Publ. Math. Debrecen
87/1-2 (2015), 23–33
DOI: 10.5486/PMD.2015.6092

Generating iterated function systems for a class of self-similar sets with complete overlap

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Abstract. It is an interesting topic to find all generating iterated function systems (IFSs) for a given self-similar set. Previous results on this topic require some separation condition. In this paper, we discuss all generating IFSs for a class of self-similar sets with complete overlap. We prove that the IFS $\{\rho x, \rho x+\rho, \rho x+1\}$ with $0<\rho<(3-\sqrt{5})/2$ is a minimal presentation, i.e. every other generating IFS with the same attractor is an iteration of this one.

1. Introduction

Let $\mathcal{F}=\{f_i=r_ix+b_i\}_{i=1}^N$ be a family of distinct functions with $0<|r_i|<1$. Hutchinson [6] proved that there exists a nonempty compact set $F\subseteq\mathbb{R}$ such that

$$F=\bigcup_{i=1}^{N} f_i(F).$$

We say that F is a self-similar set generated by the *iterated function system* (IFS) $\mathcal{F}$ or F is the attractor of $\mathcal{F}$. $\mathcal{F}$ is said to satisfy the open set condition (OSC) if there exists a nonempty bounded open set O such that $\bigcup_{i=1}^N f_i(O)\subseteq O$ with a

Mathematics Subject Classification: 28A80, 28A78.
Key words and phrases: self-similar set, iterated function system, complete overlap, minimal presentation.
The first author is supported by the National Natural Science Foundation of China #11101148 and the Fundamental Research Funds for the Central Universities in China. The second author is supported by the National Natural Science Foundation of China #11271137 and Shanghai Education Committee Project 11ZZ41.

disjoint union on the left side. $\mathcal{F}$ is said to satisfy the strong separation condition (SSC) if $f_i(F) \cap f_j(F) = \emptyset$ for all $i \neq j$.

Let $\Phi = \{\phi_i\}_{i=1}^M$ and $\Psi = \{\psi_j\}_{j=1}^N$ be two IFSs for F, we say that Ψ is an iteration of Φ if $\forall 1 \leq j \leq M, \psi_j = \phi_{i_1} \circ \cdots \circ \phi_{i_k}$ for some $1 \leq i_1, \cdots, i_k \leq N$. A generating IFS for F is call a *minimal presentation* if any IFS for F is an iteration of this one.

Finding a minimal presentation is important in applications, e.g. in the theory of tiling and image compression (see [1], [2], [7]). However, few studies have been reported on this topic. Recently, Feng and Wang [5] studied the existence of a minimal presentation for some self-similar sets in $\mathbb{R}$. Deng *et al.* [3], [8], [11] discussed the form of the IFSs for self-similar sets produced by intersecting the linear homogeneous Cantor sets with their translations. Yao [10] explored the form of orthogonal matrices in generating IFSs for certain planar self-similar sets. Elekes, Keleti and Mathe [4] gave an example of a self-similar set which satisfies the SSC but has no minimal presentation.

All the above mentioned results were obtained under the requirement of some separation properties, such as OSC and SSC. We will focus on a class of self-similar sets for which the separation properties are violated.

We consider the IFS $\{f_1, f_2, f_3\}$ where

$$f_1(x) = \rho x, \ f_2(x) = \rho x + \rho, \ f_3(x) = \rho x + 1 \quad \text{with } 0 < \rho < \frac{3-\sqrt{5}}{2}.$$

Let $E \subseteq \mathbb{R}$ be the self-similar set generated by this IFS, i.e.

$$E = f_1(E) \cup f_2(E) \cup f_3(E). \tag{1}$$

Note that $f_1 \circ f_3 = f_2 \circ f_1$, the IFS $\{f_1, f_2, f_3\}$ is of complete overlap. As one knows, E can be expressed as

$$E = \left\{ \sum_{k=1}^{\infty} x_k \rho^k : x_k \in \{0, 1, 1/\rho\} \right\}.$$

This gives a way to encode elements in E by digits from $\{0, 1, 1/\rho\}$. Let $\mathbb{N}$ be the set of positive integers. We call $(x_k)_{k=1}^\infty \in \{0, 1, 1/\rho\}^{\mathbb{N}}$ a $(0, 1, 1/\rho)$-code of x if $x = \sum_{k=1}^\infty x_k \rho^k$. Because of complete overlap, typical elements in E have multiple $(0, 1, 1/\rho)$-codes. We use $E_{\mathcal{U}}$ to denote the elements of E whose $(0, 1, 1/\rho)$-code is unique.

