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Ricci almost solitons satisfying certain conditions on the potential vector field

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Abstract. We study Ricci almost soliton with certain conditions on the potential vector field V. First, we prove that a Ricci almost soliton is a Ricci soliton if and only if the potential vector field V is an infinitesimal harmonic transformation. Next, we study Ricci almost soliton satisfying some integral inequalities.

1. Introduction

In [11], Pigola *et al.* introduced and studied the notion of Ricci almost solitons on a Riemannian manifold, where they essentially modified the definition of Ricci solitons. A Riemannian manifold (M^n, g) is said to be a Ricci almost soliton if there exist a complete vector field V and a smooth soliton function $\lambda : M^n \to \mathbb{R}$ satisfying

$$(\pounds_V g)(Y, Z) + 2S(Y, Z) = 2\lambda g(Y, Z), \tag{1.1}$$

where S stands for the 2-covariant Ricci tensor, $\pounds_V$ denotes the Lie-derivative operator in the direction of the vector field V, and Y, Z are arbitrary vector fields on M. This is said to be *expanding* if $\lambda < 0$, *steady* if $\lambda = 0$, and *shrinking* if $\lambda > 0$. If λ is constant, then (1.1) represents the so called Ricci soliton. Since the soliton function λ is not necessarily constant, most of the results will differ as compared with the Ricci soliton (see [11]). The existence of non-compact Ricci almost soliton has been established by Pigola *et al.* [11] on some certain class

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of warped product manifolds. Using Lemma 2.1 of [11] we can construct the following example of a Ricci almost soliton.

Example 1.1. Let $M^{n+1} = \mathbb{R} \times_{\cosh t} \mathbb{H}^n$ with metric $g = dt^2 + (\cosh^2 t)g_0$, where g_0 is the standard metric on the hyperbolic space $\mathbb{H}^n$. Then M^{n+1} becomes Einstein manifold with Ricci tensor $S^M = -ng$. Consequently, $(M^{n+1}, g, \nabla f, \lambda)$ is a Ricci almost Ricci soliton with $f(x,t) = \sinh t$ and $\lambda(x,t) = \sinh t - n$. Actually, M^{n+1} is a realisation of the hyperbolic space.

Following Barros and Ribeiro Jr. [1], we can construct the following formal example of a compact Ricci almost soliton on the standard sphere $S^n(c)$.

Example 1.2. Let $S^n(c)$ be a sphere with its canonical metric g_0 considered as a hypersurface of $\mathbb{R}^{n+1}$. Let $V(x)$ be a vector field on S^n such that $V(x) = a^T(x)$, where $a \in \mathbb{R}^{n+1}$ is a constant vector. Therefore, $V(x) = \nabla h(x)$, where $h : S^n \to \mathbb{R}$ is a height function given by $h(x) = \langle x, a\rangle$. A straightforward computation shows that h is a non-trivial eigenfunction corresponding to the first eigenvalue of the Laplacian $-\Delta$ on S^n. Since S^n is Einstein, its Ricci tensor is given by $S = (n-1)g_0$. Thus, taking into account $\nabla^2 h = -hg_0$ it is easy to see that $(S^n(c), V, g_0)$ is a Ricci almost soliton with $\lambda = (n-1) - h$. Next, we shall show that V is non-zero on S^n. In fact, if $V \equiv 0$, then h becomes constant. But this contradicts the fact that h is a non-trivial eigenfunction.

Some characterizations of Ricci almost soliton on a compact Riemannian manifold in terms of integral inequalities were obtained by Barros–Ribeiro Jr. [1] and Barros–Batista-Ribeiro Jr. [2]. It is interesting to note that if the potential vector field V of the Ricci almost soliton (Mg, V, λ) is Killing, then the soliton becomes trivial (i.e. Einstein) provided the dimension of $M > 2$. Moreover, if V is a non-Killing conformal vector field on a compact manifold M^n with $n \geq 3$, then M^n is isometric to the Euclidean sphere S^n. Thus the Ricci almost soliton can be considered as a generalization of Einstein metric as well as Ricci soliton. In [13] Stepanov–Shelepova proved that *"The potential vector field of V of a Ricci soliton is necessarily an infinitesimal harmonic transformation"*. So the question may naturally arise,

What happens if the potential vector field V of a Ricci almost soliton generates an infinitesimal harmonic transformation?

The organization of this paper is as follows. In Section 2, we collect some basic definitions and fundamental formulas of infinitesimal harmonic transformations. Thereafter, we study Ricci almost solitons when its potential vector field V

generates an infinitesimal harmonic transformation. In this case, the Ricci almost soliton reduces to Ricci soliton. In Section 4, we prove some triviality results of Ricci almost soliton satisfying certain integral inequalities.

