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A note on non-recurring sequences over Galois fields

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Introduction

Let F be a Galois field and $\Gamma(F)$ be the $F[D]$–module of all sequences over F, [4]. Consider an $f(D) \neq 0$ in $F[D]$. The concept of a pseudo-periodic sequence with $f(D)$ as its pseudo-characteristic polynomial was introduced in [1]. The set $\bar{\Omega}(f(D))$ of all such sequences is a submodule of $\Gamma(F)$. It was seen in [1] that in the study of the lattice $L(F)$ of these $\bar{\Omega}(f(D))$'s, $\bar{\Omega}(1)$ plays a very significant role. The possibilities of different kinds of non-recurring sequences, that one can have inside $\bar{\Omega}(1)$ seem to be too many. In the present note we give an explicit construction of an uncountable direct sum $K = \oplus \sum_\alpha F[D]S_\alpha$ and its closure in any torsion-free submodule of $\bar{\Omega}(1)$. Let E be an injective hull of $\bar{\Omega}(1)$. By using K, it is proved that for any monic irreducible polynomial $p(D) \neq D$, in $F[D]$, the $p(D)$–primary component of $E/\bar{\Omega}(1)$ is of uncountable rank. Finally we discuss the concept of a minimal sparse set associated with a member of $\bar{\Omega}(1)$.

§1. Preliminaries

Throughout F is a Galois field and $\Gamma(F)$ denotes the F–vector space of all sequences $S = (s_n)_{n\geq 0}$ over F. Consider the ring of polynomials $F[D]$, where D is an indeterminate. For any $f(D) = \sum_{i=0}^{k} a_i D^i \in F[D]$, $S = (s_n) \in \Gamma(F)$, define $f(D)S = (w_n) \in \Gamma(F)$ such that $w_n = \sum_i a_i s_{n+i}$. This makes $\Gamma(F)$ a divisible left $F[D]$–module, [4]. For any $S = (s_n) \in \Gamma(F)$ the power series $G_s(x) = \sum_{i=0}^{\infty} s_i x^i$, with x an indeterminate, is called the generating function of S, [3]. For any $f(x) = a_0 + a_1 x + \cdots + a_k x^k$, $a_k \neq 0$, the reciprocal of $f(x)$, is the polynomial $\bar{f}(x) = a_k + a_{k-1}x + \cdots + a_0 x^k$. If $f(x) = x^u g(x)$, with $g(0) \neq 0$, it is obvious that $\bar{f}(x) = \bar{g}(x)$. The definition of $\Gamma(F)$ as an $F[D]$–module gives the following.

Lemma 1.1. (a). *For any* $S,\ S' \in \Gamma(F)$ *and* $f(D) \in F[D]$ *with* $\deg f(D) = k \geq 0,\ f(D)S = S'$ *if and only if* $x^k G_{s'}(x) = g(x) + \bar{f}(x)G_s(x)$ *for some* $g(x) \in F[x]$ *with* $\deg g(x) < k$.

(b) *For any* $f(x) \in F[x]$ *with* $\deg f(x) = k \geq 0$, *and* $S \in \Gamma(F)$ $f(x)G_s(x) = \sum_{i=0}^{k-1} c_i x^i + x^k G_{s'}(x)$ *for some* $c_i \in F$, *and* $S' \in \Gamma(F)$ *such that* $\bar{f}(D)S = S'$.

