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Real hypersurfaces of non–flat complex space forms in terms of the Jacobi structure operator

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Abstract. Real hypersurfaces satisfying the condition $\phi l = l\phi$, ($l = R(.,\xi)\xi$), have been studied by many authors, under at least one more condition, since the class of these hypersurfaces is too large. Moreover the operator l has been studied satisfying other conditions, including $\nabla_\xi l = 0$ and $lA = Al$. Even more, not much work has been done on the last equation. In the present paper we study condition $\phi l = l\phi$, combined with either $\nabla_\xi l = 0$ or $lA = Al$. All conditions are restricted in subspaces of the tangent space, in order to produce larger classes.

0. Introduction

An n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called complex space form, which is denoted by $M_n(c)$. A complete simply connected complex space form is a complex projective space $\mathbb{C}P^n$ if $c > 0$, a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$, or a complex Euclidean space $\mathbb{C}^n$ if $c = 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ will be denoted by (ϕ, ξ, η, g).

Homogeneous real hypersurfaces in $\mathbb{C}P^n$, were classified by R. Takagi [16]. J. Berndt [1] classified real hypersurfaces with principal structure vector fields in $\mathbb{C}H^n$.

Another class of real hypersurfaces were studied by Okumura [14], and Montiel and Romero [12]. They classified real hypersurfaces satisfying $\phi A =$

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$A\phi$, in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ respectively. In both cases, real hypersurfaces were categorized as *type A*, described in Section 1. For more details and examples on real hypersurfaces of type A, we refer to [13].

A Jacobi field along geodesics of a given Riemannian manifold (M, g) plays an important role in the study of differential geometry. It satisfies a well known differential equation which inspires Jacobi operators. For any vector field X, the Jacobi operator is defined by $R_X : R_X(Y) = R(Y, X)X$, where R denotes the curvature tensor and Y is a vector field on M. R_X is a self-adjoint endomorphism in the tangent space of M, and is related to the Jacobi differential equation, which is given by $\nabla_{\acute{\gamma}}(\nabla_{\acute{\gamma}} Y) + R(Y, \acute{\gamma})\acute{\gamma} = 0$ along a geodesic γ on M, where $\acute{\gamma}$ denotes the velocity vector along γ on M.

In a real hypersurface M of a complex space form $M_n(c)$, $c \neq 0$, the Jacobi operator on M with respect to the structure vector field ξ, is called the Jacobi structure operator and is denoted by $lX = R_\xi(X) = R(X, \xi)\xi$.

Many authors have studied real hypersurfaces from many points of view. Certain authors have studied real hypersurfaces under the condition $\phi l = l\phi$, equipped with one or two additional conditions. U-H. Ki *et al.* [9] classified real hypersurfaces in complex space forms satisfying i) $\phi l = l\phi$ and $A^2\xi = \theta A\xi + \tau\xi$ (θ is a function, τ is constant) ii) $\phi l = l\phi$ and $Q\xi = \sigma\xi$ (where Q is the Ricci operator, σ is constant). U-Hang Ki [7] classified real hypersurfaces in complex hyperbolic space satisfying $\phi l = l\phi$ and $lQ = Ql$. U-H. Ki *et al.* [8], classified real hypersurfaces in complex space forms satisfying $\phi l = l\phi$, $lQ = Ql$, and additional conditions on the mean curvature. U-H. Ki *et al.* [10] studied real hypersurfaces in complex space forms satisfying $\phi l = l\phi$ and $lQ = lQ$ ([7], [8], [9], [10]).

Other authors have studied real hypersurfaces under the conditions $\nabla_X l = 0$ $(X \in TM)$ or $\nabla_\xi l = 0$ [5], [11], [15].

In the present paper, we consider $\phi l = l\phi$ (commuting structure Jacobi operator) and $\nabla_\xi l = 0$ (Reeb parallel structure Jacobi operator). Both conditions are restricted on the distribution on $M : \ker(\eta), (\ker(\eta)^\perp = \mathrm{span}\{\xi\})$. Namely we prove:

Theorem 0.1. *Let M be a real hypersurface of a complex space form $M_n(c)$, $(n > 2)$ $(c \neq 0)$, satisfying $\phi l = l\phi$ on $\ker(\eta)$. If $\nabla_\xi l = 0$ holds on $\ker(\eta)$ or on $\mathrm{span}\{\xi\}$, then M is a Hopf hypersurface. Furthermore, if $\eta(A\xi) \neq 0$, then M is locally congruent to a model space of type A.*

