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On some Hardy type inequalities involving generalized means

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Abstract. We discuss properties of a natural generalization of Power Means proposed in 1971 by Carlson, Meany and Nelson. For a parameters $k \in \mathbb{N}$; $s, q \in \mathbb{R}$ and a vector $v \in (0, +\infty)^n$, $n \geq k$ they are defined by $\mathcal{P}_s(\mathcal{P}_q(v_{i_1}, \dots, v_{i_k}) : 1 \leq i_1 < \dots < i_k \leq n)$ ($\mathcal{P}_s$ denotes the s-th power mean).

We determine when these means satisfy inequalities resembling the classical Hardy inequality within a large part of the parameter space. Moreover we point out a Hardy mean for which an arbitrarily small increment of the parameter q leads to means not being Hardy.

1. Introduction

The most popular family of means encountered in the literature consists of Power Means. They are parametrized by $p \in \mathbb{R} \cup \{\pm\infty\}$ and, for any all-positive-components vector $(v_1, \dots, v_n)$, $n \in \mathbb{N}$, are defined by

$$\mathcal{P}_p(v_1, \dots, v_n) := \begin{cases} \min(v_1, \dots, v_n) & \text{if } p = -\infty, \\ \left(\dfrac{1}{n} \displaystyle\sum_{i=1}^n v_i^p \right)^{1/p} & \text{if } p \neq 0, \\ \left(\displaystyle\prod_{i=1}^n v_i \right)^{1/n} & \text{if } p = 0, \\ \max(v_1, \dots, v_n) & \text{if } p = +\infty. \end{cases}$$

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Upon putting $p = 1, 0, -1$ one gets Arithmetic, Geometric, and Harmonic Mean, respectively.

Power Means have been investigated ever since their conception in the 19th century. One of the classical results concerning them is the following inequality, in its final form reproduced below due to LANDAU, [9]:

$$\sum_{n=1}^{\infty} \mathcal{P}_p(a_1, \dots, a_n) < (1-p)^{-1/p} \|a\|_1 \text{ for } p \in (0,1) \text{ and } a \in l_1(\mathbb{R}_+).$$

The constant $(1-p)^{-1/p}$ in the above inequality cannot be diminished for any p. Similar inequality (with a non-optimal constant, however) was proved one year earlier by HARDY, [6], in the course of his commenting some still earlier results of Hilbert.

In particular, for $p = \frac{1}{2}$ one has

$$\sum_{n=1}^{\infty} \mathcal{P}_{1/2}(a_1, \dots, a_n) < 4 \|a\|_1 \quad \text{for every } a \in l_1(\mathbb{R}_+). \tag{1}$$

Nowadays these results are few of many in the emerging “theory of Hardy means”. Precisely, one says that a mean $\mathfrak{A}$ is Hardy when there exists a constant C such that

$$\sum_{n=1}^{\infty} \mathfrak{A}(a_1, \dots, a_n) < C \|a\|_1 \quad \text{for every } a \in l_1(\mathbb{R}_+).$$

This definition was formally introduced by PÁLES and PERSSON in [11], but it had been felt in the air since HARDY's paper [6]. These authors proposed in [11] certain conditions sufficient for a mean to be Hardy. Those conditions are relatively mild and are satisfied by the means in a considerable number of families.

Hence it is natural to ask what other means are Hardy. In fact, this question was extensively dealt with decades before the formal definition appeared. The detailed history of the events related to, and facts implied by above inequalities is sketched in catching surveys [13], [3], [10], and in a recent book [8].

Unfortunately, for many families of means the problem if they are Hardy remains open. Such a problem for the two-parameter family of Gini means was, for instance, considered in [11], where many special subcases were done. This problem, explicitly worded three years later in [8, p. 89], was recently solved in [12].

Other interesting families which might be considered in this context are the Hamy means (cf. [4], [1, p. 364] and Corollary 2 below) and Hayashi means (cf. [5], [1, p. 365] and Corollary 3 below). These two families of means are intrinsically related with what is going to be presented in this note.