Consider any two sequences $S = (s_n)$ and $S' = (s'_n)$ in $\Gamma(F)$. Throughout the paper let N denotes the set of natural numbers. For $m<n$, $[s_m, s_n]=(s_m, s_{m+1}, \ldots, s_n)$ is called a section of S of length $n-m$ [1]. For the sake of convenience, sometime we denote the $(m-n-1)$–tuple $(s_{m+1}, s_{m+2}, \ldots, s_{n-1})$ by $]s_m, s_n[$ and call it an open section of length $n-m$. Indeed $]s_m, s_n[\ = [s_{m+1}, s_{n-1}]$. A section of the form $[s_0, s_n]$ is called an initial section. $[s_m, s_n] = 0$ will mean that $s_t = 0$ for $m \leq t \leq n$. S and S' are said to be ultimately equal if $D^r S = D^r S'$ for some $r \geq 0$. Consider an $f(D) \in F[D]$ with $\deg f(D) = k \geq 0$. Then $\Omega(f(D)) = \{S \in \Gamma(F) : f(D)S = 0\}$, [4]. $O(f(D))$ denotes the order of $f(D)$, [3]. Observe that in $f(D)S = S'$, $[s_m, s_{n+k}]$ determines $[s'_m, s'_n]$ and conversely. If an $S \in \Omega(f(D))$ has minimal polynomial $f(D)$, then for any $g(D) \neq 0$ in $F[D]$, the minimal polynomial of $g(D)S$ is $f(D)/d(D)$, where $d(D) = \gcd(f(D)), g(D))$. These observations give the following two lemmas essentially mentioned in [1].

Lemma 1.2. *Let* $S = (s_n) \in \Gamma(F)$ *and* $f(D) = D^t g(D) \in F[D]$ *with* $t \geq 0$, $\deg f(D) = k$, *and* $g(D) \in F[D]$ *with* $g(0) \neq 0$. *Further let* $f(D)S = S' = (s'_n)$. *Suppose that for some* $n - m > \deg f(D) + 1,\ [s_{m+1}, s_{n-1}] = 0$. *Then the following hold:*

(i) $[s'_{m+1}, s'_{n-1-k}] = 0$

(ii) *If* $s_m \neq 0$, *then* $s'_u \neq 0$, *for* $m - t \leq u \leq m$

(iii) *If* $s_n \neq 0$, *then* $s'_{n-t} \neq 0$.

Lemma 1.3. *Let* $S = (s_n) \in \Gamma(F)$ *and* $f(D)S = S' = (s'_n)$ *for some* $f(D) \in F(D)$ *with* $\deg f(D) = k$. *Further let* $0 \neq g(D) \in F[D]$, *and* $n > m \geq 0$. *Then the following hold:*

(i) *If* $[s'_m, s'_n]$ *is a section of a member of* $\Omega(g(D))$, *then* $[s_m, s_{n+k}]$ *is a section of a member of* $\Omega(f(D)g(D))$; *in particular if* $[s'_m, s'_n] = 0$, *then* $[s_m, s_{n+k}]$ *is a section of a member of* $\Omega(f(D))$.

(ii) *If* $n - m \geq \deg f(D)$, $g(D)$ *is monic with* $g(0) \neq 0$, *and* $[s_m, s_n]$ *is a section of a* $T \in \Gamma(F)$ *with minimal polynomial* $g(D)$, *then* $[s'_m, s'_{n-k}]$ *is a section of* $g(D)T$ *with minimal polynomial* $f(D)/d(D)$ *where* $d(D) = \gcd(f(D),\ g(D))$.

For any $f(D) \neq 0$ in $F[D]$, $\Omega(f(D)^\infty)$ denotes the submodule $U_{n \geq 0}$ $\Omega(f(D)^n)$. As remarked in [4], $\Omega(f(D)^\infty)$ is the smallest divisible submodule of $\Gamma(F)$ containing $\Omega(f(D))$. For the definition of a sparse subset of N, and of a pseudo-periodic sequence with pseudo-characteristic polynomial $f(D) \neq 0$, we refer to [1]. $\bar{\Omega}(f(D))$ denotes the submodule of $\Gamma(F)$ consisting of all the pseudo-periodic sequences with $f(D)$ as their pseudo-charasteristic polynomial. $\bar{W}(F)$, the set of all pseudo-periodic sequences in $\Gamma(F)$ is a divisible submodule of $\Gamma(F)$ and it contains $W(F)$ the divisible submodule of all ultimately periodic sequences, [1]. Any member of $\bar{\Omega}(1)$ is called an almost zero sequence. The following was proved in [1].