The set E has some curious properties previously explored by others. Firstly, for each $\rho \in (0, (3-\sqrt{5})/2)$, the IFS $\{f_1, f_2, f_3\}$ is of finite type so that $\dim_H E$

can be determined. In fact, we have $\dim_H E = \dim_B E = \frac{\log(3+\sqrt{5})-\log 2}{-\log\rho} := s$ and $0 < \mathcal{H}^s(E) < \infty$ (see Example 5.4 in [9]). Secondly, we have $\dim_H E_{\mathcal{U}} = \dim_B E_{\mathcal{U}} = \frac{\log q_c}{-\log\rho} =: \gamma$ and $0 < \mathcal{H}^\gamma(E_{\mathcal{U}}) < \infty$ where $q_c = 2.32472\ldots$ is the positive solution of the equation $x^3 - 3x^2 + 2x - 1 = 0$ (see Theorem 1.1(c) in [12]).

The following theorem tells us that the IFS $\{f_1, f_2, f_3\}$ is a minimal presentation of E.

Theorem 1.1. *Let E be defined by* (1). *Let $g(x) = \lambda x + b$ with $0 < |\lambda| < 1$ and $b \in \mathbb{R}$. If $g(E) \subseteq E$, then there exist $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, 2, 3\}$ such that $g(x) = f_{i_1} \circ \cdots \circ f_{i_n}(x)$.*

One more interesting phenomenon is that the set E is a subset of a self-similar set satisfying the SSC when $0 < \rho < \frac{1}{3}$. We consider the self-similar set F generated by the IFS $\{h_1, h_2, h_3\}$ where

$$h_1(x) = \rho x,\ h_2(x) = \rho x + 1 \quad \text{and } h_3(x) = \rho x + 2.$$

Then

$$F = h_1(F) \cup h_2(F) \cup h_3(F) \quad \text{and} \quad F = \left\{ \sum_{k=1}^{\infty} x_k \rho^k : x_k \in \{0, 1/\rho, 2/\rho\} \right\}. \tag{2}$$

Let $(x_k)_{k=1}^\infty \in \{0, 1, 1/\rho\}^{\mathbb{N}}$ be a code of $x \in E$. For each $x_k = 1$, by rewriting $x_k\rho^k = \rho^k = \frac{1}{\rho}\rho^{k+1}$, we get a $(y_k)_{k=1}^\infty \in \{0, 1/\rho, 2/\rho\}^{\mathbb{N}}$ such that $\sum_{k=1}^\infty x_k\rho^k = \sum_{k=1}^\infty y_k\rho^k$. This induces $E \subseteq F$.

When $0 < \rho < \frac{1}{3}$, the IFS $\{h_1, h_2, h_3\}$ does satisfy the SSC. So this provides us an example that a (nontrivial) self-similar set satisfying the SSC can contain a self-similar set with complete overlap. Hence we can describe E by means of $(0, 1/\rho, 2/\rho)$-codes instead of $(0, 1, 1/\rho)$-codes. More precisely, there exists a unique $\mathcal{E} \subseteq \{0, 1/\rho, 2/\rho\}^{\mathbb{N}}$ such that

$$E = \left\{ \sum_{k=1}^{\infty} y_k \rho^k : (y_k)_{k=1}^\infty \in \mathcal{E} \right\}.$$

Let q be a symbol. For all $n \in \mathbb{N}$ we denote q^{*n} the word by repeating q for n times. Similarly, let $q^{*\infty}$ and q^{*0} be the infinite word by repeating q for infinite times and the empty word, respectively. The following theorem gives a description of $\mathcal{E}$.

Theorem 1.2. *Let* $0 < \rho < \frac{1}{3}$. *Let*

$$\mathcal{E} = \left\{ (y_k)_{k=1}^{\infty} \in \{0, 1/\rho, 2/\rho\}^{\mathbb{N}} : \sum_{k=1}^{\infty} y_k \rho^k \in E \right\}.$$

Then for a $(y_k)_{k=1}^{\infty} \in \{0, 1/\rho, 2/\rho\}^{\mathbb{N}}$, $(y_k)_{k=1}^{\infty} \in \mathcal{E}$ *if and only if for any* $m, n, p \in \mathbb{N}$ *we have*

$$y_1 \cdots y_m \neq (1/\rho)^{*(m-1)}(2/\rho)^{*1} \quad \text{and} \quad y_n \cdots y_{n+p} \neq (2/\rho)^{*1}(1/\rho)^{*(p-1)}(2/\rho)^{*1}.$$

The rest of this paper is arranged as follows. Theorem 1.1 is proved in Section 2. The third Section is devoted to the proof of Theorem 1.2.