J. T. Cho and U-H. Ki in [4] classified real hypersurfaces M of a projective space satisfying $\phi l = l\phi$ and $lA = Al$ on M. In the present paper we generalize this result, studying the real hypersurfaces of any complex space form satisfying $\phi l = l\phi$ on $\ker(\eta)$ and $lA = Al$ on $\ker(\eta)$ or on $\operatorname{span}\{\xi\}$. We prove:

Theorem 0.2. *Let M be a real hypersurface of a complex space form $M_n(c)$, $(n > 2)$ $(c \neq 0)$, satisfying $\phi l = l\phi$ on $\ker(\eta)$. If $lA = Al$ holds on $\ker(\eta)$ or on $\operatorname{span}\{\xi\}$, then M is a Hopf hypersurface. Furthermore, if $\eta(A\xi) \neq 0$, then M is locally congruent to a model space of type A.*

For the case of $\mathbb{C}P^n$ in order to determine real hypersurface of type A, the technical assumption $\eta(A\xi) \neq 0$ is needed. Actually, there is a non-homogeneous tube with $A\xi = 0$ (of radius $\frac{\pi}{4}$) over a certain Kaehler submanifold in $\mathbb{C}P^n$, when its focal map has constant rank on M [3]. For Hopf hypersurfaces in $\mathbb{C}H^n$, $(n > 2)$ it is known that the associated principal curvature of ξ never vanishes [1]. However, in $\mathbb{C}H^2$ there exists a Hopf hypersurface with $A\xi = 0$ [6].

We must also notice that equation $(\nabla_\xi l)X = 0$, $X \in \ker(\eta)$, is equivalent to $(\nabla_\xi l)X = \mu\xi$, $X \in \ker(\eta)$ and μ is a real valued function. Indeed, $(\nabla_\xi l)X = 0$ implies $(\nabla_\xi l)X = \mu\xi$, where $\mu = 0$. Conversely if $(\nabla_\xi l)X = \mu\xi$ holds $\forall X \in \ker(\eta)$, then by putting $-X$ instead of X we have $-(\nabla_\xi l)X = \mu\xi$ which is combined with $(\nabla_\xi l)X = \mu\xi$ to give $\mu = 0$.

1. Preliminaries

Let M_n be a Kaehlerian manifold of real dimension 2n, equipped with an almost complex structure J and a Hermitian metric tensor G. Then for any vector fields X and Y on $M_n(c)$, the following relations hold:

$$J^2X = -X, \quad G(JX, JY) = G(X, Y), \qquad \widetilde{\nabla} J = 0$$

where $\widetilde{\nabla}$ denotes the Riemannian connection of G of M_n.

Now, let M_{2n-1} be a real $(2n-1)$-dimensional hypersurface of $M_n(c)$, and denote by N a unit normal vector field on a neighborhood of a point in M_{2n-1} (from now on we shall write M instead of M_{2n-1}). For any vector field X tangent to M we have $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX, $\eta(X)N$ is the normal component, and

$$\xi = -JN, \quad \eta(X) = g(X, \xi), \quad g = G|_M.$$

By properties of the almost complex structure J, and the definitions of η and g, the following relations hold [2]:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1 \tag{1.1}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\quad g(X,\phi Y)=-g(\phi X,Y). \tag{1.2}$$

The above relations define an *almost contact metric structure* on M denoted by (ϕ,ξ,g,η). For (ϕ,ξ,g,η), we can define a local orthonormal basis $\{V_1,V_2,\dots V_{n-1},\phi V_1,\phi V_2,\dots \phi V_{n-1},\xi\}$, called a *$\phi$-basis* on M. Furthermore, let A be the shape operator with respect to N, and denote by ∇ the Riemannian connection of g on M. Then, A is symmetric and the following equations are satisfied

$$\nabla_X\xi=\phi AX,\quad (\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi. \tag{1.3}$$

As the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given by

$$\begin{aligned}R(X,Y)Z=\frac{c}{4}[&g(Y,Z)X-g(X,Z)Y+g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y\\&-2g(\phi X,Y)\phi Z]+g(AY,Z)AX-g(AX,Z)AY,\end{aligned} \tag{1.4}$$

$$(\nabla_XA)Y-(\nabla_YA)X=\frac{c}{4}[\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi]. \tag{1.5}$$

The tangent space T_pM, for every point $p\in M$, is decomposed as following

$$T_pM=\ker(\eta)^{\perp}\oplus\ker(\eta)$$

where $\ker(\eta)^{\perp}=\mathrm{span}\{\xi\}$ and $\ker(\eta)$ is defined as following

$$\ker(\eta)=\{X\in T_pM:\eta(X)=0\}.$$

Based on the above decomposition, by virtue of (1.3), we decompose the vector field $A\xi$ in the following way

$$A\xi=\alpha\xi+\beta U \tag{1.6}$$

where $\beta=|\phi\nabla_\xi\xi|$ and $U=-\frac{1}{\beta}\phi\nabla_\xi\xi\in\ker(\eta)$, provided that $\beta\neq 0$.

