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Some formulas of Santaló type in Finsler geometry and its applications

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Abstract. In this paper, we establish two Santaló type formulas for general Finsler manifolds. As applications, we derive a universal lower bound for the first eigenvalue of the nonlinear Laplacian, two Croke type isoperimetric inequalities, and a Yamaguch type finiteness theorem in Finser geometry.

1. Introduction

In [16], [17], Santaló considered the kinematic measure and established a formula which describes the Liouville measure on the unit sphere bundle of a Riemannian manifold in terms of the geodesic flow and the measure of a hypersurface. This formula plays an important role in global Riemannian geometry. Some of its applications are universal bounds for the first eigenvalue [5], Croke's isoperimetric inequality [9] and a generalization of Berger's theorem [8]. Moreover, with Santaló's formula, Croke in [7] solved a famous isoperimetric problem in dimension 4. See [5], [7], [8], [10], [9], [11], [16], [17] for more details.

A Finsler manifold is a differentiable manifold, on which every tangent space is endowed a Minkowski norm instead of a Euclidean norm. There is only one reasonable notion of the measure for Riemannian manifolds (cf. [4]). However,

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the measure on a Finsler manifold can be defined in various ways and essentially different results may be obtained, e.g., [1], [2], [18]. Hence, it is interesting to ask whether an analogue of Santaló's formula still holds for Finsler manifolds.

Let (M, F) be a Finsler manifold. Denote by $\pi_1 : SM \to M$ the unit sphere bundle. If $F(y) = F(-y)$ for any $y \in SM$, then F is reversible. In a reversible Finsler manifold, the reverse of a geodesic is still a geodesic (see [3], [18]). In [23], Shen and Zhao considered the problem above and established a Santaló type formula for reversible Finsler manifolds.

There are infinitely many nonreversible Finsler metrics. For example, a Randers metric in the form $F = \alpha + \beta$ is non-reversible, where α is a Riemannian metric and β is a 1-form. The reverse of a geodesic in a non-reversible Finsler manifold is in general not a geodesic. Moreover, in a non-reversible Finsler manifold, the measure of a hypersurface induced by the inward normal vector field may be different from the one induced by the outward normal vector filed (see Example 1 in Section 5 below). The purpose of this paper is to establish some Santaló type formulas for general Finsler manifolds.

Let $(M, \partial M, F, d\mu)$ be a compact Finsler manifold with the boundary, where F is possibly non-reversible and $d\mu$ is a measure on M. Denote by $\mathbf{n}_+$ and $\mathbf{n}_-$ the unit inward and outward normal vector fields along ∂M, respectively. The measures on ∂M induced by $\mathbf{n}_\pm$ are defined by $d\,\mathrm{A}_\pm := i^*(\mathbf{n}_\pm \rfloor d\mu)$. Let $S^+\partial M$ and $S^-\partial M$ be the bundles of inwardly and outwardly pointing unit vectors along ∂M, i.e., $S^\pm\partial M = \{y \in SM|_{\partial M} : g_{\mathbf{n}_\pm}(\mathbf{n}_\pm, y) > 0\}$. The measures on $S^\pm\partial M$ are the product measures $d\chi_\pm(y) := d\nu_{\pi_1(y)}(y)d\,\mathrm{A}_\pm(\pi_1(y))$, where $d\nu_x(y)$ is the Riemannian measure on $S_xM := \pi_1^{-1}(x)$ induced by F. For each $y \in S^+\partial M$, set $\mathfrak{t}(y) := \sup\{t > 0 : \gamma_y(s) \in M,\ 0 < s < t\}$ and $l(y) := \min\{i(y), \mathfrak{t}(y)\}$, where $i(y)$ is the cut value of y.

Since F may be non-reversible, to investigate the asymmetry of the Finsler manifold, we introduce the reverse of F, which is defined by $\widetilde{F}(y) := F(-y)$. Clearly, $\widetilde{F}$ is a Finsler metric as well. Let $\tilde{\mathfrak{t}}(\cdot)$, $\tilde{i}(\cdot)$ and $\tilde{l}(\cdot)$ be defined as above in $(M, \partial M, \widetilde{F})$. Then we have the following Santaló type formulas.

Theorem 1.1. *For all integral function f on SM, we have*

$$\int_{\mathcal{V}_M^-} f dV_{SM} = \int_{y \in S^+\partial M} e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+, y) d\chi_+(y) \int_0^{l(y)} f(\varphi_t(y))dt, \tag{i}$$

$$\int_{\mathcal{V}_M^+} f dV_{SM} = \int_{y \in S^-\partial M} e^{\tau(y)} g_{\mathbf{n}_-}(\mathbf{n}_-, y) d\chi_-(y) \int_0^{\tilde{l}(-y)} f(\varphi_{-t}(y))dt, \tag{ii}$$

where dV_{SM} is the canonical Riemannian measure on SM, τ is the distortion *of $d\mu$, $\varphi_t(y)$ is the geodesic flow of F, $\mathcal{V}_M^- := \{y \in SM : \tilde{\mathfrak{t}}(-y) \leq \tilde{i}(-y)\}$ and $\mathcal{V}_M^+ := \{y \in SM : \mathfrak{t}(y) \leq i(y)\}$.*

One can easily see that Theorem 1.1 implies the Santaló type formulas for reversible Finsler manifolds [23] and for Riemannian manifolds [16], [17]. It is remarkable that, in a non-reversible Finsler manifold, $\mathrm{A}_-(\partial M) \neq \mathrm{A}_+(\partial M)$ and the formulas (1) and (2) contain information about $\mathrm{A}_+(\partial M)$ and $\mathrm{A}_-(\partial M)$, respectively.

Before giving some applications of Theorem 1.1, we shall recall some notions and basic facts of the first eigenvalue in the Finsler setting. The first eigenvalue $\lambda_1(M, d\mu)$ in $(M, F, d\mu)$ is defined as the smallest positive eigenvalue of the nonlinear Laplacian $\Delta_{d\mu}$ introduced by Shen (cf. [14], [18], [19]). It should be noted that both $\Delta_{d\mu}$ and $\lambda_1(M, d\mu)$ are dependent on the choice of the measure $d\mu$. Theorem 1.1 now yields the following

Theorem 1.2. *Let $(M, \partial M, F)$ be a compact Finsler n-manifold with the boundary such that every geodesic ray in (M, F) minimizes distance up to the point that it intersects ∂M. Then*

$$\lambda_1(M, d\mu) \geq \begin{cases} \dfrac{\lambda_D(\mathbb{S}_D^+)}{\Lambda_F^{4n+1}}, & d\mu \text{ is the Busemann–Hausdorff measure,} \\[2ex] \dfrac{\lambda_D(\mathbb{S}_D^+)}{\Lambda_F^{2n+1}}, & d\mu \text{ is the Holmes–Thompson measure,} \end{cases}$$

where $D := \operatorname{diam}(M)$, Λ_F is the uniform constant of F, and $\mathbb{S}_D^+$ denotes the n-dimensional Riemannian hemisphere of the constant sectional curvature sphere having diameter equal to D. The equality holds if and only if (M, F) is isometric to $\mathbb{S}_D^+$.

Note that a Finsler metric F is Riemannian if and only if $\Lambda_F = 1$. Hence, Theorem implies Croke's sharp universal lower bound for the first eigenvalue [5], [9].

Let $(M, \partial M, F)$ be as above. For each $x \in M$, set

$$\omega := \inf_{x \in M} \min\{\omega_x^+, \omega_x^-\},$$

where $\omega_x^\pm := c_{n-1}^{-1} \int_{U_x^\pm} e^{\tau(y)} d\nu_x(y)$, $U_x^\pm := \pi|_{\mathcal{V}_M^\pm}^{-1}(x)$ and $c_{n-1} = \operatorname{Vol}(\mathbb{S}^{n-1})$. Then Theorem 1.1 furnishes the following inequalities.

Theorem 1.3. *Let $(M, \partial M, F, d\mu)$ be a compact Finsler n-manifold with the boundary, where $d\mu$ is either the Busemann–Hausdorff measure or the Holmes–Thompson measure. Then*

$$\frac{\mathrm{A}_\pm(\partial M)}{\mu(M)} \geq \frac{(n-1)c_{n-1}\,\omega}{c_{n-2}\,D\,\Lambda_F^{2n+\frac{1}{2}}}, \tag{1}$$

where $D := \mathrm{diam}(M)$.

$$\frac{\mathrm{A}_{\pm}(\partial M)}{\mu(M)^{1-\frac{1}{n}}} \geq \frac{c_{n-1}\omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}}\Lambda_F^{2n+\frac{5}{2}}}, \tag{2}$$

with equality if and only if (M,F) *is a Riemannian hemisphere of a constant sectional curvature sphere.*

If F is reversible, then $\omega_+ = \omega_-$ and $\mathrm{A}_+(\partial M) = \mathrm{A}_-(\partial M)$. Hence, Theorem 1.3 implies Croke type isoperimetric inequalities for reversible Finsler manifolds [23, Theorem 1.6] and for Riemannian manifolds [9].

As an application of Theorem 1.3, we obtain a Finslerian version of Yamaguchi's finiteness theorem.

