle \varphi_{\psi(w)}(z), \varphi_{\psi(w)}(\varphi(w)) \rangle}{|\varphi_{\psi(w)}(\varphi(w))|} \tag{25}$$

for $z \in \mathbb{B}_n$. Thus $f_w \in F(p, q, s)$ and $\sup_{w \in \mathbb{B}_n} \|f_w\|_{F(p,q,s)} \leq C$. It is clear that

$$f_w(\varphi(w)) = \ln \frac{e}{1 - |\varphi(w)|^2} \rho(\varphi(w), \psi(w)), \quad f_w(\psi(w)) = 0. \tag{26}$$

By the boundedness of $J_g C_\varphi - J_h C_\psi : F(p,q,s) \to \mathcal{B}_\mu$, and using (26) and Lemma 2.3, we have

$$
\begin{aligned}
\infty &> \|(J_g C_\varphi - J_h C_\psi)f_w\|_{\mathcal{B}_\mu} \sup_{z \in \mathbb{B}_n} \mu(|z|)|\Re((J_g C_\varphi - J_h C_\psi)f_w)(z)| \\
&= \sup_{z \in \mathbb{B}_n} \mu(|z|)|f_w(\varphi(z))\Re g(z) - f_w(\psi(z))\Re h(z)| \\
&\geq \mu(|w|)|\Re g(w)| \left| \ln \frac{e}{1 - |\varphi(w)|^2} \right| \rho(\varphi(w), \psi(w))
\end{aligned}
\tag{27}
$$

for any $w \in \mathbb{B}_n$ with $\varphi(w) \neq \psi(w)$.

Note that $\rho(\varphi(w), \psi(w)) = 0$ if $\varphi(w) = \psi(w)$. Thus if follows from (27) that (22) holds.

Next we prove (24). For given $w \in \mathbb{B}_n$, we consider the test function

$$
Q_w(z) = \ln \frac{e}{1 - \langle z, \psi(w) \rangle}
$$

Similarly, by Lemma 2.6 and a direct computation, we can obtain that $Q_w \in F(p,q,s)$ with $\|Q_w\|_{F(p,q,s)} \leq C$ for a constant $C$. It follows from Lemma 2.3 that

$$
\begin{aligned}
\infty &> \|(J_g C_\varphi - J_h C_\psi)Q_w\|_{\mathcal{B}_\mu} \geq \mu(|w|)|\Re((J_g C_\varphi - J_h C_\psi)Q_w)(w)| \\
&= \mu(|w|)|Q_w(\varphi(w))\Re g(w) - Q_w(\psi(w))\Re h(w)| \\
&= \left| \Re g(w)\mu(|w|) \ln \frac{e}{1 - \langle \psi(w), \varphi(w) \rangle} - \mu(|w|)\Re h(w) \ln \frac{e}{1 - |\psi(w)|^2} \right| \\
&= |I(w) + J(w)|,
\end{aligned}
\tag{28}
$$

where

$$
I(w) = \ln \frac{e}{1 - |\varphi(w)|^2} \mu(|w|)\Re g(w) - \ln \frac{e}{1 - |\psi(w)|^2} \mu(|w|)\Re h(w).
$$

And,

$$
\begin{aligned}
J(w) &= \Re g(w)\mu(|w|) \ln \frac{e}{1 - \langle \psi(w), \varphi(w) \rangle} - \ln \frac{e}{1 - |\varphi(w)|^2} \mu(|w|)\Re g(w) \\
&= \ln \frac{e}{1 - |\varphi(w)|^2} \mu(|w|)\Re g(w) \left[ \frac{Q_w(\varphi(w))}{\ln \frac{e}{1-|\varphi(w)|^2}} - \frac{Q_w(\psi(w))}{\ln \frac{e}{1-|\psi(w)|^2}} \right].
\end{aligned}
$$

By (22) and Lemma 2.8, we conclude that

$$|J(w)| \leq C \left| \ln \frac{e}{1 - |\varphi(w)|^2} \right| \mu(|w|) |\Re g(w)| \|Q_w\|_{F(p,q,s)} \rho(\varphi(w), \psi(w))$$

$$\leq C \left| \ln \frac{e}{1 - |\varphi(w)|^2} \right| \mu(|w|) |\Re g(w)| \rho(\varphi(w), \psi(w)) < \infty.$$

Thus we obtain that $|J(w)| < \infty$ for all $w \in \mathbb{B}_n$. This combines with (28) we obtain $|I(w)| < \infty$ for all $w \in \mathbb{B}_n$. Thus the desired result (24) follows.