If the vector field $A\xi$ is expressed as $A\xi=\alpha\xi$, then ξ is called a principal vector field. Differentiation of a function f along a vector field X will be denoted by (Xf). All manifolds and vector fields of this paper are assumed to be connected and of class C^∞.

Finally, we mention the theorems of Okumura [14], and Montiel, Romero [12], who proved respectively the following theorems.

Theorem 1.1. *Let M be a real hypersurface of $\mathbb{C}P^n$, $n\geq 2$. If it satisfies*

$$g((A\phi-\phi A)X,Y)=0$$

for any vector fields X and Y, then M is a tube of radius r over one of the following Kaehlerian submanifolds:

(A_1) *a hyperplane* $\mathbb{C}P^{n-1}$, *where* $0 < r < \frac{\pi}{2}$,

(A_2) *a totally geodesic* $\mathbb{C}P^k$ $(0 < k \leq n-2)$, *where* $0 < r < \frac{\pi}{2}$.

Theorem 1.2. *Let* M *be a real hypersurface of* $\mathbb{C}H^n$, $n \geq 2$. *If it satisfies*

$$g((A\phi - \phi A)X, Y) = 0$$

for any vector fields X *and* Y, *then* M *is locally congruent to one of the following:*

(A_0) *a self – tube, that is, horosphere,*

(A_1) *a geodesic hypershere or a tube over a hyperplane* $\mathbb{C}H^{n-1}$,

(A_2) *a tube over a totally geodesic* $\mathbb{C}H^k$ $(1 \leq k \leq n-2)$.

2. Auxiliary relations

In the study of real hypersurfaces in a complex space form $M_n(c)$, $c \neq 0$, it is a crucial condition that the structure vector field ξ is principal. The purpose of this paragraph is to prove this condition.

Let V be the open subset of points p of M, where $\alpha \neq 0$ in a neighborhood of p and V_0 be the open subset of points p of M such that $\alpha = 0$ in a neighborhood of p. Since α is a smooth function on M, then $V \cup V_0$ is an open and dense subset of M.

Lemma 2.1. *Let* M *be a real hypersurface in a complex space form* $M_n(c)$ $(c \neq 0)$, *satisfying* $\phi l = l\phi$ *on* $\ker(\eta)$. *Then,* $\beta = 0$ *on* V_0.

Proof. From (1.6) we have $A\xi = \beta U$ on V_0. Then (1.4) for $X = U$ and $Y = Z = \xi$ yields

$$lU = \frac{c}{4}U + g(A\xi, \xi)AU - g(AU, \xi)A\xi = \frac{c}{4}U - g(U, A\xi)A\xi = \left(\frac{c}{4} - \beta^2\right)U \implies$$

$$\phi lU = \left(\frac{c}{4} - \beta^2\right)\phi U.$$

In the same way, from (1.4) for $X = \phi U$, $Y = Z = \xi$ we obtain

$$l\phi U = \frac{c}{4}\phi U.$$

The last two equations yield $\beta = 0$. $\square$

Remark 1. We have proved that on V_0, $A\xi = 0\xi$ i.e., ξ is a principal vector field on V_0. Now we define on V the set V' of points p where $\beta \neq 0$ in a neighborhood of p and the set V'' of points p where $\beta = 0$ in a neighborhood of p. Obviously ξ is principal on V''. In what follows we study the open subset V' of M and define the following classes

$$A = \text{hypersurfaces satisfying } \phi l = l\phi \text{ and } lA = Al \text{ on } \ker(\eta),$$

$$B = \text{hypersurfaces satisfying } \phi l = l\phi \text{ and } lA = Al \text{ on } \operatorname{span}\{\xi\},$$

$$C = \text{hypersurfaces satisfying } \phi l = l\phi \text{ and } \nabla_\xi l = \mu\xi \text{ on } \ker(\eta),$$

$$D = \text{hypersurfaces satisfying } \phi l = l\phi \text{ and } \nabla_\xi l = \mu\xi \text{ on } \operatorname{span}\{\xi\}.$$