Theorem 1.4. (i). *$\bar{\Omega}(1) = \Omega(D^{\infty}) \oplus L$, where L is a torsion-free $F[D]$–module*

(ii) *$\bar{W}(F) = W(F) + E$, where E is any injective hull of $\bar{\Omega}(1)$ in $\Gamma(F)$.*

(iii) *$\bar{\Omega}(1)$ is divisible by D.*

For any sparse set A, given $n, m \in A$, n is called a successor of m in A if $m < n$ and there is no $p \in A$ such that $m < p < n$, a finite sequence $a_1 < a_2 < \cdots < a_t$ in A is called a successor sequence in A, if each a_{i+1} is a successor of a_i in A.

§2. The module $\bar{\Omega}(1)$

Proposition 2.1. *Let $S = (s_n) \in \Gamma[D]$, and $f(D) \in F[D]$, with $f(0) \neq 0$. If for each positive integer K, there exist $n, m \in N$ with $n - m \geq K$ such that $[s_m, s_n]$ is a section of some non-zero member of $\Omega(f(D))$, then $S \notin \bar{\Omega}(1)$.*

Proof. On the contrary suppose that $S \in \bar{\Omega}(1)$. Let $(A, u, 1)$ be a companion of S and let $t =$ sparsity (A), [1]. Consider any integer $w > \max(O(f(D)), u)$. By definition there exists $a \in A$ such that for $a \leq a_1 < a_2 < \cdots < a_t$ in A, $a_t - a_1 > wt$. The hypothesis on S gives n, m such that $m \geq a$, $n - m > (w+1)t$ and that $[s_m, s_n]$ is a section of a non-zero $S' \in \Omega(f(D))$. Let $b \in A$ be largest such that $b \leq m$. Consider the successor sequence in A,

$$b = b_0 < b_1 < b_2 < \cdots < b_t\,.$$

Clearly $m < b_1$. Suppose that $b_1 - m \geq w$. As the period of S' is a factor of $O(f(D))$ and $[s_m, s_{b_1}]$ is a section of S' of length $b_1 - m > O(f(D))$ we get $[s_m, s_{b_1}] \neq 0$. So that $[s_{b_0}, s_{b_1}] \neq 0$. However $b_1 - b_0 > u$, gives $[s_{b_0}, s_{b_1}] = 0$. This is a contradiction. Hence $b_1 - m < w$. Now $b_t - b_1 > wt$. So we get smallest $i \geq 1$ such that $i < t$ and $b_{i+1} - b_i > w$. Then $[s_{b_i}, s_n] \cap [s_{b_i}, s_{b_{i+1}}]$ is a section of S' of length greater than w and hence it is non-zero. Consequently $[s_{b_i}, s_{b_{i+1}}] \neq 0$. As $b_{i+1} - b_i > w > u$, we get $[s_{b_i}, s_{b_{i+1}}] = 0$. This is a contradiction. Hence $S \notin \bar{\Omega}(1)$.

The following is well known

Lemma 2.2. *There exist an uncountable family* $\{\mathbb{N}_\alpha\}_{\alpha\in\Lambda}$ *of subsets of* $\mathbb{N}$*, the set of natural numbers such that:*

(i) *Far each* $\alpha \in \Lambda$, $\mathbb{N}_\alpha$ *and* $\mathbb{N} - \mathbb{N}_\alpha$ *are countably infinite.*

(ii) *For any two distinct* $\alpha, \beta \in \Lambda$, $\mathbb{N}_\alpha \cap \mathbb{N}_\beta$ *is finite.*

Proof. As $\mathbb{Q}$, the set of rational numbers is countable we replace $\mathbb{N}$ by $\mathbb{Q}$. For each real number α, fix a sequence $\mathbb{Q}_\alpha = \{a_n\}_{n\in\mathbb{N}}$ of rational numbers such that $a_n > a_{n+1}$ for every n, and $\alpha = \lim_{n\to\infty} a_n$. Then $\{\mathbb{Q}_\alpha\}_{\alpha\in\Lambda}$, where Λ is the set of all real numbers, is a desired family.

