Publ. Math. Debrecen
87/1-2 (2015), 35–46
DOI: 10.5486/PMD.2015.7003

The amalgamatic curvature and the orthocurvatures of three dimensional hypersurfaces in the Euclidean space

By BOGDAN D. SUCEAVĂ (Fullerton)

To the memory of Dr. Stere Ianuş (1939–2010)

Abstract. The amalgamatic curvature $A(p)$ is a natural geometric quantity whose construction parallels that of classical scalar curvature. Its role in a ladder of curvatures corresponds to the role of harmonic mean in the classical ladder of power means, i.e. to the mean of power -1. In the present work we determine lower and upper bounds for the range of the absolute mean curvature in function of the amalgamatic curvature. Then, we introduce the orthocurvatures of a three-dimensional hypersurface in Euclidean ambient space and study several inequalities for some of these new curvature invariants.

1. Introduction

A classical result in differential geometry states that any three-dimensional Einstein manifold must have constant sectional curvature. The argument in the proof uses the interplay between the properties of the Riemann–Christoffel tensor, the Ricci tensor, and the dimension of the manifold. In [1], p. 67, Brendle points out that the three dimensional case "is quite special: for example, in dimension 3, the Riemann curvature tensor is uniquely determined by the Ricci tensor." Other elementary properties of real numbers, e.g. Nesbitt's inequality, contribute to the special geometric properties that the dimension 3 objects have.

Mathematics Subject Classification: 53B25, 53B20, 53A30.
Key words and phrases: principal curvatures, scalar curvature, shape operator, amalgamatic curvature, absolutely umbilical hypersurfaces, ladder of power means, orthocurvatures.

There are many reasons why the three-dimensional geometric objects deserve a special study. This is our main motivation in pursuing the present work.

First, we present some classical notations in the differential geometry of smooth hypersurfaces. Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. Let p be a point on the hypersurface. Denote $\sigma_k(p) = \frac{\partial \sigma}{\partial x_k}$, for all k from 1 to n. Consider $\{\sigma_1(p), \sigma_2(p), \dots, \sigma_n(p), N(p)\}$, the Gauss frame of the hypersurface, where N denotes the normal vector field. We denote by $g_{ij}(p)$ the coefficients of the first fundamental form and by $h_{ij}(p)$ the coefficients of the second fundamental form. Then we have

$$g_{ij}(p) = \langle \sigma_i(p), \sigma_j(p) \rangle, \quad h_{ij}(p) = \langle N(p), \sigma_{ij}(p) \rangle.$$

The Weingarten map $L_p = -dN_p \circ d\sigma_p^{-1} : T_{\sigma(p)}\sigma \to T_{\sigma(p)}\sigma$ is linear. Denote by $(h^i_j(p))_{1 \leq i,j \leq n}$ the matrix associated to Weingarten's map, that is:

$$L_p(\sigma_i(p) = h^k_i(p)\sigma_k(p),$$

where the repeated index and upper script above indicates Einstein's summation convention. Weingerten's operator is self-adjoint, which implies that the roots of the algebraic equation

$$\det(h^i_j(p) - \lambda(p)\delta^i_j) = 0$$

are real. The eigenvalues of Weingarten's linear map are called principal curvatures of the hypersurface. They are the roots $k_1(p), k_2(p), \dots, k_n(p)$ of this algebraic equation. The mean curvature at the point p is

$$H(p) = \frac{1}{n}[k_1(p) + \cdots + k_n(p)],$$

and the Gauss–Kronecker curvature is

$$K(p) = k_1(p)k_2(p)\dots k_n(p).$$

Additionally, we consider $\bar{H}$, the absolute mean curvature, i.e.

$$\bar{H} = \frac{1}{n}\left(|k_1| + |k_2| + \cdots + |k_n|\right).$$

Recently, in [12], a new curvature invariant for hypersurfaces was introduced: the amalgamatic curvature.