Theorem 1.4. *For any* n *and positive numbers* i, V, δ*, the class of closed Finsler* n*-manifolds* (M,F) *with injectivity radius* $\mathfrak{i}_M \geq i$, $\Lambda_F \leq \delta$ *and* $\mu(M) \leq V$*, contains at most finitely many homotopy types. Here,* $\mu(M)$ *is either the Busemann–Hausdorff volume or the Holmes–Thompson volume of* M*.*

2. Preliminaries

In this section, we recall some definitions and properties about Finsler manifolds. See [3], [18] for more details.

Let (M,F) be a (connected) Finsler n-manifold with Finsler metric $F : TM \to [0,\infty)$. Let $(x,y) = (x^i, y^i)$ be local coordinates on TM. Define

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}, \quad G^i(y) := \frac{1}{4}g^{il}(y)\left\{2\frac{\partial g_{jl}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^l}(y)\right\}y^j y^k,$$

where G^i are the geodesic coefficients. A smooth curve $\gamma(t)$ in M is called a (constant speed) geodesic if it satisfies

$$\frac{d^2\gamma^i}{dt^2} + 2G^i\left(\frac{d\gamma}{dt}\right) = 0.$$

We always use $\gamma_y(t)$ to denote the geodesic with $\dot{\gamma}_y(0) = y$.

The Ricci curvature is defined by $\mathbf{Ric}(y) := \sum_{i=1}^n R^i_{\ i}(y)$, where

$$R^i_{\ k}(y) := 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

Let $\pi_1 : SM \to M$ be the unit sphere bundle, i.e., $S_xM := \{y \in T_xM : F(x,y) = 1\}$ and $SM := \cup_{x\in M} S_xM$. The measure on SM is defined by

$$\begin{aligned} dV_{SM}|_{(x,y)} &= \sqrt{\det g_{ij}(x,y)}dx^1 \wedge \cdots \wedge dx^n \wedge d\nu_x(y) \\ &= e^{\tau(y)}\pi_1^*(d\mu(x)) \wedge d\nu_x(y). \end{aligned}$$

where

$$d\nu_x(y) := \sqrt{\det g_{ij}(x,y)} \left(\sum_{i=1}^n (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \hat{dy^i} \wedge \cdots \wedge dy^n \right).$$

is the Riemannian measure on S_xM induced by F.

The reversibility λ_F and the uniformity constant Λ_F of (M,F) are defined by $\lambda_F := \sup_{x\in M} \lambda_F(x)$ and $\Lambda_F := \sup_{x\in M} \Lambda_F(x)$, where

$$\lambda_F(x) := \sup_{y\in S_xM} F(x,-y),\ \Lambda_F(x) := \sup_{X,Y,Z\in S_xM} \frac{g_X(Y,Y)}{g_Z(Y,Y)}.$$

Clearly, $\Lambda_F \geq \lambda_F^2 \geq 1$. $\lambda_F = 1$ if and only if F is reversible, while $\Lambda_F = 1$ if and only if F is Riemannian.

The dual Finsler metric F^* on M is defined by

$$F^*(\eta) := \sup_{X\in T_xM\setminus 0} \frac{\eta(X)}{F(X)}, \quad \forall \eta \in T_x^*M.$$

The Legendre transformation $\mathfrak{L} : TM \to T^*M$ is defined as

$$\mathfrak{L}(X) := \begin{cases} g_X(X,\cdot) & X \neq 0, \\ 0 & X = 0. \end{cases}$$

In particular, $F^*(\mathfrak{L}(X)) = F(X)$. Now let $f : M \to \mathbb{R}$ be a smooth function on M. The gradient of f is defined by $\nabla f = \mathfrak{L}^{-1}(df)$. Thus, $df(X) = g_{\nabla f}(\nabla f, X)$.

Let $d\mu$ be a measure on M. In a local coordinate system (x^i), express $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$. In particular, the Busemann–Hausdorff measure $d\mu_{BH}$ and the Holmes–Thompson measure $d\mu_{HT}$ are defined by

$$d\mu_{BH} = \sigma_{BH}(x)dx := \frac{\mathrm{Vol}(\mathbb{B}^n)}{\mathrm{Vol}(\{y \in T_xM : F(x,y) < 1\})} dx^1 \wedge \cdots \wedge dx^n,$$

$$d\mu_{HT} = \sigma_{HT}(x)dx := \left(\frac{1}{c_{n-1}} \int_{S_xM} \sqrt{\det g_{ij}(x,y)} d\nu_x(y) \right) dx^1 \wedge \cdots \wedge dx^n.$$

For $y \in T_xM\setminus 0$, define the distorsion of $(M,F,d\mu)$ as

$$\tau(y) := \log \frac{\sqrt{\det g_{ij}(x,y)}}{\sigma(x)}.$$

By the same argument in [21], one can show the following lemma.

Lemma 2.1. *Let (M, F) be a Finsler n-manifold with finite uniform constant Λ_F. Let $d\mu$ denote either the Busemann–Hausdorff measure or the Holmes–Thompson measure on M. Then the distortion τ of $d\mu$ satisfy $\Lambda_F^{-n} \leq e^{\tau(y)} \leq \Lambda_F^n$, for all $y \in SM$.*

The reverse of a Finsler metric F is defined by $\widetilde{F}(y) := F(-y)$. It is not hard to see that $\widetilde{G}^i(y) = G^i(-y)$ and $d\tilde{\mu} = d\mu$, where $\widetilde{G}^i$ (resp. G^i) are the geodesic coefficients of $\widetilde{F}$ (resp. F), and $d\tilde{\mu}$ (resp. $d\mu$) denotes the Busemann–Hausdorff measure or the Holmes–Thompson measure of $\widetilde{F}$ (resp. F). In particular, if γ is a geodesic of F, then the reverse of γ is a geodesic of $\widetilde{F}$.

3. Santaló type formulas

Let $(M, \partial M, F)$ be compact Finsler manifold with the boundary. Denote by $\mathbf{n}_+$ (resp. $\mathbf{n}_-$) the unit inward (resp. outward) normal vector field along ∂M. Define $\mathcal{N}_+ := \{k \cdot \mathbf{n}_+(x) : x \in \partial M,\ k \in \mathbb{R}\}$. The exponential map Exp_+ of $\mathcal{N}_+$ is defined by

$$\mathrm{Exp}_+ : \mathcal{N}_+ \to M,\ k \cdot \mathbf{n}_+(x) \mapsto \exp_x(k\mathbf{n}_+(x)).$$

We always identify ∂M with the zero section of $\mathcal{N}_+$. The same arguments as in [23, Lemma 5.1, Remark 5.1] show the following lemma.

Lemma 3.1. Exp_+ *maps a neighborhood of $\partial M \subset \mathcal{N}_+$ C^1-diffeomorphically onto a neighborhood of $\partial M \subset \overline{M}$. Hence, there exists a small $\delta > 0$ such that* $\mathrm{Exp}_+ : M_\delta \to \mathrm{Exp}_+(M_\delta)$ *is C^1-diffeomorphic, where $M_\delta := \{k \cdot \mathbf{n}_+(x) : x \in \partial M,\ 0 \leq k < \delta\}$.*

Define $\rho : \overline{M} \to \mathbb{R}_+$ by $\rho(x) = d(\partial M, x)$. Lemma 3.1 together with the proofs of [23, Lemma 5.2-5.3, Corollay 5.1] and [18, Lemma 3.2.3] yields

Lemma 3.2. *Let $\sigma(t)$, $0 \leq t < \epsilon$, be a C^1-curve with $\sigma(0) \in \partial M$ and $\sigma((0, \epsilon)) \subset M$. Then*

$$0 \leq \left.\frac{d}{dt}\right|_{t=0^+} \rho \circ \sigma(t) = g_{\mathbf{n}_+}(\mathbf{n}_+, \dot{\sigma}(0)).$$

Hence, $\nabla \rho_+(x) = \mathbf{n}_+(x)$, for any $x \in \partial M$.

Set $S^{\pm}\partial M := \{y \in SM|_{\partial M} : g_{\mathbf{n}_{\pm}}(\mathbf{n}_{\pm}, y) > 0\}$. By the Legendre transformations, one can show that $S^{\pm}\partial M$ are two submanifolds of $S\overline{M}$.

Remark 1. In general, $\mathbf{n}_+ \neq -\mathbf{n}_-$. However, by the Legendre transformations, one can easily show that $S^\pm\partial M = \{y \in SM|_{\partial M} : g_{\mathbf{n}_\mp}(\mathbf{n}_\mp, y) < 0\}$.

Set $\mathcal{Z} := \{y \in S\partial M : \exists\, t > 0 \text{ such that } \gamma_y((0,t)) \subset M\}$. Define a function $\mathfrak{t} : SM \cup S^+\partial M \cup \mathcal{Z} \to \mathbb{R}_+$ by $\mathfrak{t}(y) := \sup\{t > 0 : \gamma_y(s) \in M,\ 0 < s < t\}$, which is called the $\mathfrak{t}$-function. By the same argument as in [23, Lemma 5.4], one can show that $\mathfrak{t}$-function is low semi-continuous on $SM \cup S^+\partial M$.

