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Congruent spectrum for compact linear relations

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Abstract. This paper is a new investigation in the analysis of linear multivalued operators. We introduce the class of congruent linear relations, and we develop the spectral analysis associated with this class. Mainly, we characterize the congruent spectrum for a compact linear relation on a Banach space. Also, we prove the closeness of some classes of linear relations under compact perturbation.

1. Introduction

Throughout this paper E will denote a linear vector space. A multivalued linear operator T on E is a mapping from a subspace $D(T)$ of E, called the domain of T, into the collection of non empty subsets of E, such that $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all non zero scalars $\alpha, \beta \in \mathbb{C}$ and $x_1, x_2 \in D(T)$. The multivalued part of T is a linear subspace of E denoted by $T(0)$. It is clear that T is singlevalued if and only if $T(0) = \{0\}$. We use $R(T) := T(D(T))$ for the range of T. The subspace $T^{-1}(0) = \{x \in E;\ T(x) = T(0)\}$ is called the kernel of T and is denoted by $\mathrm{Ker}(T)$. T is said to be surjective if $R(T) = E$, and is said to be injective if $\mathrm{Ker}(T) = \{0\}$.

The graph of T is the linear subspace of $E \times E$ defined by

$$G(T) = \{(x, y) \in E \times E;\ x \in D(T),\ y \in T(x)\}.$$

The inverse of T is the linear relation T^{-1} on E with $D(T^{-1}) = R(T)$ given by

$$G(T^{-1}) = \{(y, x) \in E \times E;\ (x, y) \in G(T)\}.$$

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Let T and S be two linear relations on E. The sum of T and S is the linear relation defined by

$$(T+S)(x) = T(x) + S(x) \quad \text{for all } x \in D(T) \cap D(S).$$

The composition ST is the linear relation defined by:

$$ST(x) = S(T(x)) \quad \text{for all } x \in T^{-1}(D(S)).$$

The algebraic resolvent set of T is defined as

$$\rho(T) = \{\lambda \in \mathbb{C};\ T - \lambda I \text{ is injective and surjective}\},$$

where I denotes the identity operator on E.

Throughout the sequel, we shall be mostly interested in the everywhere defined operators on E. We use the term "linear relation" to refer to a multivalued linear operator, while the term "operator" refers to a singlevalued linear operator.

The concept of a linear relation goes back at least to R. Arens [8]. In the last few years many papers have been interested in this theme. Fundamental notions known for operators have been extended to the frame of linear relations. Some related important results have been carried out in various contexts: Spectral analysis (see [2], [4], [9]), classification (see [5], [13]), perturbation problem (see [3], [6], [7], [11]), decomposition and functional calculus (see [1], [14], [15]). A standard reference work in the field is the Cross's book [10].

The interaction between the multivalued part and the kernel plays an important role in the theory of linear relations. In the present paper, we focus on this fact. To that purpose, we introduce, in Section 2, some new notions related to a linear relation. Let T be a linear relation on E and let $s(T) := \dim(T(0)/(T(0) \cap T^{-1}(0)))$. We call upper (resp. lower) defect number of T, the quantity $s(T)$ (resp. $s(T^{-1})$). A linear relation with equal defect numbers is said to be congruent. We investigate in this section the class of congruent linear relations. Also, we introduce the spectrum associated with this class, called congruent spectrum, and we present, in this context, a spectral mapping theorem.

Section 3 treats compact linear relations on a Banach space. We investigate the structure of the set of such relations. Also, we describe the usual spectrum of a compact linear relation. We show that the resolvent set of a non-singlevalued compact relation is empty. As to a (singlevalued) compact operator, it's well known that the spectrum is at most countable, see for example ([12], Ch6 Theorem 1.8). In Section 3.2 we present the main results of this paper. We characterize the congruent spectrum, introduced in Section 2, in the context of compact linear relations. Note that in this section, it is assumed that $T(0)$ is (topologically) complemented in E. This condition is fulfilled in the context of a Hilbert space. However, for the sake of completeness, the results are being presented in the general context of a Banach space. In Section 4, we establish some stability results in the set of closed range linear relations with finite lower defect number.