Definition 1.1. Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. In the general case, the amalgamatic curvature at point p is

$$A(p) = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \frac{2|k_i||k_j|}{|k_i| + |k_j|}.$$

Observe that $A(p)$ is defined everywhere on the hypersurface. Suppose $|k_i| + |k_j|$ vanishes at some point p. The inequality $\frac{2|k_i||k_j|}{|k_i|+|k_j|} \leq |k_i| + |k_j|$ insures the existence of the limit of the function $A(p)$ at p. For a hyperplane in $\mathbb{R}^{n+1}$, we have $A(p) \equiv 0$, for all p.

We can describe the amalgamatic curvature as the arithmetic mean of the harmonic means of all the pairs of absolute values of principal curvatures. However, there is a deeper reason to consider this quantity: it is a construction similar to the construction of the classical scalar curvature. In [3] it is included a discussion of the original argument employed by Riemann when he introduced for the first time sectional curvature in 1854. Additionally, from the study in [12], a new class of hypersurfaces was obtained: the absolutely umbilical hypersurfaces.

Definition 1.2. Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. The point p on the hypersurface is called *absolutely umbilical* if all the principal curvatures satisfy $|k_1| = |k_2| = \cdots = |k_n|$. If all the points of a hypersurface are absolutely umbilical, then the hypersurface is called absolutely umbilical.

In [12] is proved the following result.

Theorem 1.1. *Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. Let $k_1, k_2, \ldots, k_n$ be the principal curvatures at p. Denote by $\bar{H}$ its absolute mean curvature, i.e.*

$$\bar{H} = \frac{1}{n} \left(|k_1| + |k_2| + \cdots + |k_n| \right).$$

Then the absolute mean curvature and the amalgamatic curvature satisfy

$$\bar{H}(p) \geq A(p),$$

with equality being satisfied at all the points where the hypersurface is absolutely umbilical.

We can view this inequality in the spirit of the ladder of power means, see e.g. [15]. Let $a_1, a_2, a_3, \ldots, a_n$ be strictly positive numbers. Denote by c_s the mean of order s, given by

$$c_s = \left(\frac{a_1^s + a_2^s + \cdots + a_n^s}{n} \right)^{1/s}.$$

The mean of order zero is associated to the geometric mean of the n positive numbers. The Ladder of Power Means theorem states that if $s < t$, then $c_s \leq c_t$. The equality $c_s = c_t$ holds if and only if the numbers $a_1, a_2, a_3, \ldots, a_n$ are all equal. Note that c_{-1} is the harmonic mean of the strictly positive numbers $a_1, a_2, a_3, \ldots, a_n$; this quantity will play an important part in the present paper.

Recently, Brzycki *et al.* [3] introduced the following geometric curvature invariants.

Definition 1.3. For every integer $r \geq 2$ the curvature of degree r at the point p of the hypersurface is

$$Q_r(p) = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \frac{|k_i|^r + |k_j|^r}{|k_i|^{r-1} + |k_j|^{r-1}}.$$

Additionally, in [3] we consider the following.

Definition 1.4. For every integer $\alpha \leq -2$ the amalgamatic curvature of degree α at the point p of the hypersurface is

$$A_\alpha(p) = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \left(\frac{|k_i|^\alpha + |k_j|^\alpha}{2} \right)^{1/\alpha}.$$

With these quantities, in [3] it is proved the following.

Theorem 1.2. *Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. Let p be a point on the hypersurface. Denote by $k_1(p), k_2(p), \ldots, k_n(p)$ the principal curvatures of the hypersurface at the point p. Let $s \geq r \geq 2$ be two integers and $-2 \geq \alpha \geq \beta$ be two other integers. The we have the following ladder of curvatures:*

$$Q_s(p) \geq Q_r(p) \geq Q_2(p) \geq \bar{H}(p) \geq A(p) \geq A_\alpha(p) \geq A_\beta(p).$$

Equality holds if and only if p is an absolutely umbilical point.

The proof of Theorem 1.2 follows in several steps. The claim $\bar{H}(p) \geq A(p)$ is proved in [12]. The assertion $Q_r(p) \geq Q_2(p) \geq \bar{H}(p)$ relies on the following.