Since $(M, \partial M, F)$ is compact, we can define a map

$$\Psi : \{(t,y) : y \in S^+\partial M,\ 0 \leq t \leq \mathfrak{t}(y)\} \to SM, (t,y) \mapsto \varphi_t(y),$$

where φ_t is the geodesic flow of F. Let $\widetilde{\mathfrak{t}}$ (resp. $\tilde{i}$) denote the $\mathfrak{t}$-function (resp. the cut value function) defined on $(M, \partial M, \widetilde{F})$, where $\widetilde{F}(y) := F(-y)$. Set

$$U_M^- := \{y \in SM : \tilde{\mathfrak{t}}(-y) < \tilde{i}(-y)\}.$$

Since $y \in SM$ implies that $\widetilde{F}(-y) = 1$, U_M^- is well-defined. In particular, we have the following

Lemma 3.3. *$\Psi|_{\mathfrak{N}_+} : \mathfrak{N}_+ \to U_M^- \backslash U_{\mathcal{Z}}$ is a one-one map. Here, $\mathfrak{N}_+ := \{(t,y) : y \in S^+\partial M,\ t \in (0, l(y))\}$, $U_{\mathcal{Z}} := \{\varphi_t(y) : y \in \mathcal{Z},\ t \in (0, l(y))\}$, and $l(y) := \min\{i(y), \mathfrak{t}(y)\}$.*

Proof. Since $\overline{M}$ is compact, for each $y \in U_M^-$, $0 < \tilde{\mathfrak{t}}(-y) < \tilde{i}(-y) < \infty$. Thus, $\widetilde{\gamma}_{-y}(t)$, $0 \leq t \leq \tilde{\mathfrak{t}}(-y)$ is a unit speed minimal geodesic in $(\overline{M}, \widetilde{F})$. Set $Y := -\dot{\widetilde{\gamma}}_{-y}(\tilde{\mathfrak{t}}(-y))$. Thus,

$$F(Y) = \widetilde{F}(-Y) = \widetilde{F}(\dot{\widetilde{\gamma}}_{-y}(\tilde{\mathfrak{t}}(-y))) = 1.$$

It follows from Lemma 3.2 that $g_{\mathbf{n}_+}(\mathbf{n}_+, Y) \geq 0$. Hence, $Y \in S^+\partial M \cup \mathcal{Z}$.

Let d (resp. $\tilde{d}$) denote the distance function induced by F (resp. $\widetilde{F}$). Let $p := \pi_1(y)$ and $q := \pi_1(Y)$. Then $L_F(\gamma_Y([0, \tilde{\mathfrak{t}}(-y)])) = \tilde{\mathfrak{t}}(-y) = \tilde{d}(p,q) = d(q,p)$, which implies that $i(Y) \geq \tilde{\mathfrak{t}}(-y)$. We claim that $i(Y) > \tilde{\mathfrak{t}}(-y)$. If not, then p is the cut point of q along γ_Y. If p is also a conjugate point of q, then there exists a non-vanishing Jacobi field $J(t)$ along $\gamma_Y(t)$ such that $J(0) = 0$ and $J(\tilde{\mathfrak{t}}(-y)) = 0$. It is easy to check that $\tilde{J}(t) := J(\tilde{\mathfrak{t}}(-y) - t)$ is a Jacobi field along $\widetilde{\gamma}_{-y}$ in $(\overline{M}, \widetilde{F})$. Hence, q is a conjugate point of p along $\widetilde{\gamma}_{-y}$ in $(\overline{M}, \widetilde{F})$, which contradicts $\tilde{\mathfrak{t}}(-y) < \tilde{i}(-y)$. Since p is not a conjugate point of q, by the proof of [3, Proposition 8.2.1], one can show that there exists another minimal geodesic from q to p. Thus, there exist two distinct minimal geodesic from p to q with the length $\tilde{\mathfrak{t}}(-y)$ in $(M,\widetilde{F})$, which also contradicts $\tilde{\mathfrak{t}}(-y)<\tilde{i}(-y)$. Hence, the claim is true, which implies that $\tilde{\mathfrak{t}}(-y)<\min\{\mathfrak{t}(Y),i(Y)\}=l(Y)$.

From above, we show that for each $y\in U^-_M$, there exist $Y\in S^+\partial M\cup\mathcal{Z}$ and $t:=\tilde{\mathfrak{t}}(-y)<l(Y)$ such that $y=\Psi(t,Y)$. Let $N_{\mathcal{Z}}:=\{(t,y):y\in\mathcal{Z},t\in(0,l(y))\}$. Then $\Psi|_{\mathfrak{N}_+\cup N_{\mathcal{Z}}}:\mathfrak{N}_+\cup N_{\mathcal{Z}}\to U^-_M$ is subjective. Since Ψ is injective, we are done by $\Psi(N_{\mathcal{Z}})=U_{\mathcal{Z}}$. □

Given any measure $d\mu$ on M, the induced volume forms on ∂M by $\mathbf{n}_\pm$ are defined by $d\,\mathrm{A}_\pm:=i^*(\mathbf{n}_\pm\rfloor d\mu)$, where $i:\partial M\hookrightarrow M$ is the inclusion map (cf. [18]). Now we have the following Santaló formula.

Theorem 3.4. *Let $(M,\partial M,F,d\mu)$ be a compact Finsler manifold with the boundary. Thus, for all integral function f on SM, we have*

$$\int_{\mathcal{V}^-_M} f dV_{SM}=\int_{y\in S^+\partial M} e^{\tau(y)}g_{\mathbf{n}_+}(\mathbf{n}_+,y)d\chi_+(y)\int_0^{l(y)} f(\varphi_t(y))dt, \tag{1}$$

$$\int_{\mathcal{V}^+_M} f dV_{SM}=\int_{y\in S^-\partial M} e^{\tau(y)}g_{\mathbf{n}_-}(\mathbf{n}_-,y)d\chi_-(y)\int_0^{\tilde{l}(-y)} f(\varphi_{-t}(y))dt, \tag{2}$$

where $\mathcal{V}^-_M:=\{y\in SM:\tilde{\mathfrak{t}}(-y)\leq\tilde{i}(-y)\}$, $\mathcal{V}^+_M:=\{y\in SM:\mathfrak{t}(y)\leq i(y)\}$ and $d\chi_\pm(y)=d\,\mathrm{A}_\pm(\pi_1(y))\wedge d\nu_{\pi_1(y)}(y)$.

Proof. (1). Given any $y\in S^+\partial M$. We identify $T_y(S^+\partial M)$ with its image in $T_{(0,y)}(\mathbb{R}\times S^+\partial M)$. Since $\Psi_{*(0,y)}(X)=X$, $\forall X\in T_y(S^+\partial M)$, we have

$$\Psi^*(d\chi_+(y))\equiv d\chi_+|_{(0,y)}\quad(\mathrm{mod}\ dt). \tag{3.1}$$

We claim that $[\Psi^*\pi_1^*d\rho]|_{(0,y)}\equiv 0\ (\mathrm{mod}\ dt)$. In fact, for each $X\in T_y(S^+\partial M)$, there exists a curve $\xi:[0,+\varepsilon)\to S^+\partial M$ with $\xi(0)=y$ and $\dot{\xi}(0)=X$. Thus,

$$\langle X,\Psi^*\pi_1^*d\rho\rangle|_{(0,y)}=\langle\pi_{1*}\left(\Psi_{*(0,y)}X\right),d\rho\rangle=\langle\pi_{1*}X,d\rho\rangle=\left.\frac{d}{ds}\right|_{s=0}\rho(\pi_1(\xi(s)))=0.$$

The claim is true. Lemma 3.2 now yields

$$\begin{aligned}[\Psi^*\pi_1^*d\rho]\,|_{(0,y)}&=\left\langle\frac{\partial}{\partial t},\Psi^*\pi_1^*d\rho\right\rangle_{(0,y)}dt\\&=\left(\left.\frac{d}{dt}\right|_{t=0^+}\rho\circ\gamma_y(t)\right)dt=g_{\mathbf{n}_+}(\mathbf{n}_+,y)dt.\end{aligned}\tag{3.2}$$

Define a function $\eta\in C^\infty(\mathbb{R}\times S^+\partial M)$ by $\Psi^*(dV_{SM})=\eta\cdot\beta$, where $\beta|_{(t,y)}=dt\wedge d\chi_+(y)$ is a $(2n-1)$ form on $\mathbb{R}\times S^+\partial M$. It is easy to check that $\eta(t,y)=\eta(0,y)$

(cf. [23, Lemma 5.6]). By the co-area formula (see [18, Theorem 3.3.1]), (3.1) and (3.2), we have

$$\begin{aligned}[\eta dt \wedge d\chi_+]|_{(0,y)} &= \Psi^*(dV_{SM}(y)) = \Psi^*[e^{\tau(y)}\pi_1^*(d\mu)(y) \wedge d\nu_{\pi_1(y)}(y)] \\ &= \Psi^*[e^{\tau(y)}\pi_1^*(d\rho \wedge d\,\mathrm{A}_+)(y) \wedge d\nu_{\pi_1(y)}(y)] \\ &= [e^{\tau(y)}g_{\mathbf{n}_+}(\mathbf{n}_+, y)dt \wedge d\chi_+]|_{(0,y)},\end{aligned}$$

that is, $\eta(0, y) = e^{\tau(y)}g_{\mathbf{n}_+}(\mathbf{n}_+, y)$. It follows from the definition of η that

$$\Psi^*(dV_{SM}(\varphi_t(y))) = e^{\tau(y)}g_{\mathbf{n}_+}(\mathbf{n}_+, y)dt \wedge d\chi, \tag{3.3}$$

which implies that Ψ is of maximal rank. Hence, Lemma 3.3 yields that $\Psi|_{\mathfrak{N}_+}$ is a diffeomorphism.