We will analyse another multi-parameter family of means. Namely, in 1971 Carlson, Meany and Nelson [2], among other things, proposed the following family, which in one time encompasses Power Means, Hamy means and Hayashi means:

$$\mathfrak{M}_{k,s,q}(v_1,\dots,v_n) := \begin{cases} \mathcal{P}_s\big(\mathcal{P}_q(v_{i_1},\dots,v_{i_k}) : 1 \le i_1 < \dots < i_k \le n\big) & \text{if } k \le n, \\ \mathcal{P}_q(v_1,\dots,v_n) & \text{if } k > n, \end{cases}$$

$v \in \bigcup_{n=1}^{\infty}(0,+\infty)^n$. Those authors were interested in certain inequalities binding the means $\mathfrak{M}_{k,s,q}$ when the order of parameters s and q was being reversed. These means are analysed here from the point of view of being or not being Hardy. Namely, we are going to prove that these means are Hardy for

$$(s,q) \in \big((-\infty,1)\times\mathbb{R}\big) \cup \big(\{1\}\times(-\infty,0]\big) \quad \text{and any } k \ge 2$$

(see Figure 1 below for a better visualisation).

Let us note that, by [14], there hold the following inequalities

$$\mathfrak{M}_{k,s,q} \le \mathfrak{M}_{k,t,p} \qquad \text{for } s \le t \text{ and } q \le p, \tag{2}$$

$$\mathfrak{M}_{k,s,q} \le \mathfrak{M}_{k-1,s,q} \quad \text{for } s > q. \tag{3}$$

2. Main result

In our main Theorem 1 we are going to prove a seemingly isolated fact that $\mathfrak{M}_{2,1,0}$ is a Hardy mean. Obviously, all means majorized by some Hardy mean (or, more generally, majorized up to some constant coefficient) are Hardy, too. Therefore, one time $\mathfrak{M}_{2,1,0}$ being Hardy, the inequalities (2) and (3) imply that $\mathfrak{M}_{k,s,q}$ are Hardy, too, for a vast family of parameters. This is precisely worded in Corollary 1 below.

That corollary is a fairy wide extension of our Theorem 1. Its Hardy-negative part subsumes regions in the parameter plane (s,q) which have until recently seemed to be a kind of challenge – see for instance the second item, and especially the subregion $s \ge k$, $q \le 0$ encompassed by that item.

Theorem 1. *$\mathfrak{M}_{2,1,0}$ is a Hardy mean and*

$$\sum_{n=1}^{\infty} \mathfrak{M}_{2,1,0}(a_1,\dots,a_n) < 4\,\|a\|_1 \quad \text{for every } a \in l^1(\mathbb{R}_+).$$

Moreover, the constant 4 in this inequality is sharp.

Proof. We will show that $\mathfrak{M}_{2,1,0}$ is majorized by $\mathcal{P}_{1/2}$. Indeed,

$$\begin{aligned}
\mathfrak{M}_{2,1,0}(a_1,\dots,a_n) &= \binom{n}{2}^{-1} \sum_{1\le i<j\le n} \sqrt{a_i a_j} = \frac{1}{n(n-1)} \sum_{1\le i<j\le n} 2\sqrt{a_i a_j}\\
&= \frac{n}{n-1}\left(\left(\frac{1}{n}\sum_{i=1}^{n}\sqrt{a_i}\right)^2 - \frac{1}{n^2}\sum_{i=1}^{n} a_i\right)\\
&= \frac{n}{n-1}\left(\mathcal{P}_{1/2}(a_1,\dots,a_n) - \frac{1}{n}\mathcal{P}_1(a_1,\dots,a_n)\right)\\
&\le \frac{n}{n-1}\left(\mathcal{P}_{1/2}(a_1,\dots,a_n) - \frac{1}{n}\mathcal{P}_{1/2}(a_1,\dots,a_n)\right)\\
&= \mathcal{P}_{1/2}(a_1,\dots,a_n).
\end{aligned} \tag{4}$$

Hence, by (1), one obtains

$$\sum_{n=1}^{\infty} \mathfrak{M}_{2,1,0}(a_1,\dots,a_n) \le \sum_{n=1}^{\infty} \mathcal{P}_{1/2}(a_1,\dots,a_n) < 4\,\|a\|_1 .$$