2. Proof of Theorem 1.1

The following lemma tells that $f_1(E)$ and $f_2(E)$ share the same points in the interval $[\rho, \rho/(1-\rho)]$.

Lemma 2.1. *Let* E *be defined by* (1). *Then* $f_1(E) \cap [\rho, \rho/(1-\rho)] = f_2(E) \cap [\rho, \rho/(1-\rho)] = f_1(E) \cap f_2(E)$.

Proof. We only need to prove that (A) $f_1(E) \cap f_2([0, 1/(1-\rho)]) \subseteq f_2(E)$ and (B) $f_2(E) \cap f_1([0, 1/(1-\rho)]) \subseteq f_1(E)$.

Part (A) can be checked directly. In fact, we have

$$\begin{aligned} f_1(E) \cap f_2([0, 1/(1-\rho)]) &= (f_{11}(E) \cup f_{12}(E) \cup f_{13}(E)) \cap f_2([0, 1/(1-\rho)]) \\ &= f_{13}(E) \cap f_2([0, 1/(1-\rho)]) = f_{21}(E) \cap f_2([0, 1/(1-\rho)]) \subseteq f_2(E). \end{aligned}$$

Now we move to part (B). Note that for $n \in \mathbb{N}$

$$f_{2^{*n}1^{*1}}(E) = f_{1^{*1}3^{*n}}(E) \subseteq f_1(E) \quad \text{and} \quad f_{2^{*n}3^{*1}}(E) \cap f_1([0, 1/(1-\rho)]) = \emptyset,$$

where the second statement is deduced from the fact that $f_{2^{*n}3^{*1}}(0) = \frac{\rho+\rho^n-2\rho^{n+1}}{1-\rho} > f_1\left(\frac{1}{1-\rho}\right) = \frac{\rho}{1-\rho}$.

Thus for any $n \in \mathbb{N}$

$$\begin{aligned} f_2(E) \cap f_1([0, 1/(1-\rho)]) &= (f_{21}(E) \cup f_{22}(E) \cup f_{23}(E)) \cap f_1([0, 1/(1-\rho)]) \\ &\subseteq f_1(E) \cup (f_{22}(E) \cap f_1([0, 1/(1-\rho)])) \\ &= f_1(E) \cup ((f_{221}(E) \cup f_{222}(E) \cup f_{223}(E)) \cap f_1([0, 1/(1-\rho)])) \end{aligned}$$

$$= f_1(E) \cup (f_{222}(E) \cap f_1([0, 1/(1-\rho)]))$$
$$= \cdots = f_1(E) \cup (f_{2*n}(E) \cap f_1([0, 1/(1-\rho)])).$$

Since $f_{2*n}(E) \cap f_1([0, 1/(1-\rho)]) \subseteq \left[\frac{\rho(1-\rho^n)}{1-\rho}, \frac{\rho}{1-\rho}\right]$ and $\frac{\rho}{1-\rho} \in f_1(E)$, we get the desired result. □

For a nonempty compact set D of $\mathbb{R}$ we use $D_{\min}$ and $D_{\max}$ to denote its smallest and largest number, respectively. For $0 < |\lambda| < 1$ and $b \in \mathbb{R}$, the following fact about E is important and will be frequently used:

$$\text{if } \lambda E + b \subseteq E \text{ then, } [(\lambda E + b)_{\min}, (\lambda E + b)_{\max}] \supseteq [\rho + \rho/(1-\rho), 1] \text{ will never happen.} \quad (3)$$

Otherwise, this forces $E = [0, 1/(1-\rho)]$, a contradiction!

Remark 2.2. Let $\lambda E + b \subseteq E$ with $(\lambda E + b)_{\max} < 1$. From Lemma 2.1 and the above fact we can draw that if $(\lambda E + b)_{\max} \le \frac{\rho}{1-\rho}$, then $\lambda E + b \subseteq \rho E$; if $(\lambda E + b)_{\min} \ge \rho$, then $\lambda E + b \subseteq \rho E + \rho$.