2. Defect numbers of linear relations

In this section, for special emphasis on the duality of T and its inverse T^{-1}, the notation $T^{-1}(0)$ is used instead of $\mathrm{Ker}(T)$.

Definition 2.1. Let T be a linear relation on E and let

$$\Delta(T) := T(0)/(T(0) \cap T^{-1}(0)).$$

(i) The upper defect number of T is defined by $s(T) := \dim(\Delta(T))$.

(ii) The lower defect number of T is defined as $s(T^{-1})$.

(iii) The defect numbers are not necessarily finite, in which case they are defined as ∞.

We list below some simple properties:

(1) If T is singlevalued, then $s(T) = 0$.

(2) If T satisfies that $T(0) \subset D(T)$ and that $T^2(0) = T(0)$, then $T(0) \subset T^{-1}(0)$ and therefore $\Delta(T) = \{0\}$, which implies that $s(T) = 0$.

(3) If T satisfies $T(0) \cap T^{-1}(0) = \{0\}$, then $\Delta(T) = T(0)$, which implies that $s(T) = \dim(T(0))$.

(4) $\Delta(T)$ describes the position of $T^{-1}(0)$ with respect to $T(0)$. Likewise, $\Delta(T^{-1})$ describes the position of $T(0)$ with respect to $T^{-1}(0)$.

(5) Let P_T be the canonical projection on the quotient space $E/(T(0) \cap T^{-1}(0))$, it is forward that $s(T) = \dim(P_T(T(0)))$.

Proposition 1. *Let T and S be two linear relations on E.*

(i) *If $S(T(0) \cap T^{-1}(0)) \subset T(0)$, then $\Delta(T + S) \subset \Delta(T)$.*

(ii) *Furthermore, if $S(T(0)) \subset T(0)$, then $\Delta(T + S) = \Delta(T)$.*

Proof. At first observe that, $S(0) \subset T(0)$ leads to $(T+S)(0) = T(0)$.

In order to prove (i), let $x \in T(0) \cap T^{-1}(0)$. Then $(T+S)(x) = T(0) + S(x)$. On the other hand, since $S(T(0) \cap T^{-1}(0)) \subset T(0)$, it follows that $S(x) \subset T(0)$, and therefore that $(T+S)(x) = T(0)$. Hence $T(0) \cap T^{-1}(0) \subset (T+S)^{-1}(0)$. We have thus proven (i).

To get (ii), all that remains to be proved is that $\Delta(T) \subset \Delta(T+S)$. Let x be in $T(0) \cap (T+S)^{-1}(0)$. Then $(T+S)(x) = (T+S)(0) = T(0)$. Also, the stability property $S(T(0)) \subset T(0)$ yields $S(x) \subset T(0)$. Thus $T(x) = T(0)$, which implies that $T(0) \cap (T+S)^{-1}(0) \subset T^{-1}(0)$. □

Corollary 2.1. *Let T be a linear relation on E and let $\lambda \in \mathbb{C}$. Then $\Delta(T+\lambda I) = \Delta(T)$, in particular, $s(T+\lambda I) = s(T)$.*

According to ([17], Lemma 6.1), if $\rho(T) \neq \varnothing$, then $T^n(0) \cap (T^n)^{-1}(0) = \{0\}$, in which case $\Delta(T^n) = T^n(0)$. It follows from ([17], Lemma 5.1) that $s(T^n) \leq ns(T)$. A further generalization is obtained in the following Proposition.