The constant 4 in the above theorem cannot be diminished. Indeed, upon taking $a_n = \frac{1}{n}$, a simple calculation yields

$$\lim_{n\to\infty} a_n^{-1}\mathfrak{M}_{2,1,0}(a_1,\dots,a_n) = 4.$$

Then the machinery originally devised for the Power Means in [7, pp. 241–242] becomes applicable. It gives, by taking N arbitrary large and considering the auxiliary sequence

$$(a_1,\dots,a_N,(N+1)^{-2},(N+2)^{-2},(N+3)^{-2},\dots),$$

that the constant cannot be smaller than 4. □

Theorem 2. *Let $k \in \mathbb{N}_+$. Then $\mathfrak{M}_{k,k,-\infty}$ is not a Hardy mean.*

Proof. Let us take any decreasing sequence $a \in l^1(\mathbb{R}_+)$. For any $n \geq k$ one obtains

$$\mathfrak{M}_{k,k,-\infty}(a_1,\dots,a_n) = \left(\binom{n}{k}^{-1} \sum_{1\leq i_1<\dots<i_k\leq n} \min(a_{i_1},\dots,a_{i_k})^k\right)^{1/k}$$
$$> \left(\binom{n}{k}^{-1} \min(a_1,\dots,a_k)^k\right)^{1/k} = \binom{n}{k}^{-1/k} a_k > n^{-1}a_k.$$

Hence $\sum_{n=1}^{\infty} \mathfrak{M}_{k,k,-\infty}(a_1,\dots,a_n) = +\infty$. □

3. Discussion of parameters

We know that for any fixed $q > 0$ and $k \in \mathbb{N}$ the inequality

$$\mathcal{P}_q(v_1,\dots,v_k) = \left(\frac{1}{k}\sum v_i^q\right)^{1/q} > \left(\frac{1}{k}\max(v_1^q,\dots,v_n^q)\right)^{1/q}$$
$$= k^{-1/q}\max(v_1,\dots,v_k) = C(k,q)\max(v_1,\dots,v_k)$$

holds for any $v \in \mathbb{R}_+^k$, with $C(k,q) := k^{-1/q}$. In particular, for any $r \in \mathbb{R}\cup\{\pm\infty\}$,

$$\mathcal{P}_q(v_1,\dots,v_k) > C(k,q)\mathcal{P}_r(v_1,\dots,v_k) \tag{5}$$

(cf. also [1, p. 237]). The above inequalities are instrumental in proving the following

Corollary 1. *For any $k \geq 2$*

- *$\mathfrak{M}_{k,s,q}$ is not a Hardy mean for any $s \geq 1$ and $q > 0$,*
- *$\mathfrak{M}_{k,s,q}$ is not a Hardy mean for any $s \geq k$ and $q \in \mathbb{R}\cup\{\pm\infty\}$,*
- *$\mathfrak{M}_{k,1,q}$ is a Hardy mean for any $q \leq 0$,*
- *$\mathfrak{M}_{k,s,q}$ is a Hardy mean for any $s < 1$ and $q \in \mathbb{R}\cup\{\pm\infty\}$*

(see Figure 1).

Proof. Let us recall that the length of vectors in $\mathcal{P}_q$ in the definition of $\mathfrak{M}_{k,s,q}$ is fixed (and equal to k, whenever $k \leq n$). Moreover, if a mean could be majorized by some Hardy mean up to a constant coefficient, then it is Hardy, too. Therefore the use of (5) is very natural to in the investigation of behaviour of $\mathfrak{M}_{k,s,q}$ while the parameter q is changed.

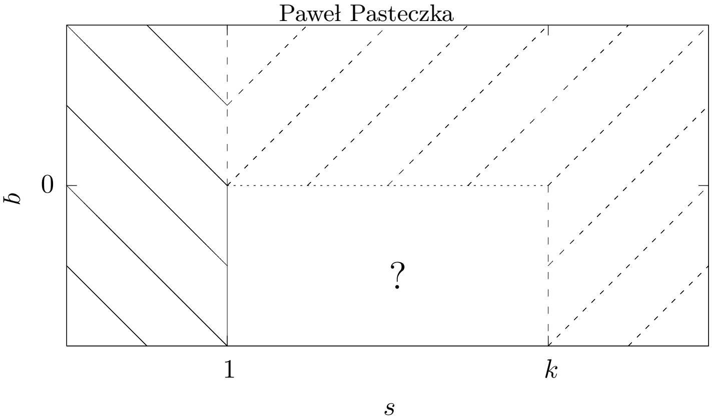

Figure 1. Space of parameters, for which the mean $\mathfrak{M}_{k,s,q}$ is Hardy (solid lines), and for which it is not Hardy (dashed lines); k is fixed.