Lemma 2.3. *Let E be defined by* (1). *If $\lambda E + b \subseteq E$ with $0 < |\lambda| < 1$, then $0 < |\lambda| \le \rho$.*

Proof. Firstly we have $b \ge 0$ since $b \in E$. Below we argue by contradiction and divide the proof into two cases.

Case 1. $\rho < |\lambda| < 1$ and $b = 0$.

Since $\lambda = \lambda \cdot 1 \in E$ we have $\lambda > 0$. We claim that $\lambda \le \rho(2-\rho)$.

Otherwise, if $\lambda > \rho(2-\rho)$, then $\frac{\lambda}{1-\rho} > \rho + \frac{\rho}{1-\rho}$. As $\frac{\lambda}{1-\rho} \in E$, we have $\frac{\lambda}{1-\rho} \in \left[1, \frac{1}{1-\rho}\right)$. According to (3), $\lambda E \subseteq E$ could not happen.

Let $k \ge 2$ be the positive integer such that $\lambda^k \le \rho < \lambda^{k-1}$.

Subcase 1-1. If $k \ge 3$ or $\rho = \lambda^2$, then $\sqrt{\rho} = \min_{k \ge 3}\{\rho^{\frac{1}{k-1}}, \sqrt{\rho}\} \le \lambda \le \rho(2-\rho)$, which is equivalent to $t^3 - 2t + 1 \le 0$ by letting $t = \sqrt{\rho}$. Define $f(t) = t^3 - 2t + 1$, then $f'(t) = 3t^2 - 2$ and so $f(t)$ is strictly deceasing for $0 < t < \frac{\sqrt{5}-1}{2} = \sqrt{\frac{3-\sqrt{5}}{2}} < \sqrt{\frac{2}{3}}$. Hence for $t \in \left(0, \frac{\sqrt{5}-1}{2}\right)$ we have $f(t) > f\left(\frac{\sqrt{5}-1}{2}\right) = 0$, leading to a contradiction.

Subcase 1-2. If $\lambda^2 < \rho < \lambda$, then $\lambda^2 E \subseteq \rho E \cup (\rho E + \rho)$ as $(\lambda^2 E)_{\max} < \frac{\rho}{1-\rho}$. It follows from Remark 2.2 that $\lambda^2 E \subseteq \rho E$, which is equivalent to $\frac{\lambda^2}{\rho} E \subseteq E$.

Let $\lambda_1 = \frac{\lambda^2}{\rho}$, then $\rho < \lambda_1 < 1$ and $\lambda_1 E \subseteq E$. Running the same argument as above yields

$$0 < \lambda_1 = \frac{\lambda^2}{\rho} \le \rho(2-\rho), \quad \frac{\lambda_1^2}{\rho} E \subseteq E \quad \text{and} \quad \rho < \frac{\lambda_1^2}{\rho} < 1.$$

One can continue this process again and again. After n times we can derive

$$0<\lambda_n=\frac{\lambda^{2^n}}{\rho^{2^n-1}}\leq\rho(2-\rho),\quad \frac{\lambda_n^2}{\rho}E\subseteq E\quad\text{and}\quad \rho<\frac{\lambda_n^2}{\rho}<1.$$

Since $\lambda>\rho$, the first statement could not happen if n is large enough.

Case 2. $\rho<|\lambda|<1$ and $b>0$.

For this case we claim that

$$\lambda E+b\subseteq f_1(E)\cup f_2(E)=\rho E\cup(\rho E+\rho),$$

This follows by (3) and the fact that $\lambda E+b\subseteq\rho E+1$ implies $0<|\lambda|\leq\rho$.

Subcase 2-1. $\lambda>0$.

Note that $\lambda(\rho E)+b\subseteq\lambda E+b\subseteq\rho E\cup(\rho E+\rho)$ and $(\lambda(\rho E)+b)_{\max}=\frac{\lambda\rho}{1-\rho}+b=\left(\frac{\lambda}{1-\rho}+b\right)+\frac{\lambda\rho}{1-\rho}-\frac{\lambda}{1-\rho}\leq\rho+\frac{\rho}{1-\rho}-\lambda<\frac{\rho}{1-\rho}$. Then $\lambda(\rho E)+b\subseteq\rho E$ by Remark 2.2, which is equivalent to $\lambda E+\frac{b}{\rho}\subseteq E$.