We now fix an $S = (s_n) \in \Gamma(F)$, such that

$$G_s(x) = \sum_{n\geq 1} x^{\lambda_n}(1 - x^{n!})$$

with $\lambda_1 = 0$, $\lambda_{n+1} - \lambda_n > 2(n!) + 4$. Observe the following

Lemma 2.3. (i) $s_{\lambda_n} = 1$, $s_{\lambda_n + n!} = 1$ *and* $s_k = 0$*, otherwise.*

(ii) *The open section* $]s_{\lambda_n+n!}, s_{\lambda_{n+1}}[= 0$ *and that* $\lambda_{n+1} - (\lambda_n + n!)$ *approaches infinity as* n *approaches infinity.*

We now fix a family $\{\mathbb{N}_\alpha\}_{\alpha\in\Lambda}$ given by (2.2). By using the S fixed above, for each $m \geq 1$, and $\alpha \in \Lambda$, we define $S_{\alpha,m}$ in $\Gamma(F)$ such that

$$G_{s_{\alpha,m}}(x) = \sum_n x^{\lambda_n}(1 - x^{n!}), \quad n \geq m \ \text{ and } \ n \in \mathbb{N}_\alpha\,.$$

Further define $\bar{S}_{\alpha,m}$ such that

$$G_{\bar{S}_{\alpha,m}}(x) = \frac{1}{1 - x^m} G_{s_{\alpha,m}}(x)$$

$$\text{i.e. } \ (1 - x^m) G_{\bar{S}_{\alpha,m}} = G_{s_{\alpha,m}}(x)\,.$$

For $m \geq 2$, by (1.1) (b) $(D^m - 1)\bar{S}_{\alpha,m}$ has generating function $\sum x^{\lambda_n - m}(1 - x^{n!})$, $n \in \mathbb{N}_\alpha$. So that $(D^m - 1)\ \bar{S}_{\alpha,m} = D^m S_{\alpha,m}$. For $m \geq n$, as $(1 - x^m)$ divides $(1 - x^{n!})$, we get

$$\frac{x^{\lambda_n}(1 - x^{n!})}{1 - x^m} = x^{\lambda_n}[1 + x^m + x^{2m} + \cdots + x^{r_n m}]$$

where $r_n = [(n!)/m - 1]$. This corresponds to a section of a non-zero member of $\Omega(D^m - 1)$. Because $r_n \to \infty$, as $n \to \infty$, by (2.1). $\bar{S}_{\alpha,m} \notin \bar{\Omega}(1)$, we write $S_\alpha = S_{\alpha,1}$. We collect these observations and their immediate consequences in the following.

Lemma 2.4. (i) $S_{\alpha,m} \in \bar{\Omega}(1),\ \bar{S}_{\alpha,m} \notin \bar{\Omega}(1),\ \bar{S}_{\alpha,m} \in \bar{\Omega}(D^m - 1)$.

(ii) *For* $m \geq 2,\ D^{\lambda_m} S_\alpha = D^{\lambda_m} S_{\alpha,m}$ *and* $(D^m - 1)\ \bar{S}_{\alpha,m} = D^m S_{\alpha,m}$.

(iii) *Given any finitely many distinct members* $\alpha_1, \alpha_2, \ldots, \alpha_u$ *of* Λ *and any positive integer* K*, there exists* $n \in \mathbb{N}_{\alpha_1}$ *such that* $n \notin \mathbb{N}_{\alpha_j}$ *for* $j \geq 2,\ n! \geq K$ *and* $\lambda_{n+1} - \lambda_n - n! \geq K$.

The following result gives an explicit uncountable direct sum in $\bar{\Omega}(1)$.

Lemma 2.5. $K = \sum_\alpha F[D] S_\alpha$ *is a direct sum in* $\bar{\Omega}(1)$