Lemma 1.1. [2] *Prove that for $a_1, \ldots, a_n > 0$, and for any integer $r \geq 2$ we have:*

$$\sum_{1 \leq i < j \leq n} \frac{a_i^r + a_j^r}{a_i^{r-1} + a_j^{r-1}} \geq \frac{n-1}{2}(a_1 + a_2 + \cdots + a_n).$$

Equality holds true if and only if $a_1 = a_2 = \cdots = a_n$.

This present development is a natural continuation of the study of new curvature invariants, as it was inspired by the ideas considered by B.-Y. Chen in [5], [6], [8], [9]. For the whole vision of this research direction, the most comprehensive surveys are in [10], [11].

2. A study of the amalgamatic curvature for three-dimensional hypersurfaces

In this section we use the fact that the dimension of the hypersurface is three. The sectional mean curvatures defined below make a lot of geometric sense in dimension three. For further developments of the geometry of hypersurfaces of dimension three in an Euclidean ambient space see e.g. [17].

Theorem 2.1. *Let $M^3 \subset \mathbb{R}^4$ be a smooth hypersurface and k_1, k_2, k_3 be its principal curvatures in the ambient space $\mathbb{R}^4$ endowed with the canonical metric. Let $p \in M$ be an arbitrary point. Suppose that at p none of the principal curvatures vanish. Denote by A the amalgamatic curvature, by H the mean curvature, $\bar{H}$ the absolute mean curvature, K the Gauss–Kronecker curvature. Introduce the sectional absolute mean curvatures defined by*

$$\bar{H}_{ij}(p) = \frac{|k_i(p)| + |k_j(p)|}{2}.$$

Then the following inequalities hold true at every point $p \in MV$:

$$A \cdot \bar{H}_{12} \cdot \bar{H}_{23} \cdot \bar{H}_{13} \geq \bar{H} \cdot |K| \geq A \cdot |K|.$$

Equality holds for absolutely umbilical points.

Proof. The second inequality is a consequence of Theorem 1.1, proved in [12]. To prove the first inequality, we show the following:

$$A \cdot \bar{H}_{12} \cdot \bar{H}_{23} \cdot \bar{H}_{13} \geq c_{-1} \cdot \bar{H}_{12} \cdot \bar{H}_{23} \cdot \bar{H}_{13} \geq \bar{H} \cdot |K|.$$

The notation c_{-1} indicates the mean power of order -1, i.e. the harmonic mean of the three principal curvatures. Because we are using c_{-1}, we have incorporating in the hypothesis the assumption that at p none of the principal curvatures vanish. To simplify the notation, we denote $|k_1| = a$, $|k_2| = b$, $|k_3| = c$.

We prove first that $A \geq c_{-1}$. This means we need to show:

$$\frac{1}{3}\left(\frac{2}{\frac{1}{a}+\frac{1}{b}} + \frac{2}{\frac{1}{b}+\frac{1}{c}} + \frac{2}{\frac{1}{c}+\frac{1}{a}}\right) \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

By using the substitutions $\frac{1}{a} = x$, $\frac{1}{b} = y$, $\frac{1}{c} = z$, the inequality we need to prove reduces to

$$2\left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}\right) \geq \frac{9}{x+y+z}.$$

This inequality is a direct consequence of Engel's Lemma:

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\geq\frac{(1+1+1)^2}{2(x+y+z)}=\frac{9}{2(x+y+z)}.$$

Next we prove $c_{-1}\cdot\bar{H}_{12}\cdot\bar{H}_{23}\cdot\bar{H}_{13}\geq\bar{H}\cdot|K|$. This can be rewritten as

$$\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\cdot\frac{a+b}{2}\cdot\frac{b+c}{2}\cdot\frac{c+a}{2}\geq\frac{a+b+c}{3}\cdot abc.$$

This claim is equivalent to:

$$\frac{3abc}{ab+bc+ca}\cdot(a+b)(b+c)(c+a)\geq\frac{8}{3}(a+b+c)\cdot abc.$$

By simplifying abc, we get:

$$9(a+b)(b+c)(c+a)\geq 8(a+b+c)(ab+bc+ca).$$

This last inequality reduces to

$$a(b-c)^2+b(c-a)^2+c(a-b)^2\geq 0. \qquad \square$$

This theorem is the first step in understanding the process we pursue in the next section.