2. Preliminaries

A vector field V is an infinitesimal harmonic transformation on a Riemannian manifold (M, g) if the local 1-parameter group of infinitesimal point transformations generated by the vector field V forms a group of harmonic transformations(see [12], [6]). An analytic characterization of such vector field is given by the equation trace$(\pounds_V \nabla) = 0$. Moreover, an interesting characterization of such vector field was given by Stepanov–Shandra in [12]. They proved that
"A vector field V generates an infinitesimal harmonic transformation on a Riemannian manifold (M, g) if and only if $\Delta V = 2QV$".
The operator Δ is known as the Laplacian and determined by the Weitzenböck formula

$$\Delta V = \nabla^* \nabla V + QV,$$

where ∇^* is the formal adjoint of ∇ and Q is the Ricci operator associated with the $(0, 2)$ Ricci tensor S. The rough Laplacian $\bar{\Delta}$ of a vector field V is defined by $\bar{\Delta} V = -tr \nabla^2 V$, where

$$(\nabla^2 V)(X, Y) = \nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V.$$

Explicitely, if $\{e_i\}$ is any orthonormal frame field, then we have

$$\bar{\Delta} V = \sum_i \{\nabla_{\nabla_{e_i} e_i} - \nabla_{e_i} \nabla_{e_i}\} V. \tag{2.1}$$

It is well-known that $\bar{\Delta} V = \nabla^* \nabla V$ and therefore, $\Delta V = \bar{\Delta} V + QV$. The known examples of harmonic transformations are:

- Any Killing vector field on a Riemannian manifold generates an infinitesimal harmonic transformation (see [12]).
- The potential vector field V of the Ricci soliton is necessarily an infinitesimal harmonic transformation (see [13]).
- Since the Reeb vector field ξ of a K-contact manifold is Killing, it generates an infinitesimal harmonic transformation (see [9]).

3. Infinitesimal harmonic transformation and Ricci almost solitons

In [13], the authors proved that *"the potential vector field V of a Ricci soliton on a Riemannian manifold (M, g) is an infinitesimal harmonic transformation on M"*. Since Ricci almost soliton generalizes the notion of Ricci soliton we are interested to study Ricci almost soliton when its potential vector field generates an infinitesimal harmonic transformation. Precisely, we prove

Theorem 3.1. *Let (M^n, g, V), $n \geq 3$, be a Ricci almost soliton. Then g is a Ricci soliton if and only if the potential vector field V is an infinitesimal harmonic transformation.*

The proof of Theorem 3.1 relies on the following Lemma. Although this latter already appears in [2] (see equation (2.9) of Lemma 2), for the sake of completeness we provide an independent proof which makes use of the Lie-derivative technique.

Lemma 3.1. *For a Ricci almost soliton (M^n, g, V), we have $\bar{\Delta}V - QV = (n-2)D\lambda$, where $D\lambda$ denotes the gradient of λ.*

Proof. First, assume that g is a Ricci almost soliton. Then taking covariant differentiation of (1.1) along X gives

$$(\nabla_X \pounds_V g)(Y,Z) + 2(\nabla_X S)(Y,Z) = 2(X\lambda)g(Y,Z). \tag{3.1}$$

Now the commutation formula (see Yano [14], p. 23)

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y), Z) - g((\pounds_V \nabla)(X,Z), Y),$$

reduces to

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y), Z) + g((\pounds_V \nabla)(X,Z), Y). \tag{3.2}$$

Making use of (3.1) in (3.2) gives

$$g((\pounds_V \nabla)(X,Y), Z) + g((\pounds_V \nabla)(X,Z), Y) + 2(\nabla_X S)(Y,Z) = 2(X\lambda)g(Y,Z). \tag{3.3}$$

Permuting (3.3) cyclically twice over $\{X, Y, Z\}$ and then taking their difference we obtain

$$g((\pounds_V \nabla)(Y,X), Z) - g((\pounds_V \nabla)(Z,X), Y) + 2(\nabla_Y S)(Z,X) - 2(\nabla_Z S)(X,Y) = 2(Y\lambda)g(X,Z) - 2(Z\lambda)g(X,Y).$$

Adding the foregoing equation with (3.3) and noting that $(\pounds_V\nabla)(X,Y) = (\pounds_V\nabla)(Y,X)$, we deduce

$$g((\pounds_V\nabla)(X,Y),Z) = (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) \\ -(\nabla_Y S)(X,Z) + (X\lambda)g(Y,Z) + (Y\lambda)g(X,Z) - (Z\lambda)g(X,Y). \quad (3.4)$$

Using the formula (see p. 23 of [14])

$$g((\pounds_V\nabla)(X,Y),Z) = g(\nabla_X\nabla_Y V - \nabla_{\nabla_X Y}V - R(X,V)Y,Z)$$

in the above equation we obtain

$$g(\nabla_X\nabla_Y V - \nabla_{\nabla_X Y}V - R(X,V)Y,Z) = (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) \\ -(\nabla_Y S)(X,Z) + (X\lambda)g(Y,Z) + (Y\lambda)g(X,Z) - (Z\lambda)g(X,Y). \quad (3.5)$$