Proof. Let for some distinct $\alpha_i \in \Lambda$ and $f_i(D) \in F[D]$,

$$\sum_{i=1}^{u} f_i(D) S_{\alpha_i} = 0\,.$$

Suppose that $f_1(D) \neq 0$. Let k be a positive integer greater than every $\deg f_i(D)$. We get an $n \in \mathbb{N}_{\alpha_1}$ such that $n > k,\ n \notin \mathbb{N}_{\alpha_j}$ for $j \geq 2$ and $\lambda_n - \lambda_{n-1} - (n-1)! > K$. The section of S_{α_1} indexed by $[\lambda_n, \lambda_n + n!]$ is of the form $(1, 0, 0, \ldots, 1)$ and that indexed by $]\lambda_{n-1} + (n-1)!,\ \lambda_n[$ is zero. For $j \geq 2$, the corresponding sections of S_{α_j} are all zeros. As $f_1(D) \neq 0$, by (1.2), the section of S_{α_1} indexed by $[\lambda_n - k,\ \lambda_n + n!]$ is non zero, but the corresponding sections of any $S_{\alpha_j},\ j \geq 2$ are zeros. This then contradicts the fact that $\sum_i f_i(D) S_{\alpha_i} = 0$. Hence $f_1(D) = 0$. Similarly every $f_i(D) = 0$. This proves the result.

Theorem 2.6. *For any* $T \in \bar{\Omega}(1)$ *and* $f(D) \in F[D]$ *with* $f(0) \neq 0$ *and* $\deg f(D) = k > 0$*, if* $f(D)T \in \sum_\alpha F[D] S_\alpha$*, then* $D^\ell T \in \sum_\alpha F[D] S_\alpha$ *for some* $\ell \geq 0$.

Proof. As $f(0) \neq 0$, we get an integer $m \geq 2$ such that $f(D)$ divides $(D^m - 1)$. Write $D^m - 1 = f(D) g(D)$. By the hypothesis $f(D)T = \sum_{i=1}^{u} f_i(D) S_{\alpha_i}$ for some distinct $\alpha_i \in \Lambda,\ f_i(D) \in F[D]$. Then by (2.4)

$$D^{\lambda_m + m} f(D) T = \sum_i f_i(D) D^{\lambda_m + m} S_{\alpha_{i,m}} =$$

$$= f(D) g(D) \sum_i f_i(D) D^{\lambda_m} \bar{S}_{\alpha_{i,m}}\,.$$

Consequently

$$D^{\lambda_m + m} T = \sum_i f_i(D) g(D) D^{\lambda_m} \bar{S}_{\alpha_{i,m}} + T' \tag{1}$$

where $T' \in \Omega(f(D))$. All the $\bar{S}_{\alpha_{i,m}}$ have common-zero sections indexed by open intervals $]\lambda_n + n!,\ \lambda_{n+1}[$ which are of arbitrarily large lengths.

Consequently by (1.2), all the $f_i(D)g(D)\bar{S}_{\alpha_{i,m}}$ have commonly indexed zero sections of arbitrarily large lengths. If $T' \neq 0$, (1) gives that T has sections of arbitrarily large lengths, which are sections of $T' \in \Omega(f(D))$. By (2.1) $D^{\lambda_m+m}T \notin \bar{\Omega}$ (1). This gives a contradiction. Hence $T' = 0$, i.e.

$$D^{\lambda_m+m}T = \sum_i f_i(D)g(D)D^{\lambda_m}\bar{S}_{\alpha_{i,m}} \tag{2}$$

Fix a positive integer s greater than every $\deg(f_i(D)g(D))$, $O(f_i(D)g(D))$. By using (2.2) we choose an $n \in \mathbb{N}_{\alpha_1}$ such that $n > m$, $r_n m > 2s$, $\lambda_n - (\lambda_{n-1} + (n-1)!) > s$, $r_n = (n!/m) - 1$ and $n \notin \mathbb{N}_{\alpha_j}$ for $j \geq 2$. Now $x^{\lambda_n}\,(1 + x^m + \cdots + x^r n^m)$ corresponds to the sections B of $\bar{S}_{\alpha_{1,m}}$ indexed by $[\lambda_n,\, \lambda_n + r_n m]$. It is a section of a member $T'' \in \Omega(D^m - 1)$ with $D^m - 1$ as its minimal polynomial. Suppose $(D^m - 1)$ does not divide $f_1(D)g(D)$. Then $f_1(D)g(D)T'' \neq 0$, and by (1.3) the section of $f_1(D)g(D)S_{\alpha_{1,m}}$ indexed by $[\lambda_n,\, \lambda_n + r_n m - s]$ is a section of $f_1(D)g(D)T''$. As its length is greater than s it is non-zero. However the corresponding section of every $f_j(D)g(D)\bar{S}_{\alpha_j,m}$, $j \geq 2$ is zero. So that $D^{\lambda_m+m}T$ has a section, which is a section of $f_1(D)g(D)T''$; by (2.2) n can be choosen as large as desired. So by (2.1) $D^{\lambda_m+m}T \notin \bar{\Omega}(1)$. This is a contradiction. Hence $(D^m - 1)$ divides $f_1(D)g(D)$. Write $f_1(D)g(D) = (D^m - 1)\; g_1(D)$. Similarly we get $f_j(D)g(D) = (D^m - 1)\; g_i(D)$, $g_i(D) \in F[D]$. Hence