3. A pinching theorem for three-dimensional hypersurfaces

In Riemannian geometry, an important example are the Berger spheres, whose study originates in the study of Hopf fibration (see e.g. [18]). Inspired by the geometry of Berger spheres, we are interested to see how the amalgamatic curvature behaves under the control of the principal curvatures. Note that the warped product metric studied in [18], e.g. p. 99, yields the sectional curvatures of the Berger spheres in the interval bounded by ε^2 and $4-3\varepsilon^2$. Due to the way the amalgamatic curvature is constructed, we focus our study on the Weingarten operator of the hypersurface and the bounds are considered for absolute values of the principal curvatures.

Theorem 3.1. *Let $M^3\subset\mathbb{R}^4$ be a smooth hypersurface and k_1,k_2,k_3 be its principal curvatures in the ambient space $\mathbb{R}^4$ endowed with the canonical metric. Let $p\in M$ be an arbitrary point. Suppose that $|k_1|,|k_2|,|k_3|\in[1,1+\varepsilon]$. Denote* *by A the amalgamatic curvature, by H the mean curvature, $\bar{H}$ the absolute mean curvature. Then the following inequalities hold true at every point $p \in M$:*

$$A \leq \bar{H} \leq \frac{1}{9}\left\{5 + \frac{2}{1+\varepsilon}[1 + (1+\varepsilon)^2]\right\} A.$$

Equality holds for absolutely umbilical points with the property $|k_1| = |k_2| = |k_3| = 1$.

Proof. The first inequality is from Theorem 1.1. For the second inequality, we plan to use the comparison with the harmonic mean c_{-1} and prove the following:

$$\bar{H} \leq \left\{5 + \frac{2}{1+\varepsilon}[1 + (1+\varepsilon)^2]\right\} c_{-1} \leq \left\{5 + \frac{2}{1+\varepsilon}[1 + (1+\varepsilon)^2]\right\} A.$$

The inequality in the right is insured by $c_{-1} \leq A$, a fact established in the proof of Theorem 2.1. We use the notations $|k_1| = a$, $|k_2| = b$, $|k_3| = c$.

We plan to prove the that $\bar{H} \leq \frac{L}{9} \cdot c_{-1}$, where $L = 5 + \frac{2}{1+\varepsilon}[1 + (1+\varepsilon)^2]$. This is due to the following fact:

$$\frac{a+b+c}{3} \leq L \cdot \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}.$$

The claim left to prove reduces to the following statement: if $a, b, c \in [1, 1+\varepsilon]$, then

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \leq 5 + \frac{2}{1+\varepsilon}[1 + (1+\varepsilon)^2].$$

By multiplying the terms in the left hand side of this expression, we obtain that we need to prove the following:

$$3 + \frac{a}{b} + \frac{b}{c} + \frac{c}{b} + \frac{b}{a} + \frac{c}{a} + \frac{a}{c} \leq 5 + \frac{2}{1+\varepsilon}[1 + (1+\varepsilon)^2]. \tag{3.1}$$

Without any loss of generality, suppose that $a \geq b \geq c$. Then $(a-b)(b-c) \geq 0$. This yields $ab + bc \geq b^2 + ac$. Dividing both sides by bc and by ab, respectively, we obtain first:

$$\frac{a}{c} + 1 \geq \frac{a}{b} + \frac{b}{c},$$

then

$$1 + \frac{c}{a} \geq \frac{c}{b} + \frac{b}{a}.$$

Adding term by term we get:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{b}+\frac{b}{a}\leq 2+\frac{a}{c}+\frac{c}{a}.$$