Lemma 2.2. *Let M be a real hypersurface of a complex space form $M_n(c)$ $(c \neq 0)$, satisfying $\phi l = l\phi$ on $\ker(\eta)$. Then the following relations hold on the set V' of classes A, B, C, D:*

$$AU = \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)U + \beta\xi, \quad A\phi U = -\frac{c}{4\alpha}\phi U \tag{2.1}$$

$$\nabla_\xi \xi = \beta\phi U, \quad \nabla_U \xi = \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\phi U, \quad \nabla_{\phi U}\xi = \frac{c}{4\alpha}U \tag{2.2}$$

$$\nabla_\xi U = W_1, \quad \nabla_U U = W_2, \quad \nabla_{\phi U} U = W_3 - \frac{c}{4\alpha}\xi \tag{2.3}$$

$$\nabla_\xi \phi U = \phi W_1 - \beta\xi, \quad \nabla_U \phi U = \phi W_2 + \left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right)\xi, \quad \nabla_{\phi U}\phi U = \phi W_3. \tag{2.4}$$

where W_1, W_2, W_3 are vector fields on $\ker(\eta)$ satisfying $W_1, W_2, W_3 \perp U$.

Proof. From (1.4) we get

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - g(AX,\xi)A\xi \tag{2.5}$$

which, for $X = U$ yields

$$lU = \frac{c}{4}U + \alpha AU - \beta A\xi. \tag{2.6}$$

The scalar products of (2.6) with U (resp. ϕU) yield

$$g(AU,U) = \frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}, \tag{2.7}$$

$$g(AU,\phi U) = \frac{1}{a}g(lU,\phi U) \tag{2.8}$$

respectively, where $\gamma = g(lU, U) = g(\phi lU, \phi U) = g(l\phi U, \phi U)$.

The second relation of (1.2) for $X = U$, $Y = lU$, the condition $\phi l = l\phi$ and the symmetry of the operator l imply:

$$g(lU, \phi U) = 0.$$

The above equation and (2.8) imply

$$g(AU, \phi U) = 0. \tag{2.9}$$

The symmetry of A and (1.6) imply

$$g(AU, \xi) = \beta. \tag{2.10}$$

From relations (2.7), (2.9) and (2.10), we obtain

$$AU = \left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)U + \beta\xi + \lambda W \tag{2.11}$$

where $W \in \operatorname{span}\{U, \phi U, \xi\}^{\perp}$ and $\lambda = g(AU, W)$. Combining (2.11) with (2.6) we obtain $lU = \gamma U + \lambda\alpha W$. Acting on this relation with the tensor field ϕ and by virtue of $\phi l = l\phi$ we take $l\phi U = \gamma\phi U + \lambda\alpha\phi W$. On the other hand by virtue of (2.5) we have $l\phi U = \frac{c}{4}\phi U + \alpha A\phi U$. From the last two relations we obtain $A\phi U = \left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right)\phi U + \lambda\phi W$.

On class A

Since $lA = Al$ holds on $\ker(\eta)$ we have $lAW = AlW$. This relation because of (2.5) and (2.11) implies $\lambda\beta A\xi = 0$ and so $\lambda = 0$. Since $\lambda = 0$, equations $lAU = AlU$, (2.6) and (2.11) yield $\gamma = 0$, therefore we have the first of (2.1). Moreover from (2.5) we have $l\phi U = \frac{c}{4}\phi U + \alpha A\phi U$ which is written as $\phi lU = \frac{c}{4}\phi U + \alpha A\phi U$ ($\phi l = l\phi$). From $\phi lU = \frac{c}{4}\phi U + \alpha A\phi U$ and $\gamma = \lambda = 0$ we obtain the second of (2.1). Using (1.3) for $X \in \{\xi, U, \phi U\}$ and by virtue of (2.1) we obtain (2.2). It is well known that:

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{2.12}$$

Let us set $\nabla_\xi U = W_1$ and $\nabla_U U = W_2$. If we use (2.2) and (2.12), it is easy to verify that $g(\nabla_\xi U, U) = 0 = \eta(\nabla_\xi U)$ and $g(\nabla_U U, U) = 0 = \eta(\nabla_U U)$ which means $W_1 \perp \{\xi, U\}$ and $W_2 \perp \{\xi, U\}$.