Let $\mathscr{N} := \{y \in SM : \tilde{\mathfrak{i}}(-y) = \tilde{i}(-y)\}$. Thus, $\mathcal{V}_M^- = U_M^- \cup \mathscr{N}$. By an argument similar to the proof of Lemma 3.3, one has $\mathscr{N} \subset \{\varphi_{l(y)}y : y \in S^+\partial M \cup \mathcal{Z},\ l(y) = i(y)\}$, which implies that $\mathscr{N}$ has measure zero with respect to dV_{SM}. Also note that $V_{SM}(U_M^- \backslash \Psi(\mathfrak{N}_+)) = V_{SM}(U_{\mathcal{Z}}) = 0$. Hence, by (3.3), we have

$$\begin{aligned}\int_{\mathcal{V}_M^-} f dV_{SM} &= \int_{U_M^-} f dV_{SM} = \int_{\Psi(\mathfrak{N}_+)} f dV_{SM} = \int_{\mathfrak{N}_+} \Psi^*(f dV_{SM}) \\ &= \int_{S^+\partial M} e^{\tau(y)}g_{\mathbf{n}_+}(\mathbf{n}_+, y)d\chi(y) \int_0^{l(y)} f(\varphi_t(y))dt.\end{aligned}$$

(2). By considering $(M, \partial M, \widetilde{F})$ and using the formula (1), we have

$$\int_{y\in\widetilde{\mathcal{V}_M^-}} f(-y)d\widetilde{V_{SM}}(y) = \int_{y\in\widetilde{S^+\partial M}} e^{\tilde{\tau}(y)}\tilde{g}_{\tilde{\mathbf{n}}_+}(\tilde{\mathbf{n}}_+, y)d\widetilde{\chi}_+(y) \int_0^{\tilde{l}(y)} f(-\tilde{\varphi}_t(y))dt,$$

where the quantities $\tilde{*}$ denote the quantities $*$ defined by $\widetilde{F}$. Note that $\tilde{\mathbf{n}}_+ = -\mathbf{n}_-$ and $-\tilde{\varphi}_t(y) = \varphi_{-t}(-y)$, $0 \le t \le \tilde{l}(y)$. The formula (2) now follows from the transformation $y \mapsto -y$. $\square$

4. A universal lower bound for the first eigenvalue of the nonlinear Laplacian

Definition 4.1 ([14], [19]). Let $(M, F, d\mu)$ be a compact Finsler manifold. Denote $\mathscr{H}_0(M, d\mu)$ by

$$\mathscr{H}_0(M, d\mu) := \begin{cases}\{f \in W_2^1(M) : \int_M f d\mu = 0\}, & \partial M = \emptyset, \\ \{f \in W_2^1(M) : f|_{\partial M} = 0\}, & \partial M \ne \emptyset.\end{cases}$$

Define the canonical energy functional $E_{d\mu}$ on $\mathscr{H}_0(M,d\mu)-\{0\}$ by

$$E_{d\mu}(u) := \frac{\int_M F^*(du)^2 d\mu}{\int_M u^2 d\mu}.$$

λ is an eigenvalue if there is a function $u \in \mathscr{H}_0(M,d\mu)-\{0\}$ such that $d_u E_{d\mu} = 0$ with $\lambda = E_{d\mu}(u)$. In this case, u is called an eigenfunction corresponding to λ. The first eigenvalue $\lambda_1(M,d\mu)$ is defined by

$$\lambda_1(M,d\mu) := \inf_{u\in\mathscr{H}_0(M,d\mu)-\{0\}} E_{d\mu}(u),$$

which is the smallest positive critical value of $E_{d\mu}$.

Remark 2. u is an eigenfunction corresponding to λ if and only if

$$\Delta_{d\mu}u + \lambda u = 0 \text{ (in the weak sense)},$$

where $\Delta_{d\mu}$ is the nonlinear Laplacian introduced by Shen [14], [18], [19]. It should be noted that $\Delta_{d\mu}$ is dependent on the choice of $d\mu$.

Proposition 4.2. *Let (M,F) be a Finsler n-manifold. Then for any $p \in M$ and $f \in C^\infty(M)$, we have*

$$F^*(df|_p)^2 \geq \frac{n}{c_{n-1}\Lambda_F^{n+1}(p)} \int_{S_pM} \langle y, df\rangle^2 d\nu_p(y), \tag{4.1}$$

with equality if and only if $F(p,\cdot)$ is a Eucildean norm.

Proof. Without loss of generality, we may suppose $df|_p \neq 0$. Set $B_pM := \{y \in T_pM : F(p,y) < 1\}$. By [21], one can choose a $g_{\nabla f}$-orthnormal basis $\{e_i\}$ of T_pM such that $e_n = \nabla f/F(\nabla f)$ and $\det g_{ij}(p,y) \leq \Lambda_F^n(p)$. Let $\{y^i\}$ denote the corresponding coordinates. By Stokes' formula, we have

$$\begin{aligned}
\int_{S_pM} \langle y, df\rangle^2 d\nu_p(y) &\leq \Lambda_F^{\frac{n}{2}}(p) F^2(\nabla f) \\
&\quad \times \int_{S_pM} (y^n)^2 \sum_{k=1}^n (-1)^{k-1} y^k dy^1\wedge\cdots\wedge\widehat{dy^k}\wedge\cdots\wedge dy^n \\
&= (n+2)\Lambda_F^{\frac{n}{2}}(p)F^2(\nabla f)\int_{B_pM}(y^n)^2 dy^1\wedge\cdots\wedge dy^n \\
&\leq (n+2)\Lambda_F^{\frac{n}{2}}(p)F^2(\nabla f)\int_{\mathbb{B}^n(\sqrt{\Lambda_F(p)})}(y^n)^2 dy^1\wedge\cdots\wedge dy^n \\
&= \frac{c_{n-1}}{n}\Lambda_F^{n+1}(p)F^2(\nabla f)
\end{aligned} \tag{4.2}$$

If equality holds in (4.1), then it follows from (4.2) that $B_pM = \mathbb{B}^n(\sqrt{\Lambda_F(p)})$. Namely, $F(y) = 1$ if and only if $g_{\nabla f}(y,y) = \Lambda_F(p)$. In particular, $1 = F(e_n) = g_{\nabla f}(e_n, e_n) = \Lambda_F(p)$, which implies that $F(p,\cdot)$ is a Eucildean norm. $\square$

Theorem 4.3. *Let $(M, \partial M, F)$ be a compact Finsler n-manifold with the boundary such that every geodesic ray in (M,F) minimizes distance up to the point that it intersects ∂M. Then*

$$\lambda_1(M, d\mu) \geq \begin{cases} \dfrac{\lambda_1(\mathbb{S}^+_D)}{\Lambda_F^{4n+1}}, & d\mu = d\mu_{BH}, \\ \dfrac{\lambda_1(\mathbb{S}^+_D)}{\Lambda_F^{2n+1}}, & d\mu = d\mu_{HT}, \end{cases} \tag{4.3}$$

where $D := \operatorname{diam}(M)$ and $\mathbb{S}^+_D$ denotes the n-dimensional Riemannian hemisphere of the constant sectional curvature sphere having diameter equal to D. The equality holds if and only if (M,F) is isometric to $\mathbb{S}^+_D$.