Thus it is always true that for $a, b, c > 0$, with $a \geq b \geq c$, we have

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{b}+\frac{b}{a}+\frac{c}{a}+\frac{a}{c}\leq 2+2\left(\frac{a}{c}+\frac{c}{a}\right). \tag{3.2}$$

We estimate the right hand side term of this expression. Since $1 \leq a \leq 1+\varepsilon$, we have $\frac{1}{1+\varepsilon} \leq \frac{1}{a} \leq 1$. From such estimates, by multiplying term by term double inequalities with strictly positive terms, we obtain

$$\frac{1}{1+\varepsilon}\leq\frac{a}{c}\leq 1+\varepsilon.$$

Denote by $x = \frac{a}{c}$. Then

$$[x-(1+\varepsilon)]\cdot\left(x-\frac{1}{1+\varepsilon}\right)\leq 0,$$

that is

$$x+\frac{1}{x}\leq 1+\varepsilon+\frac{1}{1+\varepsilon}.$$

This estimate yields the best upper bound in (3.2):

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{b}+\frac{b}{a}+\frac{c}{a}+\frac{a}{c}\leq 2+2\left(1+\varepsilon+\frac{1}{1+\varepsilon}\right).$$

Adding 3 both sides yields (3.1), which completes this proof. The equality case is obtained when equality holds in all these inequalities, which yields $a=b=c=1$. □

4. Orthocurvatures of order (a, b) of a three dimensional hypersurface in Euclidean ambient space

Our motivation in pursuing the study of the concepts introduced in the remaining part of our present work resides in Sophie Germain's words, [14], p. 25, when she invites the study of the "l'idée complète de la manière dont la courbure est distribuée autour du point donné, aussi bine que la connoissance de la distance moyenne entre les points de la surface qui environnent le point de tangence et le plan tangent [...]." Furthemore, Sophie Germain writes that "La considération des courbures moyennes est nécessairement applicable à toutes les questions où la courbure des surfaces joue le rôle d'une puissance." In 1831, this approach, inspired also from profound physical considerations, led to the definition of the mean curvature.

Sophie Germain's ideas inspire the following more general question: *how is curvature distributed around a point?* What quantities best detect deformations of the hypersurface around a point? Is this study related to various symmetric expressions in three variables?

In the last two decades, several seminal works have explored the very foundation of the concept of curvature. All the evolution started with B.-Y. Chen's construction of $\delta-$invariants (nowadays called Chen invariants) in [5], [6], [8], [9]. Two very useful surveys are available in [10], [11]. Besides the strings of Chen curvature invariants, who have many applications, there are other curvature invariants that can be explored and deserve attention, as they encode the fundamental idea of curvature. Pursuing this research direction and viewing the problem in this context, we propose the curvature invariants presented below.

We have seen in Theorem 2.1 that the absolute mean sectional curvatures play a certain geometric part in understanding the geometry of a three-dimensional hypersurface.

Definition 4.1. Let $M^3 \subset \mathbb{R}^4$ be a smooth hypersurface and k_1, k_2, k_3 be its principal curvatures in the ambient space $\mathbb{R}^4$ endowed with the canonical metric. Let $p \in M$ be an arbitrary point. Suppose at p none of the principal curvatures vanish. Then we define the absolute mean sectional curvatures by

$$\bar{H}_{ij}(p) = \frac{|k_i(p)| + |k_j(p)|}{2}.$$

Additionally, we call *the orthocurvature of order* (a, b) *of* M *at* p the quantity:

$$B^a_b = \frac{|k_1|^a}{\bar{H}^b_{23}} + \frac{|k_2|^a}{\bar{H}^b_{13}} + \frac{|k_3|^a}{\bar{H}^b_{12}}.$$

This definition is inspired by Sophie Germain's idea: *how are the curvature quantities distributed around a point?*

In the present section we study the fundamental properties of some orthocurvatures. We obtain the following.