Proposition 2. *Let T be a linear relation on E. Then $s(T^n) \leq ns(T)$, for every integer n.*

Proof. Since $T^n(0)/(\mathrm{Ker}(T^n) \cap T^n(0)) \subset T^n(0)/(\mathrm{Ker}(T) \cap T^n(0))$, it is enough to prove that

$$\dim(T^n(0)/(\mathrm{Ker}(T) \cap T^n(0))) \leq ns(T). \tag{1}$$

To do this, proceed by induction on n. The case $n = 1$ is obvious. Let $n \geq 1$ and suppose that (1) holds true. It is clear that

$$T^{n+1}(0)/(\mathrm{Ker}(T) \cap T^{n+1}(0)) \subset T^{n+1}(0)/(\mathrm{Ker}(T) \cap T(0)). \tag{2}$$

Since $\dim(T(0)/(\mathrm{Ker}(T) \cap T(0))) = s(T)$, it follows that

$$\dim(T^{n+1}(0)/(\mathrm{Ker}(T) \cap T(0))) = \dim(T^{n+1}(0)/T(0)) + s(T). \tag{3}$$

On the other hand, the map that assigns to each $\bar{x} \in T^n(0)/(\mathrm{Ker}(T) \cap T^n(0))$ the class $Q_T T(x) \in T^{n+1}(0)/T(0)$ is a vector space isomorphism. It follows from this and the induction hypothesis (1), that

$$\dim(T^{n+1}(0)/T(0)) = \dim(T^n(0)/(\mathrm{Ker}(T) \cap T^n(0))) \leq ns(T).$$

Combining this with the identity (3) yields the following inequality.

$$\dim(T^{n+1}(0)/(\mathrm{Ker}(T) \cap T(0))) \leq (n+1)s(T).$$

Finally, the inclusion (2) completes the proof. □

In the following example, it is shown that the inequality in Proposition 2 is optimal.

Example 1. Let T be an injective linear relation on E satisfying that $T(0) = \mathrm{vect}\{e_1\}$ for some non vanishing vector $e_1 \in E$, so that $s(T) = 1$. Construct a sequence of linearly independent vectors of E, by considering $T^k(e_1) = e_{k+1} + T^k(0)$. Notice that e_{k+1} is outside $T^k(0)$ since T^k is injective. Thus $T^n(0) = \mathrm{vect}\{e_1, e_2, \ldots, e_n\}$, and therefore $s(T^n) = n = ns(T)$.

Remark 2.1. Proposition 2 shows that $s(T^n)$ is finite provided that $s(T)$ is finite. However, $s(T)$ need not be finite even if $s(T^n)$ is finite for some integer $n > 1$. For instance, let $E = E_1 \oplus E_2$ where E_1 and E_2 are two infinite dimensional subspaces of E. Consider the linear relation T defined on E as follows,

$$y = (y_1, y_2) \in T(x) \quad \text{for } x = (x_1, x_2) \in E, \text{ if } y_2 = x_1.$$

Then $T(0) = E_1$, $T^{-1}(0) = E_2$ and $T^2(0) = E$. Thus $s(T^2) = 0$, but $s(T) = \infty$.

2.1. Congruent linear relations. Let linear relation T be such that $T(0)$ and $T^{-1}(0)$ are both finite dimensional. Then $\dim(T(0)) - \dim(T^{-1}(0)) = s(T) - s(T^{-1})$. On the other hand, to be injective, an operator T should satisfy $\dim(T^{-1}(0)) = \dim(T(0))$. These simple observations suggest to introduce the notion of congruent linear relation.

Definition 2.2. A linear relation T on E is said to be congruent if $s(T)$ and $s(T^{-1})$ are both finite and $s(T) = s(T^{-1})$.

We list below some immediate properties

(1) T is congruent if and only if so is T^{-1}.

(2) If $T(0) \subset T^{-1}(0))$, then T is congruent if and only if $T^{-1}(0) = T(0)$.

(3) If $\dim(T(0)) < \infty$ then T is congruent if and only if $\dim(T^{-1}(0)) = \dim(T(0))$.