On the other hand using (2.12) and the third of (2.2) we find $\eta(\nabla_{\phi U} U) = -\frac{c}{4\alpha}$ and $g(\nabla_{\phi U} U, U) = 0$ which means that $\nabla_{\phi U} U$ is decomposed as $\nabla_{\phi U} U = W_3 - \frac{c}{4\alpha}\xi$, $W_3 \perp \{U, \xi\}$. Now, by virtue of (1.3) and (2.3) for $X = \xi, Y = U$ and $X = Y = U$ and $X = \phi U, Y = U$, we get (2.4).

On class B
We analyze equation $lA\xi = Al\xi$ by virtue of (1.6), (2.6) and (2.11) and we have $\gamma U + \lambda\alpha W = 0$. Since $W \perp U$ we have $\gamma = \lambda = 0$. The rest of the proof is similar to the one in class A.

On class C
The scalar product of $(\nabla_\xi l)\phi U = 0$ with ξ, the symmetry of l and (2.12) yield $g(l\phi U, \phi U) = \gamma = 0$. In addition $(\nabla_\xi l)\phi W = 0$ holds. So, the scalar product of the previous equation with ξ, the symmetry of l and (2.12) yield $g(l\phi U, \phi W) = 0$, which, by virtue of (2.5), the second of (2.1) and $\gamma = 0$, yields $\lambda = 0$. The rest of the proof is similar to the one in class A.

On class D
We analyze $(\nabla_\xi l)\xi = 0$ and obtain $\beta l\phi U = 0 \Rightarrow l\phi U = 0$. We analyze $l\phi U = 0$ using (2.5) and $A\phi U = \big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\big)\phi U + \lambda\phi W$, and we have $\gamma\phi U + \lambda\alpha\phi W = 0$. This relation and the linear independency of the vector fields ϕU and ϕW yield $\gamma = \lambda = 0$. The rest of the proof is similar to the one in class A. $\square$

Lemma 2.3. *Let M be a real hypersurface of a complex space form $M_n(c)$ $(c \neq 0)$, of class A, B, C, or D. Then on V' we have $g(\nabla_\xi U, \phi U) = -4\alpha$ and $g(\nabla_U U, \phi U) = -4\beta + \frac{c}{4\alpha\beta}\big(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\big)$.*

Proof. Putting $X = U$, $Y = \xi$ in (1.5), we obtain

$$(\nabla_U A)\xi - (\nabla_\xi A)U = -\frac{c}{4}\phi U.$$

Combining the last equation with (1.6), and Lemma 2.2 it follows:

$$(U\alpha)\xi + (U\beta)U + \beta W_2 + \left(-\frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\frac{c}{4\alpha}\phi U$$
$$-\xi(-\frac{c}{4\alpha} + \frac{\beta^2}{\alpha})U - \left(-\frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)W_1 - (\xi\beta)\xi + AW_1 = 0.$$

Taking the scalar products of the last relation with ξ and U respectively, we obtain

$$(U\alpha) = (\xi\beta) \tag{2.13}$$

and

$$(U\beta) = \left(\xi\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\right). \tag{2.14}$$

Combining the last three equations we have

$$AW_1 = \frac{c}{4\alpha}\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right)\phi U + \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)W_1 - \beta W_2. \tag{2.15}$$

The scalar product of (2.15) with ϕW_1 yields:

$$\beta g(\phi W_1, W_2) = -g(AW_1, \phi W_1).$$

But from (2.5) we have

$$g(l\phi W_1, W_1) = g(\phi W_1, lW_1) = \alpha g(AW_1, \phi W_1).$$

Moreover $g(l\phi W_1, W_1) = g(\phi W_1, lW_1) = -g(W_1, \phi l W_1) = -g(W_1, l\phi W_1)$ which means that

$$g(l\phi W_1, W_1) = 0.$$

The above relations lead to $g(\phi W_1, W_2) = 0$ which, by virtue of (2.15) implies $g(AW_1, \phi W_2) = 0$.