Proof. Lemma 2.1 yields that

$$\int_{S_pM} e^{\tau(y)} d\nu_p(y) = c_{n-1}\frac{\sigma_{HT}(p)}{\sigma(p)} \geq \begin{cases} \dfrac{c_{n-1}}{\Lambda_F^{2n}}, & d\mu = d\mu_{BH}, \\ c_{n-1}, & d\mu = d\mu_{HT}. \end{cases} \tag{4.4}$$

Since $\mathcal{V}^+_M = SM$, Theorem 3.4 together with Proposition 4.2 and (4.4) then yields

$$\begin{aligned}
\int_M F^{*2}(df) d\mu &\geq \frac{n}{c_{n-1}\Lambda_F^{n+1}} \int_M d\mu(p) \int_{S_pM} \langle y, df\rangle^2 d\nu_p(y) \\
&= \frac{n}{c_{n-1}\Lambda_F^{n+1}} \int_{SM} e^{-\tau(y)} \langle y, df\rangle^2 dV_{SM}(y) \\
&= \frac{n}{c_{n-1}\Lambda_F^{n+1}} \int_{y\in S^-\partial M} e^{\tau(y)} g_{\mathbf{n}_-}(\mathbf{n}_-, y) d\chi_-(y) \int_{-\tilde{l}(-y)}^0 e^{-\tau(\varphi_t(y))} \langle \varphi_t(y), df\rangle^2 dt \\
&\geq \frac{n}{c_{n-1}\Lambda_F^{2n+1}} \int_{y\in S^-\partial M} e^{\tau(y)} g_{\mathbf{n}_-}(\mathbf{n}_-, y) d\chi_-(y) \int_{-\tilde{l}(-y)}^0 \left(\frac{d}{dt} f(\gamma_y(t))\right)^2 dt \\
&\geq \frac{n}{c_{n-1}\Lambda_F^{2n+1}} \int_{y\in S^-\partial M} e^{\tau(y)} g_{\mathbf{n}_-}(\mathbf{n}_-, y) d\chi_-(y) \int_{-\tilde{l}(-y)}^0 \left(\frac{\pi}{\tilde{l}(-y)}\right)^2 f^2(\gamma_y(t)) dt \\
&\geq \frac{n}{c_{n-1}\Lambda_F^{2n+1}} \left(\frac{\pi}{D}\right)^2 \int_{SM} f^2(\pi_1(y)) dV_{SM}(y)
\end{aligned}$$

$$\geq \begin{cases} \dfrac{\lambda_1(\mathbb{S}^+_D)}{\Lambda_F^{4n+1}} \int_M f^2 d\mu, & d\mu = d\mu_{BH}, \\ \dfrac{\lambda_1(\mathbb{S}^+_D)}{\Lambda_F^{2n+1}} \int_M f^2 d\mu, & d\mu = d\mu_{HT}. \end{cases} \tag{4.5}$$

If we have equality in (4.3), then (4.5) together with Proposition 4.2 implies $\Lambda_F = 1$. Hence, we obtain that $\lambda_1(M) = \lambda_1(\mathbb{S}^+_D)$ and (M, F) is a Riemannian manifold. By the standard argument (see [5, p.131] or [9]), one can show that (M, F) is isometric to $\mathbb{S}^+_D$. $\square$

In [19], Shen shows that the first eigenvalue of a forward metric ball is bounded from above by a constant depending only on the dimension and lower bounds on the Ricci curvature and the S-curvature. From Theorem 4.3, we obtain a lower bound for the first eigenvalue of a forward metric ball.

Corollary 4.4. *Let $(M, F, d\mu)$ be a forward complete Finsler n-manifold of injectivity radius $\mathfrak{i}_M$. For any $0 < r < \mathfrak{i}_M/(1 + \sqrt{\Lambda_F}\,)$ and any $p \in M$, we have*

$$\lambda_1(B^+_p(r)) \geq \begin{cases} \dfrac{\lambda_1\big(\mathbb{S}^+_{2\sqrt{\Lambda_F}r}\big)}{\Lambda_F^{4n+1}}, & d\mu = d\mu_{BH}, \\ \dfrac{\lambda_1\big(\mathbb{S}^+_{2\sqrt{\Lambda_F}r}\big)}{\Lambda_F^{2n+1}}, & d\mu = d\mu_{HT}. \end{cases}$$

with equality if and only if $B^+_p(r)$ is isometric to $\mathbb{S}^+_{2r}$.

5. Croke type isoperimetric inequalities

In this section, we shall establish Theorem 1.3 and give some applications.

Lemma 5.1. *For each $x \in \partial M$, we have*

$$\int_{S^\sharp_x \partial M} g_{\mathbf{n}_\sharp}(\mathbf{n}_\sharp, y) e^{\tau(y)} d\nu_x(y) \leq \frac{c_{n-2}}{n-1} \Lambda_F^{2n+\frac{1}{2}}(x),$$

with equality if and only if $F(x, \cdot)$ is a Euclidean norm. Here, "$\sharp$" denotes either "+" or "−", and $S^\sharp_x \partial M := \{y \in S_x M : g_{\mathbf{n}_\sharp}(\mathbf{n}_\sharp, y) > 0\}$.

Proof. Suppose $\sharp = +$. By [21], one can choose a $g_{\mathbf{n}_+}$-orthnormal basis $\{e_i\}$ of $T_x M$ such that $e_n = \mathbf{n}_+$ and $\det g_{ij}(x, y) \leq \Lambda^n_F(x)$. Let $\{y^i\}$ be the corresponding coordinates. Set $\|\cdot\| := \sqrt{g_{\mathbf{n}_+}(\cdot, \cdot)}$. Define

$$B^+_x := \{y \in T_x M : F(y) < 1,\ y^n > 0\}, \quad B^+_{x,r} := \{y \in T_x M : F(y) = 1,\ y^n = r\}$$

$$\mathbb{B}_{x,r}(s) := \{y \in T_xM : y^n = r,\ \|y^\alpha e_\alpha\| < s\}, \quad \varpi := g_{\mathbf{n}_+}(\mathbf{n}_+, y)e^{\tau(y)}d\nu_p(y).$$

For each $y \in B_x^+$, $y^n = g_{\mathbf{n}_+}(\mathbf{n}_+, y) \leq F(\mathbf{n}_+)F(y) \leq 1$. Stokes' formula together with Lemma 2.1 then yields

$$\begin{aligned}
&\int_{S_x^+\partial M} \varpi \leq \Lambda_F^{3n/2}(x) \int_{S_x^+\partial M} y^n \sum_{k=1}^n (-1)^{k-1} y^k dy^1 \wedge \cdots \wedge \widehat{dy^k} \wedge \cdots \wedge dy^n \\
&= (n+1)\Lambda_F^{\frac{3n}{2}}(x) \int_{B_x^+} y^n dy^1 \wedge \cdots \wedge dy^n = (n+1)\Lambda_F^{\frac{3n}{2}}(x) \int_0^1 \mathrm{Vol}(B^+_{x,y^n}) y^n dy^n \\
&\leq (n+1)\Lambda_F^{\frac{3n}{2}}(x) \int_0^{\sqrt{\Lambda_F(x)}} \mathrm{Vol}\left(\mathbb{B}_{x,y^n}(\sqrt{\Lambda_F(x) - (y^n)^2})\right) y^n dy^n \\
&= \frac{c_{n-2}}{n-1} \Lambda_F^{2n+\frac{1}{2}}(x),
\end{aligned}$$

with equality if and only if $\Lambda_F(x) = 1$, i.e., $F(x, \cdot)$ is a Euclidean norm.

Suppose $\sharp = -$. Note that $\Lambda_F(x) = \Lambda_{\widetilde{F}}(x)$. Using the same method as in Theorem 3.4, one can get the formula. □

Given any point $x \in M$, let (r, y) denote the polar coordinates about x. Set $\mathscr{F}(r, y) = e^{\tau(\gamma_y(r))}\hat{\sigma}_x(r, y)$, where $d\mu|_{(r,y)} =: \hat{\sigma}_x(r, y)dr \wedge d\nu_x(y)$. Then we have the following inequality of Berger–Kazdan type [23, Theorem 1.3]

Lemma 5.2 ([23]). *Let (M, F) be a compact Finsler n-manifold. For each $y \in SM$ and $0 < t \leq l \leq i_y$, we have*

$$\int_0^l dr \int_0^{l-r} \mathscr{F}(t, \varphi_r(y))\, dt \geq \frac{\pi c_n}{2c_{n-1}} \left(\frac{l}{\pi}\right)^{n+1},$$

with equality if and only if

$$R_{\dot{\gamma}_y(t)}(\cdot, \dot{\gamma}_y(t))\dot{\gamma}_y(t) = \left(\frac{\pi}{l}\right)^2 \mathrm{id}, \quad 0 \leq t \leq l,$$

where R is the (Riemannian) curvature tensor acting on $\dot{\gamma}_y(t)^\perp$.

Now we have the following theorem.

Theorem 5.3. *Let $(M, \partial M, F, d\mu)$ be a compact Finsler n-manifold with the boundary, where $d\mu$ is either the Busemann–Hausdorff measure or the Holmes–Thompson measure. Set*

$$\omega := \inf_{x\in M} \min\{\omega_x^+, \omega_x^-\} = \min\left\{\inf_{x\in M} \omega_x^+, \inf_{x\in M} \omega_x^-\right\},$$

where $\omega_x^{\pm} := c_{n-1}^{-1} \int_{U_x^{\pm}} e^{\tau(y)} d\nu_x(y)$ *and* $U_x^{\pm} := \pi|_{\mathcal{V}_M^{\pm}}^{-1}(x)$. *Then*

(1)
$$\frac{\mathrm{A}_{\pm}(\partial M)}{\mu(M)} \geq \frac{(n-1)c_{n-1}\,\omega}{c_{n-2}\,D\,\Lambda_F^{2n+\frac{1}{2}}},$$

where $D := \operatorname{diam}(M)$.