(ii) $\implies$ (iii). Assume that (22) and (24) hold, we need only to show (23) hold. Note that the pseudohyperbolic metric $\rho$ is less than 1. Then we have that

$$\left| \ln \frac{e}{1 - |\psi(z)|^2} \right| \mu(|z|) |\Re h(z)| \rho(\varphi(z), \psi(z))$$

$$\leq \ln \frac{e}{1 - |\varphi(z)|^2} \mu(|z|) |\Re g(z)| \rho(\varphi(z), \psi(z))$$

$$+ \left| \ln \frac{e}{1 - |\varphi(z)|^2} \mu(|z|) |\Re g(z)| - \ln \frac{e}{1 - |\psi(z)|^2} \mu(|z|) |\Re h(z)| \right| \rho(\varphi(z), \psi(z))$$

From which, and employing (22) and (24), we can get (23).

(iii) $\implies$ (i). Assume that (23) and (24) hold. By Lemma 2.3, Lemma 2.4, Lemma 2.7 and Lemma 2.8, for any $f \in F(p, q, s)$, we have

$$\mu(|z|) |\Re((J_g C_\varphi - J_h C_\psi)f)(z)| = \mu(|z|) |f(\varphi(z)) \Re g(z) - f(\psi(z)) \Re h(z)|$$

$$\leq C \|f\|_{F(p,q,s)} \left| \mu(|z|) |\Re g(z)| \ln \frac{e}{1 - |\varphi(w)|^2} - \ln \frac{e}{1 - |\psi(z)|^2} \mu(|z|) |\Re h(z)| \right|$$

$$+ C \|f\|_{F(p,q,s)} \mu(|z|) |\Re h(z)| \rho(\varphi(z), \psi(z)) \ln \frac{e}{1 - |\psi(z)|^2} \leq C \|f\|_{F(p,q,s)} < \infty.$$

from which it follows that $J_g C_\varphi - J_h C_\psi : F(p, q, s) \to \mathcal{B}_\mu$ is bounded. The whole proof is complete. $\square$

Let $h = 0$ in the above theorem, we have the following corollary.

**Corollary 3.3.** *Let* $0 < p, s < \infty$, $-n - 1 < q < \infty$, $q + s > -1$ *and* $(n + 1 + q)/p = 1$. *Suppose that* $\mu$ *is normal on* $[0, 1)$, $\varphi \in S(\mathbb{B})$ *and* $g \in H(\mathbb{B})$. *Then* $J_g C_\varphi : F(p, q, s) \to \mathcal{B}_\mu$ *is bounded if and only if*

$$\sup_{z \in \mathbb{B}_n} \mu(|z|) |\Re g(z)| \left| \ln \frac{e}{1 - |\varphi(z)|^2} \right| < \infty.$$

Next, we discuss the essential norm of $J_g C_\varphi - J_h C_\psi : F(p,q,s) \to B_\mu$.

$$\limsup_{|\varphi(z)| \to 1} \mu(|z|)|\Re g(z)| \left| \ln \frac{e}{1-|\varphi(z)|^2} \right| \rho(\varphi(z), \psi(z)). \qquad (29)$$

$$\limsup_{|\psi(z)| \to 1} \mu(|z|)|\Re h(z)| \left| \ln \frac{e}{1-|\psi(z)|^2} \right| \rho(\varphi(z), \psi(z)). \qquad (30)$$

$$\limsup_{\min\{|\varphi(z)|, |\psi(z)|\} \to 1} \left| \mu(|z|)\Re g(z) \ln \frac{e}{1-|\varphi(z)|^2} - \mu(|z|)\Re h(z) \ln \frac{e}{1-|\psi(z)|^2} \right|. \qquad (31)$$

**Theorem 3.4.** *Let $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$ and $(n+1+q)/p = 1$. Suppose that $\mu$ is normal on $[0,1)$, $\varphi, \psi \in S(\mathbb{B}_n)$, $g \in H(\mathbb{B}_n)$, $g(0) = 0$, $h(0) = 0$ and $\max\{\|\varphi\|_\infty, \|\psi\|_\infty\} = 1$. If $J_g C_\varphi, J_h C_\psi : F(p,q,s) \to \mathcal{B}_\mu$ are bounded, then the essential norm $\|J_g C_\varphi - J_h C_\psi\|_{e, F(p,q,s) \to \mathcal{B}_\mu}$ is equivalent to the maximum* (29)–(31).