Let $b_1=\frac{b}{\rho}$, then $\lambda E+b_1\subseteq E$ and $b_1>0$. Replacing b by b_1 in the above process yields $\lambda E+b_2\subseteq E$ with $b_2=\frac{b}{\rho^2}$. Adopting the above procedure for n times gives rise to $\lambda E+\frac{b}{\rho^n}\subseteq E$. Recall that $\rho<1$ and $b>0$, this is impossible if n is large enough.

Subcase 2-2. $\lambda<0$.

As before, we have $\lambda(\rho E)+b\subseteq\lambda E+b\subseteq\rho E\cup(\rho E+\rho)$. Besides,

$$(\lambda(\rho E)+b)_{\min}=\frac{\lambda\rho}{1-\rho}+b\geq\frac{\lambda\rho}{1-\rho}-\frac{\lambda}{1-\rho}=-\lambda>\rho\ \left(\text{note that}\frac{\lambda}{1-\rho}+b\in E\right).$$

Thus $\lambda(\rho E)+b\subseteq\rho E+\rho$ by Remark 2.2. In other words,

$$\lambda E+b_1\subseteq E\quad\text{with } b_1=\frac{b-\rho}{\rho}>0.$$

Otherwise, $b_1=0$ induces $\lambda>0$. We continue this process inductively to obtain

$$\lambda E+b_n\subseteq E\quad\text{with}\quad b_n=\left(b-\frac{\rho}{1-\rho}\right)\frac{1}{\rho^n}+\frac{\rho}{1-\rho}.$$

Note that $b+\frac{\lambda}{1-\rho}\geq0$, so $b\geq\frac{-\lambda}{1-\rho}>\frac{\rho}{1-\rho}$, giving $\lim_{n\to\infty}b_n=+\infty$. This is impossible. □

Lemma 2.4. *Let E be defined by* (1)*. If $\lambda E+b\subseteq E$, then $\lambda=\rho^n$ for some $n\in\mathbb{N}$.*

PROOF. By Lemma 2.3 we have $0 < |\lambda| \le \rho$. Suppose first that $\lambda > 0$ and $\lambda \neq \rho^n$ for all $n \in \mathbb{N}$, then there exists $k \in \mathbb{N}$ such that $\rho^{k+1} < \lambda < \rho^k$. We will show a contradiction by the following two cases:

Case 1. $b \in \rho E + 1$.

Then $\lambda E + b \subseteq \rho E + 1$ and so

$$\frac{\lambda}{\rho} E + \frac{b-1}{\rho} \subseteq E. \tag{4}$$

Case 2. $b \in \rho E \cup (\rho E + \rho)$.

Then $\frac{\lambda}{1-\rho} + b \le \rho + \frac{\rho}{1-\rho}$ by (3).

(I) If $\frac{\lambda}{1-\rho} + b \le \frac{\rho}{1-\rho}$, then $\lambda E + b \subseteq \rho E$ follows by Remark 2.2, i.e.

$$\frac{\lambda}{\rho} E + \frac{b}{\rho} \subseteq E. \tag{5}$$

(II) If $\frac{\rho}{1-\rho} < \frac{\lambda}{1-\rho} + b \le \rho + \frac{\rho}{1-\rho}$, then $\lambda + b > \rho$ and $\lambda(\rho E + 1) + b \subseteq \lambda E + b \subseteq \rho E \cup (\rho E + \rho)$, so $\lambda \rho E + \lambda + b \subseteq \rho E + \rho$ according to Remark 2.2. Thus

$$\lambda E + \frac{\lambda + b - \rho}{\rho} \subseteq E \quad \text{and} \quad \frac{\lambda + b - \rho}{\rho} > b.$$

Define $f(x) = \frac{\lambda + x - \rho}{\rho}$, note that

$$f^n(b) = \frac{b}{\rho^n} + \frac{(\rho - \lambda)(1 - \frac{1}{\rho^n})}{1-\rho} = \left(b - \frac{\rho - \lambda}{1-\rho}\right)\frac{1}{\rho^n} + \frac{\rho - \lambda}{1-\rho} \nearrow +\infty,$$

there must exist a positive integer N such that $f^N(b) \in \rho E + 1$. This means that by repeating the above process for N times one can get $\lambda E + f^N(b) \subseteq E$ with $f^N(b) \in \rho E + 1$. Then the case is reduced to Case 1.