In what follows we define the following functions:

$$\kappa_1 = g(W_1, \phi U) \quad \kappa_2 = g(W_2, \phi U), \quad \kappa_3 = g(W_3, \phi U).$$

Putting $X = \phi U$, $Y = \xi$ in (1.5), we obtain

$$\begin{aligned} A\phi W_1 = &\left[\frac{3\beta c}{4\alpha} + \alpha\beta - (\phi U\alpha)\right]\xi - \left[(\phi U\beta) + \frac{c}{4\alpha}\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right) - \beta^2\right]U \\ &+ \frac{c}{4\alpha^2}(\xi\alpha)\phi U - \frac{c}{4\alpha}\phi W_1 - \beta W_3. \end{aligned} \tag{2.16}$$

The scalar product of (2.16) with ξ implies

$$(\phi U\alpha) = \frac{3\beta c}{4\alpha} + \alpha\beta + \kappa_1\beta. \tag{2.17}$$

Using the A is symmetric and ϕ is skew-symmetric, by taking the scalar product of (2.16) with U we have

$$g(\phi W_1, AU) = -(\phi U\beta) - \frac{c}{4\alpha}\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right) + \beta^2 + \frac{c}{4\alpha}g(W_1, \phi U)$$

which, eventually (with the aid of Lemma 2.2 and the definition of κ_1) yields

$$(\phi U\beta) = \frac{c}{4\alpha}\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right) + \beta^2 + \kappa_1\frac{\beta^2}{\alpha}. \tag{2.18}$$

Due to (2.5), (2.15) and (2.16) the condition $\phi l W_1 = l\phi W_1$ implies

$$\begin{aligned} \beta^2\phi W_1 - \alpha\beta\phi W_2 + \alpha\left[(\phi U\alpha) - \frac{3\beta c}{4\alpha} - \alpha\beta\right]\xi + \alpha[(\phi U\beta) - \beta^2]U + \alpha\beta W_3 \\ = \kappa_1\beta A\xi + \frac{c}{4\alpha}(\xi\alpha)\phi U. \end{aligned}$$

Taking the scalar product of the last relation with U we have

$$-2\kappa_1\beta^2 + \alpha\beta\kappa_2 + \alpha(\phi U\beta) - \alpha\beta^2 = 0.$$

If in the above relation we replace the term κ_1 using (2.17) we obtain

$$-2\beta(\phi U\alpha) + \frac{3\beta^2 c}{2\alpha} + \alpha\beta^2 + \alpha\beta\kappa_2 + \alpha(\phi U\beta) = 0. \tag{2.19}$$

The relation $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -\frac{c}{2}\xi$, using Lemma 2.2 implies

$$\begin{aligned} &\frac{c}{4\alpha^2}(U\alpha)\phi U + \left[\frac{c}{2\alpha}\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right) + \beta^2 - (\phi U\beta)\right]\xi \\ &\quad + \left[-\frac{3\beta c}{4\alpha} + \frac{\beta^3}{\alpha} + \left(\phi U\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right)\right)\right]U - \frac{c}{4\alpha}\phi W_2 - A\phi W_2 \\ &\quad + AW_3 + \left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right)W_3 = 0. \end{aligned} \tag{2.20}$$

The scalar product of the above relation with U yields

$$\frac{\kappa_2\beta^2}{\alpha} - \frac{3\beta c}{4\alpha} + \frac{\beta^3}{\alpha} + \phi U\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right) = 0.$$

Expanding the last relation and by virtue of (2.19) we get

$$\left(-\frac{3\beta^2}{\alpha^2} + \frac{c}{4\alpha^2}\right)(\phi U\alpha) + \frac{3\beta}{\alpha}(\phi U\beta) + \frac{3\beta^3 c}{2\alpha^3} + \frac{3\beta c}{4\alpha} = 0.$$

Combining the last equation with (2.17) and (2.18) we obtain $\kappa_1 = -4\alpha$. The scalar product of (2.15) with ϕU because of $\kappa_1 = -4\alpha$, yields $\kappa_2 = -4\beta + \frac{c}{4\alpha\beta}\left(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\right)$. □

Lemma 2.4. *Let M be a real hypersurface of a complex space form $M_n(c)$ ($c \neq 0$), of class A, B, C, or D. Then the structure vector field ξ is principal on M.*

Proof. The scalar products of (2.16) and (2.20) with ϕU, yield $(\xi\alpha) = \frac{4\alpha^2\beta}{c}\kappa_3$ and $(U\alpha) = \frac{4\alpha\beta^2}{c}\kappa_3$. Combining the last two relations with (2.13) and (2.14) we have

$$(\xi\alpha) = \frac{4\alpha^2\beta}{c}\kappa_3, \quad (U\alpha) = (\xi\beta) = \frac{4\alpha\beta^2}{c}\kappa_3, \quad (U\beta) = \left(\beta + \frac{4\beta^3}{c}\right)\kappa_3. \tag{2.21}$$