$$D^{\lambda_m+m}T = \sum_i g_i(D)D^{m+\lambda_m}S_{\alpha_{i,m}} =$$

$$= \sum_i g_i(D)D^{m+\lambda_m}S_{\alpha_i} \in \sum_\alpha F[D]S_\alpha\,.$$

This proves the result.

Given submodule L of a torsion-free $F[D]$–module M, the closure $\mathrm{cl}(L)$ of L in M is the set of those $x \in M$ such that $f(D)x \in L$ for some $f(D) \neq 0$ in $F[D]$. Let $K = \sum_\alpha F[D]S_\alpha$. In general closure of a direct sum is not equal to the direct sum of closures. Here we explicity describe the closure of K, in any torsion-free submodule of $\bar{\Omega}(1)$. By using (1.4) and the fact that $\bar{W}(F)$ is injective, we get $\bar{\Omega}(1) = \Omega(D^\infty) \oplus L'$ with $K \subseteq L'$.

Theorem 2.7. *Let $K = \oplus \sum_\alpha F[D]S_\alpha$ and L' be a submodule of $\bar{\Omega}(1)$ containing K such that $\bar{\Omega}(1) = \Omega(D^\infty) \oplus L'$. Then $\mathrm{cl}_{L'}(K) = \{T \in L' : D^\lambda T \in K$ for some $\lambda \geq 0\}$ and $\mathrm{cl}_{L'}(K) = \oplus \sum_\alpha \mathrm{cl}_{L'}(F[D]S_\alpha)$.*

Proof. The first part is an immediate consequence of (2.6) and the remark just above this theorem. We have an injective hull E of $\bar{\Omega}(1)$ in $\bar{W}(F)$ such that $E = \Omega(D^\infty) \oplus E_1 \oplus E_2$ where $E_1 \oplus E_2$ and E_1 are injective hulls of L' and K respectively in E. As $F[D]$ is noetherian, we have $E = \oplus \sum E_\alpha$, where E_α is the injective hull of $L_\alpha = F[D]S_\alpha$ in E_1. It follows from (2.6) that $\mathrm{cl}_{L'}(K)/K$ is the D–primary component of E_1/K. However $E_1/K \cong \oplus \sum_\alpha E_\alpha/L_\alpha$. So that $\mathrm{cl}_{L'}(K)/K$ is direct sum of D–primary components of E_α/L_α. This in view of (2.6) yields.

$$\mathrm{cl}_{L'}(K) = \oplus \sum \mathrm{cl}_{L'}(F[D]S_\alpha).$$

Theorem 2.8. *Let E be any injective hull of $\bar{\Omega}(1)$ in $\bar{W}(F)$. Then for any monic irreducible polinomial $p(D) \neq D$, in $f[D]$, the $p(D)$–primary component of $E/\bar{\Omega}(1)$ is of uncountable rank.*

Proof. We have $E = \Omega(D^\infty) \oplus E_1 \oplus E_2$ where $E_1 \oplus E_2$ is an injective hull of a submodule L' of $\bar{\Omega}(1)$ containing K such that $\bar{\Omega}(1) = \Omega(D^\infty) \oplus L'$ and E_1 is an injective hull of K in E. Let $K_\alpha = cl_{L'}(F[D]S_\alpha)$. Then for $K' = \mathrm{cl}_{L'}(K)$, $K' = E_1 \cap \bar{\Omega}(1)$ and

$$[E_1 + \bar{\Omega}(1)]/\bar{\Omega}(1) \cong \oplus \textstyle\sum_\alpha E_\alpha/K_\alpha.$$