It follows from (4) and (5) that one can always get $\frac{\lambda}{\rho} E + b_1 \subseteq E$ for some $b_1 \in \mathbb{R}$. By carrying out the above procedure for k times one can establish

$$\frac{\lambda}{\rho^k} E + b_k \subseteq E \quad \text{for some } b_k \in \mathbb{R}.$$

In the case when $\lambda < 0$, we can assume $-\rho^k \le \lambda < -\rho^{k+1}$ for some $k \in \mathbb{N}$. Applying nearly the same process as above also yields $\frac{\lambda}{\rho^k} E + b_k \subseteq E$ for some $b_k \in \mathbb{R}$.

In a word, we have $\rho < \left|\frac{\lambda}{\rho^k}\right| < 1$ or $\lambda = -\rho^k$ for some $k \in \mathbb{N}$. The first case contradicts Lemma 2.3, while the latter one is contrary to the fact that E is not symmetric. The lemma is thus proved. $\square$

Proof of Theorem 1.1. By Lemma 2.4, we have $\lambda = \rho^n$ for some positive integer n. The proof of Therem 1.1 is by induction on n.

When $\lambda = \rho^1$, we claim that $b \in \{f_1(0), f_2(0), f_3(0)\}$.

(I) If $b \in \rho E + 1$, then $\rho E + b \subseteq \rho E + 1$, which leads to $b = 1 = f_3(0)$.

(II) If $b \in \rho E \cup \rho E + \rho$, then $b \le \rho + \frac{\rho}{1-\rho}$.

(II-1) When $\rho \le b \le \rho + \frac{\rho}{1-\rho}$, we have $\rho E + b \subseteq \rho E + \rho$, hence $b = \rho = f_2(0)$.

(II-2) When $0 \le b < \rho$, we will prove that $b = 0 = f_1(0)$. Otherwise, note that $\rho^2 E + b = \rho(\rho E) + b \subseteq E$ and $(\rho^2 E + b)_{\max} = \frac{\rho^2}{1-\rho} + b < \frac{\rho}{1-\rho}$, we have $\rho^2 E + b \subseteq \rho E$ by Remark 2.2, i.e. $\rho E + \frac{b}{\rho} \subseteq E$. Then $\frac{b}{\rho} \le \rho + \frac{\rho}{1-\rho}$ follows from $\frac{b}{\rho} < 1$.

If $\rho \le \frac{b}{\rho} \le \rho + \frac{\rho}{1-\rho}$, then we have $b = \rho^2$ by (II-1).

If $0 < \frac{b}{\rho} < \rho$, repeating the above process yields $\rho E + \frac{b}{\rho^2} \subseteq E$ with $\frac{b}{\rho^2} < 1$. By induction, one can find a positive integer $n \ge 2$ such that $\rho E + \frac{b}{\rho^n} \subseteq E$ and $\rho \le \frac{b}{\rho^n} \le \rho + \frac{\rho}{1-\rho}$. So we have $b = \rho^{n+1}$ by (II-1). Consequently,

$$\rho E + \rho^m \subseteq \rho E \cup (\rho E + \rho) \quad \text{for some } m \ge 2.$$

Note that $\rho(\rho E+\rho)+\rho^m \subseteq \rho E \cup (\rho E+\rho)$ and $(\rho(\rho E+\rho)+\rho^m)_{\max} = \frac{\rho^2}{1-\rho}+\rho^2+\rho^m \le \frac{\rho^2}{1-\rho} + 2\rho^2 \le \frac{\rho}{1-\rho}$. This gives that $\rho(\rho E + \rho) + \rho^m \subseteq \rho E$ by Remark 2.2. Hence $\rho E + \rho + \rho^{m-1} \subseteq E$. However, since

$$(\rho E + \rho + \rho^{m-1})_{\min} = \rho + \rho^{m-1} \le 2\rho < 1$$

and

$$(\rho E + \rho + \rho^{m-1})_{\max} = \frac{\rho}{1-\rho} + \rho + \rho^{m-1} > \frac{\rho}{1-\rho} + \rho,$$

we have $E = [0, 1/(1-\rho)]$, this could not happen.

Suppose that $b \in \{f_{\mathbf{i}}(0) : \mathbf{i} \in \{1,2,3\}^n\}$ if $\lambda = \rho^n$. We will prove that $b \in \{f_{\mathbf{i}}(0) : \mathbf{i} \in \{1,2,3\}^{n+1}\}$ if $\lambda = \rho^{n+1}$.