First item follows from the fact that $\mathfrak{M}_{k,1,1}$ is an arithmetic mean. So it is not Hardy. But $\mathfrak{M}_{k,s,q} \geq \mathfrak{M}_{k,1,q} \geq C\mathfrak{M}_{k,1,1}$ for some constant C, hence $\mathfrak{M}_{k,s,q}$ is not Hardy, too.

Second item is an immediate corollary from Theorem 2. Third one is implied by Theorem 1 and (2).

Fourth item is being proved in two steps. First, with no loss of generality we may assume that $s \in (0,1)$. We know that $\mathfrak{M}_{k,s,s} = \mathcal{P}_s$, so it is a Hardy mean. Second, applying (5), one gets $\mathfrak{M}_{k,s,s} > C(k,s)\mathfrak{M}_{k,s,q}$. So $\mathfrak{M}_{k,s,q}$ is Hardy as well. □

Corollary 2. *For any $k \geq 2$ the Hamy mean $\mathfrak{ha}^{[k]} := \mathfrak{M}_{k,1,0}$ (cf.* [4], [1, pp. 364–365] *for more details) is a Hardy mean.*

Corollary 3. *For any $k \geq 2$ the Hayashi mean $\mathfrak{hy}^{[k]} := \mathfrak{M}_{k,0,1}$ (cf.* [5] *and* [1, pp. 365–366] *for more details) is a Hardy mean.*

3.1. Remaining cases. The problem whether $\mathfrak{M}_{k,s,q}$ is a Hardy mean for $k \geq 2$, $s \in (1,k)$ and $q \leq 0$ remains open (see the central part in Figure 1).

References

[1] P. S. Bullen, Handbook of Means and Their Inequalities, Vol. 560, *Kluwer Acad. Publ., Dordrecht*, 2003.

[2] B. C. Carlson, R. K. Meany and S. A. Nelson, Mixed arithmetic and geometric means, *Pacif. J. Math.* **38** (1971), 343–349.

[3] J. Ducan and C. M. McGregor, Carleman's Inequality, *Amer. Math. Monthly* **110** (2003), 424–431.

[4] M. Hamy, Sur le théorème de la moyenne, *Bull. Sci. Math.* **14** (1890), 103–104.

[5] T. Hayashi, Un théorème relatif aux valeurs moyennes, *Nouv. Ann. Math.* **5** (1905), 355–357.

[6] G. H. Hardy, Note on a theorem of Hilbert, *Math. Zeitschrift* **6** (1920), 314–317.

[7] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, *Cambridge University Press, Cambridge*, 1934.

[8] A. Kufner, L. Maligranda and L.-E. Persson, The Hardy Inequality: About its History and Some Related Results, *Vydavatelský Servis*, *Pilsen*, 2007.

[9] E. Landau, A note on a theorem concerning series of positive terms, *J. London Math. Soc.* **1** (1921), 38–39.

[10] J. A. Oguntuase and L.-E. Persson, Hardy type inequalities via convexity – the journey so far, *Aust. J. Math. Anal. Appl.* **7** (2011), 1–19.

[11] Zs. Páles and L.-E. Persson, Hardy-type inequalities for means, *Bull. Austral. Math. Soc.* **70** (2004), 521–528.

[12] P. Pasteczka, On negative results concerning Hardy means, *Acta Math. Hungar.* (*to appear*).

[13] J. Pečarič and K. B. Stolarsky, Carleman's inequality: history and new generalizations, *Aequationes Math.* **61** (2001), 49–62.

[14] R. Kh. Sadikova, Comparison of discrete mixed means containing symmetric functions, *Math. Notes* **80** (2006), 254–260, English translation.

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