(4) Congruent operators are those that are injective.

Observe that, for operators on a finite dimensional linear space, the surjectivity and the injectivity are equivalent properties provided that the domain is the whole space. The following two propositions make the connection between the surjectivity and the congruent property for a linear relation.

Proposition 3. *Let T be an everywhere defined linear relation on a finite dimensional vector space E. Then T is congruent if and only if it is surjective.*

Proof. Since T is everywhere defined, it follows from ([17], Lemma 4.1), that
$$\dim(E) + \dim(T(0)) = \dim(R(T)) + \dim((T^{-1}(0))).$$
The result follows from the property (3) above. $\square$

Proposition 4. *Let T be an everywhere defined linear relation on E. Assume that $T(0)$ has finite codimension. Then T is congruent if and only if it is surjective.*

Proof. Since T is everywhere defined, it follows from ([17], Lemma 3.1), that
$$\dim(E/T^{-1}(0)) = \dim(R(T)/T(0)) \leq \dim(E/T(0)).$$
Then $T^{-1}(0)$ and hence $T(0) \cap T^{-1}(0)$ must have a finite codimension. This implies that $s(T)$ and $s(T^{-1})$ are both finite. On the other hand, according to ([17], Lemma 2.1),
$$\begin{aligned}\dim(E/T(0) \cap T^{-1}(0)) &= \dim(E/T(0)) + \dim(T(0)/T(0) \cap T^{-1}(0)) \\ &= \dim(E/T^{-1}(0)) + \dim(T^{-1}(0)/T(0) \cap T^{-1}(0)).\end{aligned}$$
Then
$$\begin{aligned}s(T^{-1}) - s(T) &= \dim(E/T(0)) - \dim(E/T^{-1}(0)) \\ &= \dim(E/T(0)) - \dim(R(T)/T(0)) = \dim(E/R(T)).\end{aligned}$$
The last identity follows from ([17], Lemma 2.1). This completes the proof. $\square$

Proposition 5. *Let $(T_i)_{i=1,\ldots,k}$ be a family of everywhere defined linear relations from E onto itself. Assume that $T_i(0) = T_j(0)$ and that $T_i(0) \cap \operatorname{Ker}(T_i) = T_j(0) \cap \operatorname{Ker}(T_j)$, for all $0 \leq i, j \leq k$. If T_i is congruent for all $i = 1, \ldots, k$ then $T_1T_2 \ldots T_k$ is congruent.*

Proof. First observe that for all $1 \leq i, j \leq k$,
$$s(T_i) = s(T_i^{-1}) = s(T_j) = s(T_j^{-1}) := s < \infty.$$
Then $T_1(0)$ decomposes as follows
$$T_1(0) = T_1(0) \cap \operatorname{Ker}(T_1) \oplus E_1 \quad \text{with } \dim(E_1) = s. \tag{4}$$
Also, combining the identities $T_1(0) = T_2(0)$ with $T_1(0) \cap \operatorname{Ker}(T_1) = T_2(0) \cap \operatorname{Ker}(T_2)$ implies that $E_1 \cap \operatorname{Ker}(T_2) = \{0\}$. Since T_2 is everywhere defined, it follows by (4) that $T_2(T_1(0)) = T_2(0) \oplus E_2$, with $\dim(E_2) = s$. Thus
$$T_2(T_1(0)) = T_1(0) \cap \operatorname{Ker}(T_1) \oplus E_1 \oplus E_2.$$

Likewise, since T_1 is onto E, it follows that

$$T_1^{-1}(T_2^{-1}(0)) = T_1(0) \cap \mathrm{Ker}(T_1) \oplus F_1 \oplus F_2, \quad \text{with } \dim(F_1) = \dim(F_2) = s.$$