(I) If $b \in \rho E + 1$, then $\rho^{n+1} E + b \subseteq \rho E + 1$ and so $\rho^n E + \frac{b-1}{\rho} \subseteq E$. Thus we have $\frac{b-1}{\rho} = f_{\mathbf{i}}(0)$ for some $\mathbf{i} \in \{1,2,3\}^n$, which gives rise to $b = f_3(f_{\mathbf{i}}(0))$.

(II) If $b \in \rho E \cup \rho E + \rho$, then $\left(\rho^{n+1} E + b\right)_{\max} = \frac{\rho^{n+1}}{1-\rho} + b \le \rho + \frac{\rho}{1-\rho}$. Otherwise, we have $E = [0, \frac{1}{1-\rho}]$.

(II-1) If $\frac{\rho^{n+1}}{1-\rho} + b \le \frac{\rho}{1-\rho}$, then $\rho^{n+1}E + b \subseteq \rho E$ by Remark 2.2. Thus $\rho^n E + \frac{b}{\rho} \subseteq E$ and so $\frac{b}{\rho} = f_{\mathbf{i}}(0)$ for some $\mathbf{i} \in \{1,2,3\}^n$, which yields $b = f_1(f_{\mathbf{i}}(0))$.

(II-2) If $\frac{\rho^{n+1}}{1-\rho}+b > \frac{\rho}{1-\rho}$, then $(\rho^{n+1}E+b)_{\min} = b > \frac{\rho-\rho^{n+1}}{1-\rho} \geq \frac{\rho-\rho^2}{1-\rho} = \rho$. Hence $\rho^{n+1}E + b \subseteq \rho E + \rho$ by Remark 2.2. This gives that $\rho^n E + \frac{b-\rho}{\rho} \subseteq E$ and so $\frac{b-\rho}{\rho} = f_{\mathbf{i}}(0)$ for some $\mathbf{i} \in \{1,2,3\}^n$, which leads to $b = f_2(f_{\mathbf{i}}(0))$. □

3. Proof of Theorem 1.2

We adopt the convention that $x_0 \in \{0, 1/\rho\}$. Let $(x_k)_{k=1}^{\infty} \in \{0,1,1/\rho\}^{\mathbb{N}}$ be a $\{0,1,1/\rho\}$-code of $x \in E$. We call $(x_\ell,\dots,x_{\ell+k})$ a 1-block of length $k+1$ with $k+1 \in \mathbb{N}$ if $(x_\ell,\dots,x_{\ell+k}) = 1^{*(k+1)}$ and $x_{\ell-1}, x_{\ell+k+1} \in \{0,1/\rho\}$. We call $(x_\ell, x_{\ell+1},\dots)$ a 1-block of length ∞ if $(x_\ell, x_{\ell+1},\dots) = 1^{*\infty}$ and $x_{\ell-1} \in \{0,1/\rho\}$. Recall that $E \subseteq F$.

Lemma 3.1. *Let $0 < \rho < 1/3$. Let E and F be defined by (1) and (2), respectively. Let $(x_k)_{k=1}^{\infty} \in \{0,1,1/\rho\}^{\mathbb{N}}$ be a $\{0,1,1/\rho\}$-code of $x \in E$. Then x, as a point of F, has a unique $\{0,1/\rho,2/\rho\}$-code $(y_k)_{k=1}^{\infty} \in \{0,1/\rho,2/\rho\}^{\mathbb{N}}$ defined as follows.*

For each finite 1-block $(x_\ell,\dots,x_{\ell+k})$ with $k \in \mathbb{N}\cup\{0\}$,

$$(y_\ell,\dots,y_{\ell+k},y_{\ell+k+1}) = \begin{cases} (0(1/\rho)^{*(k+1)}) & \text{if } x_{\ell+k+1} = 0 \\ (0(1/\rho)^{*k}(2/\rho)) & \text{if } x_{\ell+k+1} = 1/\rho, \end{cases}$$

where $(1/\rho)^0$ denotes the empty word. For each infinite 1-block $(x_\ell, x_{\ell+1},\dots)$,

$$(y_\ell, y_{\ell+1},\dots) = 0(1/\rho)^{*\infty},$$

and $y_k = x_k$ for all remaining k.