(2)
$$\frac{\mathrm{A}_{\pm}(\partial M)}{\mu(M)^{1-\frac{1}{n}}} \geq \frac{c_{n-1}\omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}}\,\Lambda_F^{2n+\frac{5}{2}}}, \tag{5.1}$$

with equality if and only if (M,F) *is a Riemannian hemisphere of a constant sectional curvature sphere.*

Proof. (1) Theorem 1.1 together with Lemma 5.1 furnishes

$$\begin{aligned} c_{n-1}\omega\mu(M) &\leq c_{n-1}\int_M \omega_x^{\mp} d\mu(x) = \int_{x\in M} d\mu(x)\int_{U_x^{\mp}} e^{\tau(y)} d\nu_x(y) = V_{SM}(\mathcal{V}_M^{\mp}) \\ &\leq D\int_{S^{\pm}\partial M} e^{\tau(y)} g_{\mathbf{n}_{\pm}}(\mathbf{n}_{\pm},y)d\chi_{\pm}(y) \leq D\,\mathrm{A}_{\pm}(\partial M)\frac{c_{n-2}}{n-1}\Lambda_F^{2n+\frac{1}{2}}. \end{aligned}$$

(2) For each $y \in S^+\partial M$, $l(\varphi_t(y)) \geq l(y) - t$, for any $0 \leq t \leq l(y)$. By Theorem 1.1, Lemma 2.1, Theorem 5.2 and Hölder's inequality, we have

$$\begin{aligned} \mu^2(M) &= \int_M d\mu(x)\int_{S_xM} d\nu_x(y)\int_0^{l(y)} \hat{\sigma}_x(r,y)dr \\ &= \int_{SM} dV_{SM}(y)\int_0^{l(y)} e^{-\tau(y)}\hat{\sigma}_x(r,y)dr \geq \int_{\mathcal{V}_M^-} dV_{SM}(y)\int_0^{l(y)} e^{-\tau(y)}\hat{\sigma}_x(r,y)dr \\ &= \int_{S^+\partial M} e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+,y)d\chi_+(y)\int_0^{l(y)} dt\int_0^{l(\varphi_t(y))} e^{-\tau(\varphi_t(y))}\hat{\sigma}_x(r,\varphi_t(y))dr \\ &\geq \Lambda_F^{-2n}\int_{S^+\partial M} e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+,y)d\chi_+(y)\int_0^{l(y)} dt\int_0^{l(y)-t} \mathscr{F}(r,\varphi_t(y))dr \\ &\geq \frac{c_n}{2c_{n-1}\pi^n\Lambda_F^{2n}}\int_{S^+\partial M} l(y)^{n+1}e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+,y)d\chi_+(y) \\ &\geq \frac{c_n}{2c_{n-1}\pi^n\Lambda_F^{2n}}\left(\int_{S^+\partial M} l(y)e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+,y)d\chi_+(y)\right)^{n+1} \\ &\times \left(\int_{S^+\partial M} e^{\tau(y)} g_{\mathbf{n}_+}(\mathbf{n}_+,y)d\chi_+(y)\right)^{-n} \\ &\geq \frac{c_n}{2c_{n-1}\pi^n\Lambda_F^{2n}} V_{SM}(\mathcal{V}_M^-)^{n+1}\left(\frac{n-1}{c_{n-2}\,\mathrm{A}_+(\partial M)\Lambda_F^{2n+\frac{1}{2}}}\right)^n \end{aligned}$$

$$\geq \frac{(c_{n-1})^n \omega^{n+1} \mu(M)^{n+1}}{(c_n/2)^{n-1}\, \mathrm{A}_+^n(\partial M) \Lambda_F^{(2n+\frac{5}{2})n}}. \tag{5.2}$$

That is,

$$\frac{\mathrm{A}_+(\partial M)}{\mu(M)^{1-\frac{1}{n}}} \geq \frac{c_{n-1}\omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}} \Lambda_F^{2n+\frac{5}{2}}}. \tag{5.3}$$

Let $\widetilde{\mathrm{A}}$ and $\tilde{\omega}$ be define as before on $(M, \widetilde{F})$. It is easy to check that $\widetilde{\mathrm{A}}_\pm(\partial M) = \mathrm{A}_\mp(\partial M)$ and $\tilde{\omega} = \omega$. From above, we obtain

$$\frac{\mathrm{A}_-(\partial M)}{\mu(M)^{1-\frac{1}{n}}} = \frac{\widetilde{\mathrm{A}}_+(\partial M)}{\tilde{\mu}(M)^{1-\frac{1}{n}}} \geq \frac{c_{n-1}\omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}} \Lambda_F^{2n+\frac{5}{2}}}. \tag{5.4}$$

(5.3) together with (5.4) yields (5.1).

Suppose that equality holds in (5.1). Then we have equality in (5.3) or (5.4). It follows from (5.2) and Lemma 5.1 that $1 = \Lambda_F = \Lambda_{\widetilde{F}}$. Hence, F is an Riemannian metric and (5.1) becomes

$$\frac{\mathrm{A}(\partial M)}{\mu(M)^{1-\frac{1}{n}}} = \frac{c_{n-1}\omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}}}.$$

Since $\mathcal{V}_M^- = SM$, $\mathfrak{t}(y) \leq i_y$, for all $y \in S\overline{M}$. Hölder's inequality implies $l(y)$ is constant, say, equal to l, on all of $S^+\partial M$. Hence, $\mathfrak{t}(y) = l$, for all $y \in S^+\partial M$. And Theorem 5.2 yields M has constant sectional curvature equal to $(\pi/l)^2$, i.e., M is a hemisphere. $\square$

From above, it is easy to see that Theorem 5.3 becomes Croke's isoperimetric inequality [9] if $\Lambda_F = 1$. In the Finslerian case, the upper bound on Λ_F in Theorem 5.3 is very important as the following example shows.

Example 1. Let $\mathbb{B}^n$ be the unit open ball in $\mathbb{R}^n$ equipped with a Funk metric F, that is,

$$F(x,y) = \frac{\sqrt{(1-|x|^2)|y|^2 + (x\cdot y)^2} + x\cdot y}{1-|x|^2},$$

where "$|\cdot|$" (resp. "$\cdot$") denote the Euclidean norm (resp. inner product). For $r \in (0,1)$, set $\Omega_r := \{x \in \mathbb{B}^n : |x| < r\}$. Then $(\Omega_r, \partial\Omega_r, F|_{\overline{\Omega_r}})$ is a compact Finsler manifold. By directly computing, we have $\mu_{BH}(\Omega_r) = \frac{c_{n-1}}{n} r^n$ and $\mathrm{A}_\pm(\partial\Omega_r) = c_{n-1}(1\pm r)r^{n-1}$, where $d\,\mathrm{A}_\pm$ are induced by $d\mu_{BH}$. Clearly,

$$\lim_{r\to 1} \frac{\mathrm{A}_+(\partial\Omega_r)}{\mathrm{A}_-(\partial\Omega_r)} = +\infty.$$

Note that
$$\Lambda_{F|_{\overline{\Omega}_r}} = \left(\frac{1+r}{1-r}\right)^2, \ \operatorname{diam}(\Omega_r) = \log\left(\frac{1+r}{1-r}\right).$$
For any $x \in \Omega_r$,
$$\omega_x^{\pm} = \frac{1}{(1-|x|^2)^{\frac{n+1}{2}}} \geq 1, \ \text{i.e.,}\ \omega = 1.$$
Hence, we have
$$\frac{\mathrm{A}_{\pm}(\partial\Omega_r)}{\mu_{BH}(\Omega_r)} > \frac{(n-1)c_{n-1}\,\omega}{c_{n-2}\operatorname{diam}(\Omega_r)\,\Lambda_{F|_{\overline{\Omega}_r}}^{2n+\frac{1}{2}}}, \frac{\mathrm{A}_{\pm}(\partial\Omega_r)}{\mu_{BH}(\Omega_r)^{1-\frac{1}{n}}} > \frac{c_{n-1}\omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}}\Lambda_{F|_{\overline{\Omega}_r}}^{2n+\frac{5}{2}}}.$$
In particular,
$$\lim_{r\to 1}\Lambda_{F|_{\overline{\Omega}_r}} = +\infty, \ \lim_{r\to 1}\frac{\mathrm{A}_{-}(\partial\Omega_r)}{\mu_{BH}(\Omega_r)} = \lim_{r\to 1}\frac{\mathrm{A}_{-}(\partial\Omega_r)}{\mu_{BH}(\Omega_r)^{1-\frac{1}{n}}} = 0.$$

Before giving some applications of Theorem 5.3, we introduce the definitions of the Sobolev constant, Cheeger's constant and the isoperimetric constant of a closed Finsler manifold.

Definition 5.4. Let $(M, F, d\mu)$ be a closed Finsler manifold. The Sobolev constant $\mathcal{S}(M, d\mu)$ is defined as
$$\mathcal{S}(M, d\mu) := \inf_{f\in C^\infty(M)} \frac{\left\{\int_M F^*(df)d\mu\right\}^n}{\inf_{\alpha\in\mathbb{R}}\left\{\int_M |f-\alpha|^{\frac{n}{n-1}}d\mu\right\}^{n-1}}.$$
Cheeger's constant $\mathbb{h}(M, d\mu)$ and the isoperimetric constant $\mathbb{I}(M, d\mu)$ are defined by
$$\mathbb{h}(M, d\mu) := \inf_{\Gamma}\frac{\min\{\mathrm{A}_{\pm}(\Gamma)\}}{\min\{\mu(M_1), \mu(M_2)\}},$$
$$\mathbb{I}(M, d\mu) := \inf_{\Gamma}\frac{\min\{\mathrm{A}_{\pm}(\Gamma)\}^n}{\{\min\{\mu(M_1),\, \mu(M_2)\}\}^{n-1}},$$
where Γ varies over compact $(n-1)$-dimensional submanifolds of M which divide M into disjoint open submanifolds M_1, M_2 of M with common boundary $\partial M_1 = \partial M_2 = \Gamma$.