Let $\{e_i : i = 1,2,\dots,n\}$ be an orthonormal frame of the tangent space of M. Then setting $X = Y = e_i$ and using the definition of rough Laplacian it follows that

$$-\bar{\Delta}V + QV = -(n-2)D\lambda. \quad (3.6)$$

This completes the proof. □

Proof of Theorem 3.1. First, assume that V generates an infinitesimal harmonic transformation. Then, by [12], [13] $\Delta V = 2QV$. Therefore, by applying Weitzenböck's formula :

$$\Delta V = \bar{\Delta}V + QV, \quad (3.7)$$

we have $\bar{\Delta}V = QV$. Utilizing this in (3.6) we see that λ is constant. Hence, the Ricci almost soliton becomes a Ricci soliton. The converse was proved in [13]. □

4. Triviality results under integral inequalities

In [1], the authors proved that a compact Ricci almost soliton (M^n, g, V), $n > 2$ is trivial if it satisfies $\int_M\{S(V,V) + (n-2)g(D\lambda,V)\}dM \le 0$. This result extends the corresponding result of Petersen–Wylie [10] for compact Ricci solitons. Here we present another generalization of Petersen–Wylie's result satisfying an integral inequality on the Ricci tensor which is slightly different from that of Barros–Ribeiro Jr. [1].

Theorem 4.1. *Let (M^n, g, V), $n \geq 2$ be a compact Ricci almost soliton with non-zero potential vector field V. If it satisfies $\int_M \{2S(V,V) + (n-2)g(D\lambda, V)\}dM \leq 0$, then the 1-form associated to V is harmonic and M is Ricci flat.*

Proof. First, we recall the following integral formula (see Yano [14], p. 41)

$$\int_M \left\{ g(\triangle V, V) - \frac{1}{2}|dV^\flat|^2 - (\operatorname{div} V)^2 \right\} dM = 0,$$

where $V^\flat$ is the 1-form associated with the potential vector field V. Now, using (3.6) and (3.7) we get

$$\triangle V - (n-2)D\lambda = 2QV. \tag{4.1}$$

Making use of (4.1) the foregoing integral formula reduces to

$$\int_M \{2S(V,V) + (n-2)g(D\lambda, V)\}dM = \int_M \left\{ \frac{1}{2}|dV^\flat|^2 + (\operatorname{div} V)^2 \right\} dM. \tag{4.2}$$

By hypothesis the left hand side integral of (4.2) is less or equal to zero, while the integrand on the right hand side integral of (4.2) is greater or equal to zero. Hence, we have

$$\int_M \left\{ \frac{1}{2}|dV^\flat|^2 + (\operatorname{div} V)^2 \right\} dM = 0.$$

This implies $dV^\flat = 0$ and $\operatorname{div} V = 0$. Consequently $V^\flat$ is harmonic. So, we can integrate the Bochner formula (which holds for any harmonic 1-form) to obtain

$$\int_M \{S(V,V) + |\nabla V^\flat|^2\} dM = \int_M \frac{1}{2}\Delta |V^\flat|^2 dM = 0. \tag{4.3}$$

Since V is divergence free, $\operatorname{div}(\lambda V) = g(D\lambda, V) + \lambda(\operatorname{div} V) = g(D\lambda, V)$. Integrating this over M we achieve $\int_M g(D\lambda, V)dM = 0$. On the other hand, from the harmonicity of $V^\flat$ and (4.2), it follows

$$\int_M \{2S(V,V) + (n-2)g(D\lambda, V)\}dM = 0.$$

Therefore $\int_M S(V,V)dM = 0$ and using this information into (4.3) proves that V is parallel. Thus, equation (1.1) shows that M is Einstein (i.e. $S = \lambda g$) and hence λ is constant. Since $\nabla V = 0$, we have $R(X,Y)V = 0$, and hence $S(X,V) = 0$. From which it follows that $\lambda|V|^2 = 0$. Since λ is constant and $|V|$ is a non-zero constant, we have $\lambda = 0$. Hence, M is Ricci flat. This completes the proof. □

Finally, using Lemma 2 of [1] we prove the following.

Proposition 4.1. *Let* (M^n, g, V)*,* $n > 2$ *be a compact Ricci almost soliton with non-zero potential vector field* V*. If it satisfies* $\int_M \{g(\triangle V, V) + (n-2)g(D\lambda, V)\}dM \leq 0$*, then* M *is Ricci flat.*

Proof. Indeed, using (4.1) and equation (1) of Lemma 2 of [1] yields

$$\int_M \{g(\triangle V, V) + (n-2)(g(D\lambda, V)\}dM$$
$$= 2\int_M \{S(V,V) + (n-2)g(D\lambda, V)\}dM = 2\int_M g(\nabla V, \nabla V)dM.$$

Thus, making use of our hypothesis it follows that the potential vector field V is parallel. Hence M is Einstein. The rest of the proof follows from the last Theorem. This completes the proof. □

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