Proof. Note that if $(x_\ell,\dots,x_{\ell+k})$ is a finite 1-block of x, then

$$\sum_{n=\ell}^{\ell+k+1} x_n\rho^n = \sum_{n=\ell}^{\ell+k} \frac{1}{\rho}\rho^{n+1} + x_{\ell+k+1}\rho^{\ell+k+1} = \sum_{n=\ell}^{\ell+k+1} y_n\rho^n$$

and if $(x_\ell, x_{\ell+1},\dots)$ is an infinite 1-block of x, then

$$\sum_{n=\ell}^{\infty} x_n\rho^n = \sum_{n=\ell}^{\infty} \frac{1}{\rho}\rho^{n+1} = \sum_{n=\ell}^{\infty} y_n\rho^n.$$

This finishes the proof. □

Proof of Theorem 1.2. The necessity is an immediate consequence of Lemma 3.1.

Blow we prove the sufficiency. Obviously $(y_k)_{k=1}^{\infty} \in \mathcal{E}$ if all $y_k \in \{0, 1/\rho\}$, so in the following we assume that $\{k : y_k = 2/\rho\} \neq \emptyset$.

Let $n_1 = \min\{k : y_k = 2/\rho\}$ and let $\ell_1 = \max\{1 \leq i \leq n_1 : y_i = 0\}$. Then $y_1 y_2 \cdots y_{n_1}$ can be written as

$$y_1 y_2 \cdots y_{n_1} = y_1 \cdots y_{\ell_1 - 1} 0 (1/\rho)^{*(n_1 - \ell_1 - 1)} (2/\rho)$$

with $(y_1, \cdots, y_{\ell_1 - 1}) \in \{0, 1/\rho\}^{\ell_1 - 1}$.

By letting $x_1 \dots x_{\ell_1 - 1} = y_1 \dots y_{\ell_1 - 1}$ and $x_{\ell_1} \dots x_{n_1} = 1^{*(n_1 - \ell_1)} (1/\rho)$ we have

$$(x_1, \dots, x_{n_1}) \in \{0, 1, 1/\rho\}^{n_1} \quad \text{and} \quad \sum_{k=1}^{n_1} x_k \rho^k = \sum_{k=1}^{n_1} y_k \rho^k.$$

Let $n_2 = \min\{k : k > n_1 \text{ and } y_k = 2/\rho\}$. Note that $y_{n_1} = y_{n_2} = 2/\rho$. As before, there exists a positive integer $\ell_2 = \max\{n_1 \leq i \leq n_2 : y_i = 0\}$ such that $y_{n_1+1} \cdots y_{n_2}$ can be written as

$$y_{n_1+1} \cdots y_{n_2} = y_{n_1+1} \cdots y_{\ell_2 - 1} 0 (1/\rho)^{*(n_2 - \ell_2 - 1)} (2/\rho)$$

with $(y_{n_1+1}, \cdots, y_{\ell_2 - 1}) \in \{0, 1/\rho\}^{\ell_2 - n_1 - 1}$.

By letting $x_{n_1+1} \dots x_{\ell_2 - 1} = y_{n_1+1} \cdots y_{\ell_2 - 1}$ and $x_{\ell_2} \dots x_{n_2} = 1^{*(n_2 - \ell_2)} (1/\rho)$ we have

$$(x_{n_1+1}, \dots, x_{n_2}) \in \{0, 1, 1/\rho\}^{n_2 - n_1} \quad \text{and} \quad \sum_{k=n_1+1}^{n_2} x_k \rho^k = \sum_{k=n_1+1}^{n_2} y_k \rho^k.$$

Thus one can finish the proof by repeating the above process. □

When $\rho = \frac{1}{3}$, we have $F = [0, 3]$ and its generating IFS $\{h_1, h_2, h_3\}$ satisfies the OSC. Then each point of F can be encoded by digits from $\{0, 3, 6\}$. If the codes ended by $6^{*\infty}$ are prohibited, then each point of F has a unique $\{0, 3, 6\}$-code and Theorem 1.2 is also true for $\rho = 1/3$. However, Theorem 1.2 is not true when $\frac{1}{3} < \rho < \frac{3-\sqrt{5}}{2}$. For example, let $\rho \in \left(0, \frac{3-\sqrt{5}}{2}\right)$ be the unique root of $2x^3 + x^2 + 2x - 1 = 0$. Then $\rho = 2\rho^2 + \rho^3 + 2\rho^4 = \frac{2}{\rho}\rho^3 + \frac{1}{\rho}\rho^4 + \frac{2}{\rho}\rho^5$, i.e. $\rho \in E$ has a $\{0, 1/\rho, 2/\rho\}$-code $0^{*2}(2/\rho)(1/\rho)(2/\rho)0^{*\infty}$.

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(Received February 25, 2014)