Remark 3. By using the co-area formula (cf. [18, Theorem 3.3.1]) and the same argument as in [13], one can obtain a Cheeger type inequality

$$\lambda_1(M,d\mu)\geq \frac{\mathbb{h}^2(M,d\mu)}{4\lambda_F^2}.$$

And we also have a Federer-Fleming type inequality (see Proposition 6.1 below), i.e.,

$$\mathbb{I}(M,d\mu)\leq \mathcal{S}(M,d\mu)\leq 2\mathbb{I}(M,d\mu).$$

Corollary 5.5. *Let $(M,F,d\mu)$ be a closed Finlser n-manifold with* $\mathbf{Ric}\geq (n-1)k$*, where $d\mu$ denotes either the Busemann–Hausdorff measure or the Holmes–Thompson measure. Then*

$$\lambda_1(M,d\mu)\geq \left[\frac{(n-1)\mu(M)}{4c_{n-2}\Lambda_F^{4n+1}\operatorname{diam}(M)\int_0^{\operatorname{diam}(M)}\mathfrak{s}_k^{n-1}(t)dt}\right]^2,$$

$$\mathcal{S}(M,d\mu)\geq \frac{\mu(M)^{n+1}}{4c_{n-1}(c_n)^{n-1}\Lambda_F^{4n^2+\frac{9n}{2}}\left(\int_0^{\operatorname{diam}(M)}\mathfrak{s}_k^{n-1}(t)dt\right)^{n+1}}.$$

Hence, both $\lambda_1(M,d\mu)$ and $\mathcal{S}(M,d\mu)$ can be bounded from below in terms of the diameter, volume, uniform constant and a lower bound for the Ricci curvature.

Proof. *Step 1.* Let Γ be any $(n-1)$-dimensional compact submanifold of M dividing M into two open submanifolds M_1 and M_2, such that $\partial M_1=\partial M_2=\Gamma$. Given $x\in M_1$, let

$$O_x:=\{q\in M:\exists\, y\in U_x^-\text{ such that } q=\widetilde{\gamma}_{-y}(t),\ t\in(0,\tilde{i}(-y)]\},$$

where $\widetilde{\gamma}_{-y}(t)$ is the geodesic in $(M,\widetilde{F})$ with $\dot{\widetilde{\gamma}}_{-y}(0)=-y$.

For any $q\in M_2$, there exists a minimal unit speed geodesic, say $\widetilde{\gamma}_X(t)$, from x to q. Clearly, $\widetilde{\gamma}_X(t)$ must hit the boundary and therefore, $\tilde{\mathfrak{t}}(X)\leq\tilde{i}(X)$. Since $F(-X)=\widetilde{F}(X)=1$, $q\in O_x$ which implies that $M_2\subset O_x$.

Note that $\widetilde{\mathbf{Ric}}\geq(n-1)k$, $\Lambda_{\widetilde{F}}=\Lambda_F$ and $d\tilde{\mu}=d\mu$. Hence, by Lemma 2.1 and the volume comparison theorem (cf. [23, Theorem 3.1]), we have

$$\begin{aligned}\mu(M_2)=\tilde{\mu}(M_2)\leq\tilde{\mu}(O_x)&=\int_{y\in U_x^-}d\tilde{\nu}_x(-y)\int_0^{\tilde{i}(-y)}\widetilde{\hat{\sigma}}_x(r,-y)dr\\ &\leq\Lambda_F^n\int_{y\in U_x^-}d\tilde{\nu}_x(-y)\int_0^{\tilde{i}(-y)}\mathfrak{s}_k^{n-1}(r)dr\end{aligned}$$

$$\leq c_{n-1}\Lambda_F^{2n}\omega_1^-(x)\int_0^{\operatorname{diam}(M)} \mathfrak{s}_k^{n-1}(r)dr.$$

That is,

$$\omega_i^- \geq \frac{\mu(M_j)}{c_{n-1}\Lambda_F^{2n}\int_0^{\operatorname{diam}(M)} \mathfrak{s}_k^{n-1}(r)dr}, \quad i \neq j.$$

Set $O'_x := \{q \in M : \exists\, y \in U_x^+ \text{ such that } q = \gamma_y(t),\ t \in (0, i(y)]\}$. It is easy to see that $M_2 \subset O'_x$. By the similar argument, one can show that

$$\omega_i^+ \geq \frac{\mu(M_j)}{c_{n-1}\Lambda_F^{2n}\int_0^{\operatorname{diam}(M)} \mathfrak{s}_k^{n-1}(t)dt}, \quad i \neq j.$$

Step 2. The inequalities above together with Theorem 5.3 yield

$$\mathbb{h}(M, d\mu) \geq \frac{(n-1)\mu(M)}{2c_{n-2}\Lambda_F^{4n+\frac{1}{2}}\operatorname{diam}(M)\int_0^{\operatorname{diam}(M)} \mathfrak{s}_k^{n-1}(t)dt},$$

$$\mathbb{I}(M, d\mu) \geq \frac{\mu(M)^{n+1}}{4c_{n-1}(c_n)^{n-1}\Lambda_F^{4n^2+\frac{9n}{2}}\left(\int_0^{\operatorname{diam}(M)} \mathfrak{s}_k^{n-1}(t)dt\right)^{n+1}}.$$

Corollary now follows from Remark 3. $\square$

Corollary 5.6. *Let $(M, F, d\mu)$ be a closed Finsler n-manifold, where $d\mu$ is either the Busemann–Hausdorff measure or the Holmes–Thompson measure. Then for any $x \in M$ and $0 < r < \mathfrak{i}_M/(1+\sqrt{\Lambda_F})$, we have*

$$\mu(B_x^+(r)) \geq \frac{C(n,\Lambda_F)}{n^n} r^n,\ \mathrm{A}_\pm(S_x^+(r)) \geq \frac{C(n,\Lambda_F)}{n^{n-1}} r^{n-1}.$$

Proof. The similar argument as in Lemma 3.3 shows $\mathfrak{i}_M = \tilde{\mathfrak{i}}_M$, where $\tilde{\mathfrak{i}}_M$ is the injectivity radius of $(M, \widetilde{F})$. Hence, $U_p^\pm = S_pM$ for all $p \in B_x^+(r)$. By Theorem 5.3 and (4.4), we have

$$\frac{\frac{d}{dr}\mu(B_x^+(r))}{\mu(B_x^+(r))^{1-\frac{1}{n}}} = \frac{\mathrm{A}_-(S_x^+(r))}{\mu(B_x^+(r))^{1-\frac{1}{n}}} \geq C(n,\Lambda_F),$$

which implies that

$$\mu(B_x^+(r)) \geq \frac{C(n,\Lambda_F)}{n^n} r^n. \tag{5.5}$$

Theorem 5.3 together with (5.5) yields

$$\mathrm{A}_\pm(S_x^+(r)) \geq \frac{C(n,\Lambda_F)}{n^{n-1}} r^{n-1}. \qquad \square$$

In order to establish Theorem 1.5, let us recall some definitions and properties of general LGC spaces first. Refer to [20], [24] for more details.

Definition 5.7 ([20], [24]). A general metric space is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}^+ \cup \{\infty\}$, called a metric, is a function, satisfying the following two conditions: (a) $d(x, y) \geq 0$, with equality $\Leftrightarrow x = y$; (b) $d(x, y) + d(y, z) \geq d(x, z)$. The reversibility λ_X of a general metric space (X, d) is defined by $\lambda_X := \sup_{x \neq y} \frac{d(x,y)}{d(y,x)}$.

A contractibility function $\rho : [0, r) \to [0, +\infty)$ is a function satisfying: (a) $\rho(0) = 0$, (b) $\rho(\epsilon) \geq \epsilon$, (c) $\rho(\epsilon) \to 0$, as $\epsilon \to 0$, (d) ρ is nondecreasing. A general metric space X is LGC(ρ) for some contractibility function ρ, if for every $\epsilon \in [0, r)$ and $x \in X$, the forward ball $B^+_x(\epsilon)$ is contractible inside $B^+_x(\rho(\epsilon))$.

Lemma 5.8 ([24]). *Fix a function $N : (0, \alpha) \to (0, \infty)$ with*

$$\limsup_{\epsilon \to 0^+} \epsilon^n N(\epsilon) < \infty$$

and a contractibility function $\rho : [0, r) \to [0, \infty)$. The class

$$\mathscr{C}(N, \rho) := \{X \in \mathcal{M}^\delta : X \text{ is LGC}(\rho),\ \mathrm{Cov}(X, \epsilon) \leq N(\epsilon) \text{ for all } \epsilon \in (0, \alpha)\}$$

contains only finitely many homotopy types. Here, $\mathcal{M}^\delta$ denotes the collection of compact general metric spaces with reversibility $\leq \delta$ and $\mathrm{Cov}(X; \epsilon)$ denotes the minimum number of forward ϵ-balls it takes to cover X.