The same procedure applied many times yields

$$\begin{aligned} &T_k T_{k-1} \dots T_1(0) = T_1(0) \cap \mathrm{Ker}(T_1) \oplus \tilde{E}_k && \text{with } \dim(\tilde{E}_k) = ks, \text{ and} \\ &T_1^{-1} T_2^{-1} \dots T_k^{-1}(0) = T_1(0) \cap \mathrm{Ker}(T_1) \oplus \tilde{F}_k && \text{with } \dim(\tilde{F}_k) = ks. \end{aligned}$$

The result follows since $\dim(\tilde{E}_k) = \dim(\tilde{F}_k)$. □

Corollary 2.2. *Let T be an everywhere defined congruent linear relation onto E. Then T^n is congruent for all integer n.*

2.2. Congruent spectrum.

Definition 2.3. The congruent resolvent set of T is defined by

$$\rho_s(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda I - T \text{ is congruent and surjective}\}.$$

The congruent spectrum of T is defined as $\sigma_s(T) = \mathbb{C} \backslash \rho_s(T)$.

Notice that for (singlevalued) operators, the congruent spectrum is nothing but the usual spectrum ($\sigma(T) = \mathbb{C} \setminus \rho(T)$).

Theorem 2.3. *Let T be an everywhere defined linear relation on E, and let $P(X)$ be a complex polynomial. Then*

$$\sigma_s(P(T)) \subset P(\sigma_s(T)).$$

Proof. Let $\lambda \in \mathbb{C}$ be outside $P(\sigma_s(T))$. Then

$$P(X) = \prod_{i=1}^{k} (X - \alpha_i), \text{ with } \alpha_i \in \rho_s(T), \quad \forall i = 1, \dots, k.$$

Consider the family of linear relations $\{T_i = T - \alpha_i,\, i = 1, \dots, k\}$. Then each T_i is congruent linear relation from E onto itself. Moreover

$$T_i(0) = T(0) \text{ and } T_i(0) \cap \mathrm{Ker}(T_i) = T(0) \cap \mathrm{Ker}(T), \quad \text{for all } i = 1, \dots, k.$$

It follows, from Proposition 5, that $P(T) - \lambda = \prod_{i=1}^{k} T_i$ is congruent. On the other hand, since each T_i is surjective, the $\prod_{i=1}^{k} T_i$ is surjective. This implies that λ is outside $\sigma_s(P(T))$. □

The example in Remark 2.1 shows that the reverse inclusion fails to hold.

3. Compact linear relation

Throughout this section E will denote a Banach space. A linear relation T on E is naturally associated with the (singlevalued) linear operator $Q_T T$ from E to the quotient space $E/\overline{T(0)}$ defined by $Q_T T(x) = \widehat{y}$, where $\widehat{y} := y + \overline{T(0)}$ denotes the class of some $y \in T(x)$. The norm of T is defined as the operator norm of $Q_T T$.

Definition 3.1. Let T be a linear relation on E. Then

(i) T is said to be bounded if $D(T) = E$ and $\| T \| < \infty$. The set of all bounded linear relations on E is denoted by $\mathcal{BR}(E)$.

(ii) T is said to be closed if $G(T)$ is closed. The set of all bounded and closed relations on E is denoted by $\mathcal{CR}(E)$.

(iii) T is said to be compact if $Q_T T$ is compact. The set of all bounded compact relations on E is denoted by $\mathcal{KR}(E)$.

Notice that an everywhere defined linear relation, that is closed or compact, is necessary bounded, see ([10], II.5.1, V.2.3). Also, it follows from ([10], II.1.7, II.3.13) that $\mathcal{BR}(E)$ is an algebra.

Proposition 6. *Let T and S be two linear relations on E.*

(i) *If T and S are both compact then so is $T + S$.*

(ii) *If S is bounded and T is compact then ST is compact.*

(iii) *$\mathcal{KR}(E)$ is a left-ideal of $\mathcal{BR}(E)$.*

Proof. The statements (i) and (ii) follow from ([10], IV.2.12, V.2.2, V.2.11). Consequently, (iii) follows immediately. □

Definition 3.2. A linear operator A is called a selection (or singlevalued part) of a linear relation T if $T = A + T - T$ and $D(A) = D(T)$. If A is a selection of T, then for all $x \in D(T)$, $Tx = Ax + T(0)$.