Theorem 4.1. *Let $M^3 \subset \mathbb{R}^4$ be a smooth hypersurface and k_1, k_2, k_3 be its principal curvatures in the ambient space $\mathbb{R}^4$ endowed with the canonical metric. Let $p \in M$ be an arbitrary point. Suppose that at p none of the principal* *curvatures vanish. Denote by A the amalgamatic curvature, by H the mean curvature, $\bar{H}$ the absolute mean curvature, K the Gauss–Kronecker curvature. Introduce the absolute mean sectional curvatures defined by*

$$\bar{H}_{ij}(p) = \frac{|k_i(p)| + |k_j(p)|}{2}.$$

Then the following inequalities hold true at every point $p \in M$:

$$3 \leq \frac{|k_1|}{\bar{H}_{23}} + \frac{|k_2|}{\bar{H}_{13}} + \frac{|k_3|}{\bar{H}_{12}} = B_1^1(p) \tag{4.1}$$

$$2\sqrt{2} < \sqrt{\frac{|k_1|}{\bar{H}_{23}}} + \sqrt{\frac{|k_2|}{\bar{H}_{13}}} + \sqrt{\frac{|k_3|}{\bar{H}_{12}}} = B_{0.5}^{0.5}(p) \tag{4.2}$$

$$3\bar{H} \leq \frac{|k_1|^2}{\bar{H}_{23}} + \frac{|k_2|^2}{\bar{H}_{13}} + \frac{|k_3|^2}{\bar{H}_{12}} = B_1^2(p). \tag{4.3}$$

$$\frac{3}{\bar{H}} \leq \frac{|k_1|}{\bar{H}_{23}^2} + \frac{|k_2|}{\bar{H}_{13}^2} + \frac{|k_3|}{\bar{H}_{12}^2} = B_2^1(p). \tag{4.4}$$

Proof. For a simpler notation, we denote $|k_1| = a$, $|k_2| = b$, $|k_3| = c$. Note that $a, b, c > 0$. To prove (4.1), we note that this inequality is equivalent to Nesbitt's inequality (see e.g. [4], p. 16, or [16]):

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2},$$

which can be proved by a direct argument via Cauchy–Schwarz inequality.

To prove the inequality in (4.2), we use an idea suggested in [13]:

$$\sqrt{\frac{a}{b+c}} \geq \frac{2a}{a+b+c}.$$

We first prove this inequality for any $a, b, c > 0$. By squaring both sides, we get

$$\frac{a}{b+c} \geq \frac{4a^2}{(a+b+c)^2}.$$

Since $a > 0$, we are left to show that $(a+b+c)^2 \geq 4a(b+c)$, which is true since $(b+c-a)^2 \geq 0$. Equality holds true if and only if $a = b+c$. By adding together three such inequalities we obtain

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \geq \frac{2(a+b+c)}{(a+b+c)} = 2.$$

However, equality could take place if and only if we have simultaneously $a = b+c$, $b = c + a$, $c = a + b$. That is not possible if $a, b, c > 0$. This shows that $B^{0.5}_{0.5}(p) > 2\sqrt{2}$.

We now prove (4.3). As assumed before, $|k_1| = a$, $|k_2| = b$, $|k_3| = c$, with $a, b, c > 0$. Without loss of generality we assume (by employing a classical technique, see e.g. [13], p. 333) that $a \geq b \geq c$, which implies immediately that

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}.$$

A direct application of Chebyshev's inequality (see e.g. [16]) yields:

$$\frac{2a^2}{b+c} + \frac{2b^2}{c+a} + \frac{2c^2}{a+b} \geq \frac{2}{3}(a^2+b^2+c^2)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right).$$

From the Cauchy–Schwarz inequality we get that $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$, which can be used in the right hand side term in the previous inequality to obtain:

$$\frac{2a^2}{b+c} + \frac{2b^2}{c+a} + \frac{2c^2}{a+b} \geq \frac{2}{9}(a+b+c)^2\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right).$$

To estimate the second factor in the right hand side term, we use the AM-HM inequality:

$$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \geq \frac{9}{(b+c)+(c+a)+(a+b)} = \frac{9}{2(a+b+c)}.$$

This yields (4.3).