Consider the quotient field $\mathbb{Q}$ of $F[D]$ and $M = \{a \in \mathbb{Q} : D^\lambda a \in F[D] \text{ for some } \lambda \geq 0\}$. Then $M/F[D]$ is the D–primary component of $\mathbb{Q}/F[D] \cong E_\alpha/F[D]S_\alpha$. For any monic irreducible polynomial $p(D) \neq D$ over F, the $p(D)$–primary component of $\mathbb{Q}/F[D]$ is isomorphic to that of $\mathbb{Q}/M$. However $\mathbb{Q}/M \cong E_\alpha/K_\alpha$. As $|\Lambda|$ is uncountable, and $\mathbb{Q}/M$ has non-zero $p(D)$–primary component, the result follows.

We end this paper by discussing the notion of minimal sparse sets associated with members of $\bar{\Omega}(1)$. Consider two sparse sets A and A'. Call A and A' to be equivalent sparse sets, if there exist finite subsets B and B' of A and A' respectively such that $A \setminus B = A' \setminus B'$. This is an equivalence relation. Consider any $S \in \bar{\Omega}(1)$ that is not ultimately zero. Let $(A, u, 1)$ be a companion of S. Then A is said to be a minimal sparse set relative to u, associated with S if for any sparse set $A' \subseteq A$, such that $(A', u, 1)$ is a companion of S, A' is equivalent to A.

Theorem 2.9. *Let S be any member of $\bar{\Omega}(1)$ that is not ultimately zero. Let $(A, u, 1)$ be a companion of S. Then there exists a sparse set $A' \subseteq A$ such that A' is a minimal sparse set relative to u, associated with S.*

Proof. Let $S = (s_k)$. We now construct $A' = (m_i)$, $m_i < m_{i+1}$. Choose m_0 any member of A. For some $i \geq 0$, suppose that we have already constructed.

$$m_0 < m_1 < \ldots, m_i$$

in A'. Let n be the successor of m_i in A. If $[s_{m_i}, s_n] \neq 0$, by definition $n - m_i < u$; in this case choose m_{i+1}, the largest member of A such that $m_{i+1} - m_i < u$. Let $[s_{m_i}, s_n] = 0$. As S is not ultimately zero we can find largest member of A, that is taken as m_{i+1}, such that $[s_{m_i}, s_{m_{i+1}}] = 0$. It is immediate that $(A', u, 1)$ is a companion of S. Let $(A'', u, 1)$ be a companion of S such that $A'' \subseteq A'$. We get smallest integer r, such that $m_r \in A''$. We prove that A'' consists of all $m_i,\ i \geq r$. Suppose for some $i \geq r$, we have already proved that $m_i \in A''$. Let m be the successor of m_i in A''. Then $m_i < m_{m_{i+1}} \leq m$. If $[s_{m_i}, s_{m_{i+1}}] \neq 0$ then $[s_{m_i}, s_m] \neq 0$, gives $m - m_i < u$. As m_{i+1} is the largest member of A satisfying $m_{i+1} - m_i < u$, we get $m = m_{i+1}$. If $[s_{m_i}, s_m] = 0$, once again the choice of m_{i+1} gives $m = m_{i+1}$. This proves the result.

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References

[1] H. Al-Zaid and S. Singh, A class of non-recurring sequences over a Galois Field, (*to appear in Publicationes Mathematicae Debrecen*).

[2] C. Faith, Algebra II, Ring Theory, vol. 91, *Springer Verlag*, 1976.

[3] R. Lidl and H. Niederreiter, Finite Fields, Encyclopedia Math. Appl., *Addison Wesley*, 1983.

[4] S. Singh, A note on linear recurring sequences, *Linear Algebra Appl.* **104** (1988), 97–101.

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