Proposition 7. *Let S be a linear relation with a bounded selection. If $T \in \mathcal{KR}(E)$ then TS is compact.*

Proof. Let A be a bounded selection of S. Then $S = A + S - S$, which implies that $TS = T(A + S - S)$. Since T is everywhere defined, it follows from ([10], I.4.2) that $TS = TA + T(S - S)$. On one hand, since A is singlevalued and bounded, it follows from ([10], V.2.12) that TA is compact. On the other hand, observe that $T(S - S) = TS - TS$ is compact. It follows from (Proposition 6, (i)) that $TS = TA + T(S - S)$, is compact. □

Corollary 3.1. *Let H be a Hilbert space. Then $\mathcal{KR}(H)$ is a two-sided ideal of $\mathcal{BR}(H)$.*

Proof. It follows from ([10], II.4.6) that each bounded linear relation has a bounded selection. Hence, the result follows from Propositions 6 and 7. $\square$

Lemma 3.2. *Let A be a selection of a linear relation T on E. If A is compact then T is compact.*

Proof. The result follows easily from the identity $T = A + (T - T)$ $\square$

In general, a selection of a compact linear relation need not be compact. For instance, let T be the relation defined on E by $T(x) = E$ for all $x \in D(T)$. Every linear operator on E with domain $D(T)$ is a selection of T.

3.1. Spectral analysis of compact relations. This section deals with linear relations, on a Banach space E, that are everywhere defined and closed. Recall that if $T \in \mathcal{CR}(E)$, then $T(0)$ is closed. Also, as the algebraic resolvent, $\rho(T) = \{\lambda \in \mathbb{C};\ (\lambda - T)^{-1}$ is everywhere defined and singlevalued$\}$ is called the (usual) resolvent set of T. The (usual) spectrum of T is the set $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

Observe that: if $T \in \mathcal{CR}(E)$ and $\overline{E}$ is not isomorphic to E, the resolvent set $\rho(T) = \emptyset$. In particular, if $\dim(E) < \infty$, then $\rho(T) = \emptyset$, for every non-singlevalued operator $T \in \mathcal{CR}(E)$. A. Sandovici et al present in [16] some structure theorems for the spectrum of a linear relation on a finite-dimensional space. In this section, we investigate the structure of some spectra of compact relations in $\mathcal{CR}(E)$. First, recall the following

Definition 3.3. Let T be a linear operator from E into a Banach space F.

(1) T is said to be upper semi-Fredholm if $R(T)$ is closed and the nullity $n(T) := \dim(\mathrm{Ker}(T))$ is finite;

(2) T is said to be lower semi-Fredholm if the deficiency $d(T) := \operatorname{codim} R(T)$ is finite;

(3) T is said to be Fredholm if $n(T)$ and $d(T)$ are both finite. The index of T is given by $i(T) := n(T) - d(T)$.

A linear relation $T \in \mathcal{CR}(E)$ is said to be Φ, Φ_+ and Φ_--relation, respectively, if $Q_T T$ is Fredholm, upper semi-Fredholm and lower semi-Fredholm. As usual, the corresponding spectrum is defined respectively by:

$$\sigma_{\Phi,\Phi_+,\Phi_-}(T) = \{\lambda \in \mathbb{C} \text{ such that } T - \lambda I \text{ fails to be } \Phi, \Phi_+, \Phi_-\text{-relation}\}.$$

According to ([18], Theorems 5.10, 5.22, 5.28), the class of Fredholm, upper semi-Fredholm and lower semi-Fredholm bounded operators between two Banach spaces is closed under compact perturbation, with stability of the index. Notice that for upper (resp. lower) semi-Fredholm the index is assumed to be $+\infty$ (resp. $-\infty$).