Finally, the argument to prove (4.4) is a direct application of Cauchy–Schwarz inequality in:

$$(a+b+c)\left(\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2}\right) \geq \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)^2 \geq \frac{9}{4}.$$

The last inequality is also due to Nesbitt's inequality, used previously in the proof of this theorem. □

The author thanks to Bryan Brzycki for his very useful feedback on Theorem 3.1 and for the discussions related to [2].

References

[1] S. Brendle, Ricci Flow and the Sphere Theorem, *American Mathematical Society, Providence, RI, USA*, 2010.

[2] B. Brzycki and B. D. Suceavă, Proposed problem, (submitted).

[3] B. Brzycki, M. Giesler, K. Gomez, L. H. Odom and B. D. Suceavă, A Ladder of curvatures for hypersurfaces in Euclidean ambient space, *Houston Math. J.* **40** (2014), 1347–1356.

[4] R. Bulajich Manfrino, J. A. Gómez Ortega and R. Valdez Delgado, Inequalities, A Mathematical Olympiad Approach, *Birkhäuser*, *Boston, Basel, Berlin*, 2009.

[5] B.-Y. Chen, Some pinching and classification theorems for minimal submanifolds, *Arch. Math.* **60** (1993), 568–578.

[6] B.-Y. Chen, A Riemannian invariant and its applications to submanifold theory, *Results Math.* **27** (1995), 17–26.

[7] B.-Y. Chen, Mean curvature and shape operator of isometric immersions in real-space–forms, *Glasgow Math. J.* **38** (1996), 87–97.

[8] B.-Y. Chen, Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimension, *Glasgow Math. J.* **41** (1999), 33–41.

[9] B.-Y. Chen, Some new obstructions to minimal and Lagrangian isometric immersions, *Japanese J. Math.* **26** (2000), 105–127.

[10] B.-Y. Chen, Riemannian Submanifolds, Handbook of Differential Geometry, vol. I, (F. J. E. Dillen and L. C. A. Verstraelen, eds.), *North Holland*, *Amsterdam*, 2000, 187–418.

[11] B.-Y. Chen, Pseudo-Riemannian Geometry, δ-Invariants and Applications, *World Scientific*, *Singapore*, 2011.

[12] C. Conley, R. Etnyre, B. Gardener, L. H. Odom and B. D. Suceavă, New curvature inequalities for hypersurfaces in the Euclidean ambient space, *Taiwanese J. Math.* **17** (2013), 885–895.

[13] Z. Cvetkovski, Inequalities, Theorems, Techniques and Selected Problems, *Springer-Verlag*, *New York, Heidelberg, Berlin*, 2012.

[14] S. Germain, Mémoire sur la courbure des surfaces, *Journal für die reine und angewandte Mathematik, Herausgegeben von A. L. Crelle* **7** (1831), 1–29.

[15] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, (second edition), *Cambridge University Press (Cambridge Mathematical Library)*, *New York, USA*, 1988.

[16] J. M. Steele, The Cauchy–Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities, *The Mathematical Association of America*, *Washington, D.C., USA*, 2004.

[17] B. D. Suceavă, An estimate for the Gauss–Kronecker curvature of 3-dimensional smooth hypersurfaces in $\mathbb{E}^4$, *Inter. El. J. Geometry* **7** (2014), 1–6.

[18] P. Petersen, Riemannian Geometry, (second edition), *Springer-Verlag*, *New York, Heidelberg, Berlin*, 2006.

BOGDAN D. SUCEAVĂ
DEPARTMENT OF MATHEMATICS
CALIFORNIA STATE UNIVERSITY, FULLERTON
FULLERTON, CA 91834-6850
USA

E-mail: bsuceava@fullerton.edu

(Received March 7, 2014; revised December 8, 2014)