Using (1.5) for $X = \phi W_2$, $Y = \xi$ we have

$$\nabla_{\phi W_2} A\xi - A\nabla_{\phi W_2}\xi - \nabla_\xi A\phi W_2 + A\nabla_\xi \phi W_2 = \frac{c}{4} W_2,$$

which, from (1.6) is further decomposed as

$$(\phi W_2 \alpha)\xi + \alpha\phi A\phi W_2 + (\phi W_2\beta)U + \beta\nabla_{\phi W_2} U - A\phi A\phi W_2 \\ - \nabla_\xi A\phi W_2 + A\nabla_\xi \phi W_2 = \frac{c}{4} W_2.$$

Taking the scalar product with ξ and by using (1.6), (2.12), (2.21), Lemmas 2.2, 2.3 and $W_1 \perp \phi W_2$ we obtain

$$(\phi W_2\alpha) = \kappa_3\left(\frac{16\alpha\beta^3}{c} + \beta(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})\right). \tag{2.22}$$

On the other hand from (1.5) we get

$$\nabla_{W_3} A\xi - A\nabla_{W_3}\xi - \nabla_\xi AW_3 + A\nabla_\xi W_3 = -\frac{c}{4}\phi W_3$$

which, by virtue of (1.6) is further decomposed as

$$(W_3\alpha)\xi + \alpha\phi AW_3 + (W_3\beta)U + \beta\nabla_{W_3} U - A\nabla_{W_3}\xi - \nabla_\xi AW_3 + A\nabla_\xi W_3 = -\frac{c}{4}\phi W_3.$$

Taking the scalar product of the last equation with ξ and by making use of Lemma 2.2, (2.12) and (2.21)we obtain

$$(W_3\alpha) = 3\beta\left(\frac{c}{4\alpha} - \alpha\right)\kappa_3. \tag{2.23}$$

In a similar way equation (1.5) yields $(\nabla_{\phi W_1} A)U - (\nabla_U A)\phi W_1 = 0$, which by virtue of Lemma 2.2 is further analyzed as

$$\left(\phi W_1\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\right)U + \left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\nabla_{\phi W_1} U \\ + (\phi W_1\beta)\xi + \beta\phi A\phi W_1 - A\nabla_{\phi W_1} U - \nabla_U A\phi W_1 + A\nabla_U \phi W_1 = 0.$$

The scalar product of the above equation with ξ and using $g(\phi W_1, W_2) = 0$, (2.21) and Lemmas 2.2, 2.3, leads to

$$(\phi W_1\beta) = 4\alpha\kappa_3\left(\beta + \frac{4\beta^3}{c}\right). \tag{2.24}$$

Now the calculation of Lie bracket $[\phi U, \xi]\beta$, by virtue of (2.18), Lemma 2.3 and (2.21), results to

$$[\phi U, \xi]\beta = \phi U(\xi\beta) + \kappa_3\left[-\frac{\beta c}{2\alpha} + \frac{24\alpha\beta^3}{c}\right].$$

On the other hand from Lemma 2.2, (2.21) and (2.24) we obtain

$$[\phi U, \xi]\beta = (\nabla_{\phi U}\xi - \nabla_\xi \phi U)\beta = \beta\kappa_3\left[\frac{c}{4\alpha} + \frac{\beta^2}{\alpha} - 4\alpha - \frac{12\alpha\beta^2}{c}\right].$$

Equalizing the above two relations we get

$$\phi U(\xi\beta) = \beta\kappa_3\left[\frac{3c}{4\alpha} + \frac{\beta^2}{\alpha} - 4\alpha - \frac{36\alpha\beta^2}{c}\right]. \tag{2.25}$$

In a similar way, combining (2.18), (2.21), (2.22), (2.23) and Lemmas 2.2, 2.3, the Lie bracket $[\phi U, U]\alpha$ yields

$$[\phi U, U]\alpha = \phi U(U\alpha) + 3\beta\kappa_3\left[\alpha + \frac{8\alpha\beta^2}{c} - \frac{c}{4\alpha}\right]$$
$$[\phi U, U]\alpha = (\nabla_{\phi U}U - \nabla_U\phi U)\alpha = \beta\kappa_3\left[\frac{c}{\alpha} - 5\alpha - \frac{12\alpha\beta^2}{c} - \frac{\beta^2}{\alpha}\right].$$

From the above equations we obtain

$$\phi U(U\alpha) = \beta\kappa_3\left[\frac{7c}{4\alpha} - 8\alpha - \frac{36\alpha\beta^2}{c} - \frac{\beta^2}{\alpha}\right]. \tag{2.26}$$