PROOF. The proof of this theorem follows the same idea as the proof of Theorem 3.2 with minor modifications, we omit the detail. $\qquad \square$

From this theorem, we obtain immediately the following corollary.

**Corollary 3.4.** *Let $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$ and $(n+1+q)/p = 1$. Suppose that $\mu$ is normal on $[0,1)$, $\varphi \in S(\mathbb{B})$ and $g \in H(\mathbb{B})$. If $J_g C_\varphi : F(p,q,s) \to \mathcal{B}_\mu$ is bounded, then $J_g C_\varphi : F(p,q,s) \to \mathcal{B}_\mu$ is compact if and only if*

$$\limsup_{|\varphi(z)| \to 1} \mu(|z|)|\Re g(z)| \left| \ln \frac{e}{1-|\varphi(z)|^2} \right| = 0.$$

**3.3. The case for $(n+1+q)/p < 1$.** Denote three quantities $M_{10}, M_{11}, M_{12}$ as follows:

$$M_{10} := \sup_{z \in \mathbb{B}_n} \mu(|z|)|\Re g(z)| \rho(\varphi(z), \psi(z)) < \infty. \qquad (32)$$

$$M_{11} := \sup_{z \in \mathbb{B}_n} \mu(|z|)|\Re h(z)| \rho(\varphi(z), \psi(z)) < \infty. \qquad (33)$$

$$M_{12} := \sup_{z \in \mathbb{B}_n} |\mu(|z|)\Re g(z) - \mu(|z|)\Re h(z)| < \infty. \qquad (34)$$

Similar to the discussion for the case $(n+1+q)/p < 1$, with minor modifications, we can obtain the corresponding theorems. For the sake of completeness we state them here, and leave the details to the interested reader.

**Theorem 3.5.** *Let $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$ and $(n+1+q)/p < 1$. Suppose that $\mu$ is normal on $[0,1)$, $\varphi, \psi \in S(\mathbb{B}_n)$, $g, h \in H(\mathbb{B}_n)$ and $g(0) = 0, h(0) = 0$. Then the following statements are equivalent.*

(i)  $J_g C_\varphi - J_h C_\psi : F(p, q, s) \to \mathcal{B}_\mu$ is bounded.

(ii)  The conditions (32) and (34) hold.

(iii)  The conditions (33) and (34) hold.

Denote

$$M_{13} := \limsup_{|\varphi(z)| \to 1} \mu(|z|)|\Re g(z)|\rho(\varphi(z), \psi(z)). \tag{35}$$

$$M_{14} := \limsup_{|\psi(z)| \to 1} \mu(|z|)|\Re h(z)|\rho(\varphi(z), \psi(z)). \tag{36}$$

$$M_{15} := \limsup_{\min\{|\varphi(z)|, |\psi(z)|\} \to 1} \left| \mu(|z|)\Re g(z) - \mu(|z|)\Re h(z) \right|. \tag{37}$$

**Theorem 3.6.** Let $0 < p, s < \infty$, $-n - 1 < q < \infty$, $q + s > -1$ and $(n + 1 + q)/p < 1$. Suppose that $\mu$ is normal on $[0, 1)$, $\varphi, \psi \in S(\mathbb{B}_n)$, $g, h \in H(\mathbb{B}_n)$ and $g(0) = 0$, $h(0) = 0$, $\max\{\|\varphi\|_\infty, \|\psi\|_\infty\} = 1$. If $J_g C_\varphi, J_h C_\psi : F(p, q, s) \to \mathcal{B}_\mu$ are bounded, then the essential norm $\|J_g C_\varphi - J_h C_\psi\|_{e, F(p,q,s) \to \mathcal{B}_\mu}$ is equivalent to $M = \max\{M_{13}, M_{14}, M_{15}\}$.