Corollary 5.6 together with Lemma 5.8 yields the following

Theorem 5.9. *For any n and positive numbers i, V, δ, the class of closed Finsler n-manifolds (M, F) with injectivity radius $\mathfrak{i}_M \geq i$, $\Lambda_F \leq \delta$ and $\mu(M) \leq V$, contains at most finitely many homotopy types. Here, $\mu(M)$ is either the Busemann–Hausdorff volume or the Holmes–Thompson volume of M.*

Proof. Let c_M denote the contractibility radius of (M, F) (cf. [24]). Since $c_M \geq \mathfrak{i}_M \geq i$, (M, F) is LGC(ρ), where ρ is the identity map of $[0, i)$. Corollary 5.6 implies that $\mu(B^+_p(\epsilon)) \geq C(n, \delta)\epsilon^n$ for all $p \in M$ and $\epsilon < i/(1 + \sqrt{\delta}\,)$. It follows from [20, Proposition 3.11] that

$$\mathrm{Cov}(M, \epsilon) \leq \frac{\mu(M)}{C(n, \delta)(\epsilon/(2\sqrt{\delta}))^n} = C'(n, \delta, V)\epsilon^{-n}.$$

Define the covering function $N(\epsilon) := C'(n, \delta, V)\epsilon^{-n}$, $\epsilon \in (0, i/(1 + \sqrt{\delta}\,))$. The conclusion now follows from Lemma 5.8. □

One can easily see that Theorem 5.9 implies Yamaguchi's finiteness theorem [22] and [24, Theorem 1.3].

6. Appendix

Proposition 6.1. *Let $(M, F, d\mu)$ be a closed Finsler manifold. Then*

$$\mathbb{I}(M, d\mu) \leq \mathcal{S}(M, d\mu) \leq 2\mathbb{I}(M, d\mu).$$

Proof. Fix Γ with $\mu(M_1) \leq \mu(M_2)$. Define a Lipschitz function f_ϵ^+ by

$$f_\epsilon^+(x) := \begin{cases} 1, & x \in M_1,\ d(\Gamma, x) \geq \epsilon, \\ \frac{1}{\epsilon} d(\Gamma, x), & x \in M_1,\ d(\Gamma, x) < \epsilon, \\ 0, & x \in M_2. \end{cases}$$

By letting $\epsilon \to 0^+$, we obtain that

$$\begin{aligned} \inf_{\alpha \in \mathbb{R}} \left(\int_M |f_\epsilon^+ - \alpha|^{\frac{n}{n-1}} d\mu \right)^{n-1} &\geq \inf_{\alpha \in \mathbb{R}} \left\{ |1 - \alpha|^{\frac{n}{n-1}} \mu(M_1) + |\alpha|^{\frac{n}{n-1}} \mu(M_2) \right\}^{n-1} \\ &\geq \mu(M_1)^{n-1} \inf_{\alpha \in \mathbb{R}} \left\{ |1 - \alpha|^{\frac{n}{n-1}} + |\alpha|^{\frac{n}{n-1}} \right\}^{n-1} \geq \mu(M_1)^{n-1}/2. \end{aligned}$$

Set $\rho_+(x) = d(\Gamma, x)$, $x \in \overline{M_1}$. Lemma 3.2 yields that $\nabla \rho_+|_\Gamma = \mathbf{n}_+$, where $\mathbf{n}_+$ denotes the unit inward normal vector field along $\partial M_1 = \Gamma$. By the co-area formula (cf. [18, Theorem 3.3.1]), we see that

$$\int_M F^*(df_\epsilon^+) d\mu = \frac{1}{\epsilon} \int_0^\epsilon dt \int_{\rho_+^{-1}(t)} d\,\mathrm{A}_+ \to \mathrm{A}_+(\Gamma).$$

Hence, $2\,\mathrm{A}_+(\Gamma)^n \geq \mathcal{S}(M, d\mu) \cdot \mu(M_1)^{n-1}$. Similarly, define a Lipschitz function f_ϵ^- by

$$f_\epsilon^-(x) := \begin{cases} 0, & x \in M_2,\ d(\Gamma, x) > \epsilon \\ \frac{1}{\epsilon} d(\Gamma, x) - 1, & x \in M_2,\ d(\Gamma, x) \leq \epsilon, \\ -1, & x \in M_1. \end{cases}$$

Then one can show $2\,\mathrm{A}_-(\Gamma)^n \geq \mathcal{S}(M, d\mu) \cdot \mu(M_1)^{n-1}$. Therefore, $\mathcal{S}(M, d\mu) \leq 2\mathbb{I}(M, d\mu)$.

Given $f \in C^\infty$, let α_0 be a median of f. Set $M_1 := \{x : f(x) < \alpha_0\}$ and $M_2 := \{x : f(x) > \alpha_0\}$. Then $\mu(M_i) \leq \mu(M)/2$, for $i = 1, 2$. Let $h := f - \alpha_0$ and $h_i := h|_{M_i} \in C_c^\infty(M_i)$, $i = 1, 2$.

We claim that

$$\int_{M_2} F^*(dh_2)d\mu \geq \mathbb{I}(M_2)^{\frac{1}{n}} \|h_2\|_{n/(n-1)},$$

where $\mathbb{I}(M_i)$ is defined by

$$\inf_{\Omega} \frac{\min\{\mathrm{A}_{\pm}(\partial\Omega)\}^n}{\mu(\Omega)^{n-1}},$$

where Ω range over all open submanifolds of M_i with compact closures in M_i and smooth boundary. Clearly, $\mathbb{I}(M_i) \neq 0$.

Set $M_t := \{x : h_2(x) > t\}$. Since $\mu(M_t)$ is decreasing, we have

$$\begin{aligned}\frac{d}{ds}\left(\int_0^s \mu(M_t)^{\frac{n-1}{n}} dt\right)^{\frac{n}{n-1}} &= \frac{n}{n-1}\left(\int_0^s \mu(M_t)^{\frac{n-1}{n}} dt\right)^{\frac{1}{n-1}} \mu(M_s)^{\frac{n-1}{n}} \\ &\geq \frac{n}{n-1} s^{\frac{1}{n-1}} \mu(M_s),\end{aligned}$$

which implies

$$\left(\int_0^s \mu(M_t)^{\frac{n-1}{n}} dt\right)^{\frac{n}{n-1}} \geq \int_0^s \mu(M_t) dt^{\frac{n}{n-1}}.$$

Note that ∇h_2 is the inward normal vector field along ∂M_t. Thus,

$$\begin{aligned}\int_{M_2} F^*(dh_2)d\mu &= \int_0^\infty \mathrm{A}_+(\partial M_t)dt \geq \mathbb{I}(M_2)^{\frac{1}{n}} \int_0^\infty \mu(M_t)^{\frac{n-1}{n}} dt \\ \geq \mathbb{I}(M_2)^{\frac{1}{n}}\left(\int_0^\infty \mu(M_t) dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} &= \mathbb{I}(M_2)^{\frac{1}{n}}\left(-\int_0^\infty t^{\frac{n}{n-1}} d\mu(M_t)\right)^{\frac{n-1}{n}} \\ = \mathbb{I}(M_2)^{\frac{1}{n}}\left(\int_0^\infty t^{\frac{n}{n-1}} dt \int_{\partial M_t} \frac{d\,\mathrm{A}_{\nabla h_2}}{F^*(dh_2)}\right)^{\frac{n-1}{n}} &= \mathbb{I}(M_2)^{\frac{1}{n}}\left(\int_M h_2^{\frac{n}{n-1}} d\mu\right)^{\frac{n-1}{n}}.\end{aligned}$$

Likewise, one can show that $\int_{M_1} F^*(dh_1)d\mu \geq \mathbb{I}(M_1)^{\frac{1}{n}} \|h_1\|_{n/(n-1)}$. Since $\mu(M_i) \leq \mu(M)/2$, $\mathbb{I}(M_i) \geq \mathbb{I}(M, d\mu)$. Let χ_i be the characteristic function of M_i, $i = 1, 2$. Thus,

$$\begin{aligned}\int_M F^*(df)d\mu = \int_M F^*(dh)d\mu &= \sum_j \int_M F^*(d(\chi_j f))d\mu \\ &\geq \mathbb{I}(M, d\mu)^{\frac{1}{n}} \sum_j \left\{\int_M \chi_j |f - \alpha_0|^{\frac{n}{n-1}}\right\}^{\frac{n-1}{n}} \\ &\geq \mathbb{I}(M, d\mu)^{\frac{1}{n}} \|f - \alpha_0\|_{\frac{n}{n-1}} \geq \mathbb{I}(M, d\mu)^{\frac{1}{n}} \inf_{\alpha \in \mathbb{R}} \|f - \alpha\|_{\frac{n}{n-1}},\end{aligned}$$

which implies that $\mathcal{S}(M, d\mu) \geq \mathbb{I}(M, d\mu)$. $\square$

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