Because of (2.13) from (2.25) and (2.26) we obtain

$$\frac{\beta}{\alpha}[c - 4\alpha^2 - 2\beta^2]\kappa_3 = 0.$$

Let us assume there is a point $p \in V'$ such that $\kappa_3 \neq 0$ in a neighborhood around p. Then we have $c = 4\alpha^2 + 2\beta^2$. Differentiating the last equation along ξ and by virtue of (2.21) and $\kappa_3 \neq 0$ we take $2\alpha^2 + \beta^2 = 0$ which is a contradiction. So $\kappa_3 = 0 \Rightarrow (U\alpha) = (\xi\alpha) = 0 \Rightarrow [U, \xi]\alpha = 0$. But the last equation, because of Lemma 2.2 yields

$$\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)(\phi U\alpha) - (W_1\alpha) = 0. \tag{2.27}$$

On the other hand from (1.5) for $X = W_1$, $Y = \xi$, taking the scalar product with ξ, using the Lemmas 2.2, 2.3 we have $(W_1\alpha) = \beta|W_1|^2 - \beta(4\alpha^2 + 3c)$. The last relation, (2.27), (2.17) and Lemma 2.3 lead to

$$12(5\alpha^2 + \beta^2)c + 64\alpha^4 = 16\alpha^2(|W_1|^2 + 3\beta^2) + 3c^2. \tag{2.28}$$

Because of (2.28) $f(\omega) = 64\omega^2 + 60c\omega + 12c\beta^2$, where $\omega = \alpha^2$, is positive for every ω, β. This holds if and only if the discriminant of $f(\omega)$ is negative for all β, c. But this is not true, hence we have a contradiction. Therefore V' is empty and the real hypersurface M consists only of V_0 and V'' i.e., the Reeb vector field ξ is principal and M is a Hopf hypersurface. □

3. Proof of theorems

From Lemma 2.4:

$$A\xi = \alpha\xi, \quad \alpha = g(A\xi, \xi). \tag{3.1}$$

We consider a ϕ-*basis* $\{V_i, \phi V_i, \xi\}$, $(i = 1, 2, \dots n-1)$. From (2.5) and (3.1) we obtain

$$lX = \frac{c}{4}\big[X - \eta(X)\xi\big] + \alpha AX - \eta(X)\alpha^2\xi. \tag{3.2}$$

(3.2) for $X = V_i$ implies

$$lV_i = \frac{c}{4}V_i + \alpha AV_i. \tag{3.3}$$

Applying ϕ to (3.3) we obtain

$$\phi lV_i = \frac{c}{4}\phi V_i + \alpha\phi AV_i, \quad i = 1, \dots, n-1. \tag{3.4}$$

The relation (3.2) for $X = \phi V_i$ yields

$$l\phi V_i = \frac{c}{4}\phi V_i + \alpha A\phi V_i. \tag{3.5}$$

Comparing (3.4) with (3.5), and by making use of the condition $\phi l = l\phi$ we have

$$(A\phi - \phi A)V_i = 0, \quad i = 1, \dots, n-1. \tag{3.6}$$

On the other hand the action of ϕ on (3.5) yields

$$\phi(l\phi V_i) = \frac{c}{4}\phi^2 V_i + \alpha\phi A\phi V_i, \tag{3.7}$$

which, by virtue of (1.1), is written in the form

$$(\phi l)\phi V_i = -\frac{c}{4}V_i + \alpha(\phi A)\phi V_i, \tag{3.8}$$

Moreover, the calculation of $(l\phi)\phi V_i$ by virtue of (1.1) and (3.3) yields:

$$\begin{aligned}(l\phi)\phi V_i &= l\phi^2 V_i = -lV_i = -\frac{c}{4}V_i - \alpha A V_i = -\frac{c}{4}V_i + \alpha A\phi^2 V_i \\ &= -\frac{c}{4}V_i + \alpha A\phi\phi V_i \iff (l\phi)\phi V_i = -\frac{c}{4}V_i + \alpha(A\phi)\phi V_i.\end{aligned} \tag{3.9}$$

Comparing (3.8) and (3.9), and by making use of the condition $\phi l = l\phi$ we have

$$(A\phi - \phi A)\phi V_i = 0 \tag{3.10}$$

for every $i = 1, \dots, n-1$. But from (1.1) and (3.1) we also have

$$(A\phi - \phi A)\xi = 0. \tag{3.11}$$

So, (3.6), (3.10) and (3.11) imply that $A\phi = \phi A$. This result and the Theorems 1.1 and 1.2 complete the proof of Theorems 0.1 and 0.2. □

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