**Corollary 3.5.** Let $0 < p, s < \infty$, $-n - 1 < q < \infty$, $q + s > -1$ and $(n + 1 + q)/p < 1$. Assume that $\mu$ is normal on $[0, 1)$, $\varphi \in S(\mathbb{B}_n)$ and $g \in H(\mathbb{B}_n)$. Then the following statements are equivalent.

(i)  $J_g C_\varphi : F(p, q, s) \to \mathcal{B}_\mu$ is bounded.

(ii)  $J_g C_\varphi : F(p, q, s) \to \mathcal{B}_\mu$ is compact.

(iii)  $g \in \mathcal{B}_\mu$.

PROOF. (ii) $\Longrightarrow$ (i). This implication is obvious.

(i) $\Longrightarrow$ (iii). From the boundedness of $J_g C_\varphi$, it is easy to show that (iii) follows by taking $f(z) = 1$.

(iii) $\Longrightarrow$ (ii). Suppose (iii) holds. By Lemma 2.7, we know $f \in \mathcal{B}^{(n+1+q)/p}$ for every $f \in F(p, q, s)$, and $\|f\|_{\mathcal{B}^{(n+1+q)/p}} \leq C\|f\|_{F(p,q,s)}$. Note that $\frac{n+1+q}{p} < 1$, it follows from Lemma 2.4 that $|f(z)| \leq C\|f\|_{\mathcal{B}^{(n+1+q)/p}} \leq C\|f\|_{F(p,q,s)}$ for any $z \in \mathbb{B}_n$. Then, for any $f \in F(p, q, s)$ we obtain that

$$\mu(|z|)|\Re(J_g C_\varphi f)(z)| = \mu(|z|)|\Re g(z)||f(\varphi(z))| \leq C\mu(|z|)|\Re g(z)|\|f\|_{\mathcal{B}^{(n+1+q)/p}}$$

$$\leq C\mu(|z|)|\Re g(z)|\|f\|_{F(p,q,s)} \leq \|g\|_{\mathcal{B}_\mu}\|f\|_{F(p,q,s)}.$$

So $J_g C_\varphi : F(p, q, s) \to \mathcal{B}_\mu$ is bounded.

Let $\{f_k\}_{k\in N}$ be a bounded sequence in $F(p,q,s)$ and $f_k \to 0$ on compact subsets of $\mathbb{B}_n$ as $k \to \infty$. Note that $(n+1+q)/p < 1$ again, by Lemma 2.9 we get $\lim_{k\to\infty} \sup_{z\in\mathbb{B}_n} |f_k(z)| = 0$. Since

$$\|J_g C_\varphi f_k\|_{\mathcal{B}_\mu} \asymp \sup_{z\in\mathbb{B}_n} \mu(|z|)|\Re g(z)||f_k(\varphi(z))|$$

$$\leq \sup_{z\in\mathbb{B}_n} \mu(|z|)|\Re g(z)| \sup_{z\in\mathbb{B}_n} |f_k(\varphi(z))| \leq C\|g\|_{\mathcal{B}_\mu} \sup_{z\in\mathbb{B}_n} |f_k(\varphi(z))| \to 0, \quad k \to \infty.$$

It follows that $\lim_{k\to\infty} \|J_g C_\varphi f_k\|_{\mathcal{B}_\mu} = 0$. The result (ii) is obtained by Lemma 2.1. The proof of this theorem is complete. $\qquad\square$

## References

[1] A. ALEMAN and J. A. CIMA, An integral operator on $H^p$ and Hardy's inequality, *J. Anal. Math.* **85** (2001), 157–176.

[2] A. ALEMAN and A. G. SISKAKIS, An integral operator on $H^p$, *Complex Variables. Theory and Application* **28** (1995), 149–158.

[3] A. ALEMAN and A. G. SISKAKIS, Integration operators on Bergman spaces, *Indiana Univ. Math.* **46** (1997), 337–356.

[4] C. C. COWEN and B.D. MACCLUER, Composition Operators on Spaces of Analytic Functions, *CRC Press, Boca Raton*, 1995.

[5] J. N. DAI and C. H. OUYANG, Difference of weighted composition operators on $H_\alpha^\infty(B_N)$, *J. Inequal. Appl.* **2009** (2009), 1–19.

[6] Z. S. FANG and Z. H. ZHOU, Extended Cesàro operators on BMOA spaces in the unit ball, *J. Math. Inequal.* **4** (2010), 27–36.

[7] Z. S. FANG and Z. H. ZHOU, Extended Cesàro operators from weighted Bloch spaces to Zygmund spaces in the unit ball, *J. Math. Anal. Appl.* **359** (2009), 499–507.

[8] Z. J. HU, Extended Cesàro operators on mixed norm spaces, *Proc. Amer. Math. Soc.* **131** (2003), 2171–2179.

[9] K. HELLER, B. D. MACCLUER and R. J. WEIR, Compact differences of composition operators in several variables, *Integr. Equ. Oper. Theory* **69** (2011), 247–268.

[10] T. HOSOKAWA and S. OHNO, Differences of weighted composition operators acting from Bloch space to the space $H^\infty$, *Trans. Amer. Math. Soc.* **363** (2011), 5321–5340.

[11] Z. J. JIANG and S. STEVIĆ, Compact differences of weighted composition operators from weighted Bergman spaces to weighted-type spaces, *Appl. Math. Comput.* **217** (2010), 3522–3530.

[12] S. X. LI, Riemann-Stieltjes operators from $F(p,q,s)$ spaces to $\alpha$-Bloch spaces on the unit ball, *J. Inequal. Appl.* **2006** (2006), 1–14.

[13] YU-XIA LIANG and Z. H. ZHOU, Some integral-type operators from $F(p,q,s)$ spaces to mixed-norm spaces on the unit ball, *Math. Nachr.* **287** (2014), 1298–1311.

[14] S. Ohno, K. Stroethoff and R. Zhao, Weighted composition operators between Bloch–type spaces, *Rocky Mt. J. Math.* **33** (2003), 191–215.

[15] W. Rudin, Function Theory in the Unit Ball of $C^n$, *Springer*, *New York, NY, USA*, 1980.

[16] J. H. Shapiro, Composition operators and classical function theory, *Springer-Verlag*, *New York, NY, USA*, 1993.

[17] S. Stević, Essential norm of differences of weighted composition operators between weighted-type spaces on the unit ball, *Appl. Math. Comput.* **217** (2010), 1811–1824.

[18] J. Xiao, Composition operators associated with Bloch-type spaces, *Complex Variables* **46** (2001), 109–121.

[19] J. Xiao, Riemann-Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball, *J. London Math. Soc.* **70** (2004), 199–214.

[20] J. Xiao, Riemann–Stieltjes operators between weighted Bergman spaces, *Complex and harmonic analysis* **2007** (2007), 205–212.

[21] Z. H. Zhou and R. Y. Chen, Weighted composition operators fom $F(p, q, s)$ to Bloch type spaces, *Int. J. Math.* **19** (2008), 899–926.

[22] K. H. Zhu, Operator Theory in Function Spaces, *Marcel Dekker. Inc*, *New York*, 1990.

[23] K. H. Zhu, Spaces of Holomorphic Functions in the Unit Ball, *Springer*, *New York*, 2005.

[24] Z. H. Zhou and J. H. Shi, Compactness of composition operators on the Bloch space in classical bounded symmetric domains, *Michigan Math. J.* **50** (2002), 381–405.

[25] H. G. Zeng and Z. H. Zhou, An estimate of the essential norm of a composition operator from $F(p, q, s)$ to $\mathcal{B}^\alpha$ in the unit ball, *J. Inequal. Appl.* **2010** (2010), 1–18.

LIANG ZHANG
SCHOOL OF MARINE SCIENCE
AND TECHNOLOGY
TIANJIN UNIVERSITY
TIANJIN 300072
P.R. CHINA

*E-mail:* 168zhangliang2011@163.com

ZE-HUA ZHOU
DEPARTMENT OF MATHEMATICS
TIANJIN UNIVERSITY
TIANJIN 300072
P.R. CHINA

*E-mail:* zhzhou@tju.edu.cn,
        zehuazhoumath@aliyun.com