Theorem 3.3. *Let $T \in \mathcal{CR}(E)$ be compact. Then*

(1) $\rho(T) = \emptyset$ *if and only if* $T(0) \neq \{0\}$.

(2) $\sigma_{\Phi_-}(T) = \{0\}$

(3) *If* $T(0)$ *is finite dimensional, then* $\sigma_{\Phi}(T) = \{0\}$

(4) *If* $T(0)$ *is infinite dimensional, then* $\sigma_{\Phi}(T) = \mathbb{C}$.

(5) $\sigma_{\Phi_+}(T) = \sigma_{\Phi}(T)$.

Proof. For $\beta \in \mathbb{C}$, consider the bounded operator $T_\beta = Q_T(\beta I - T)$ from E to $\overline{E}$. If $\beta \neq 0$ then

$$Q_T = -\frac{1}{\beta}T_\beta + \frac{1}{\beta}Q_T T. \tag{5}$$

To prove the first statement, suppose that there exists $\beta \neq 0 \in \rho(T)$, so that T_β is invertible (injective and surjective). Since $Q_T T$ is compact, it follows, by (5), that Q_T is a Fredholm operator with null index (see [18], Theorem 5.10). On the other hand Q_T is surjective, and therefore it is injective. Thus $\mathrm{Ker}(Q_T) = T(0) = \{0\}$. We have thus proven the first statement.

Now, observe that Q_T is already lower semi-Fredholm. Moreover, Q_T is upper semi-Fredholm if and only if $\dim(T(0)) < \infty$. Thus, the other statements follow immediately from (5). □

3.2. Congruent spectrum of compact relations. A well-known result of functional analysis is that, the spectrum of a compact operator is at most countable. In the next theorem we characterize the congruent spectrum for compact linear relation. We show, in this context, that compact linear relations in $\mathcal{CR}(E)$ behave almost like compact operators. We shall need the following lemma.

Lemma 3.4. *Let T be a linear relation on E satisfying that $T(0)$ is closed and complemented in E. Let E_1 be a complement of $T(0)$, that is a closed subspace being so that $E = T(0) \oplus E_1$. Denote by P be the projection on E_1 with kernel $T(0)$.*
Then $H_T := PQ_T^{-1}$ is a bounded operator from $\overline{E}$ to E satisfying:

(i) $H_T Q_T = P$.

(ii) $Q_T H_T = I_{\overline{E}}$.

(iii) *If* $T(0) \subset \mathrm{Ker}(T)$, *then* $Q_T T H_T$ *does not depend on the choice of the complement of* $T(0)$.

Moreover, PT *is a selection of* T *that is compact if so is* T.

PROOF. It is easy to check that H_T is a bounded operator satisfying $H_T Q_T = P$ and $Q_T H_T = I_{\overline{E}}$. All that remains is to show (iii). Suppose that $T(0) \subset \mathrm{Ker}(T)$, and let E_2 be a complement of $T(0)$. Denote by P_2 the projection on E_2 associated with the decomposition $E = T(0) \oplus E_2$. Consider $G_T : \overline{E} \to E : \overline{x} \mapsto P_2(x)$, so that $G_T Q_T = P_2$ and $Q_T G_T = I_{\overline{E}}$. Hence

$$Q_T T G_T = Q_T T P G_T + Q_T T(I - P)G_T = Q_T T H_T + Q_T T(I - P)G_T.$$

On the other hand, since $R((I - P)G_T) \subset T(0) \subset \mathrm{Ker}(T)$, it follows immediately that $Q_T T(I - P)G_T = 0$, which implies that $Q_T T G_T = Q_T T H_T$. Hence, (iii) has been shown.

Finally, it follows from ([10], I.5.2) that PT is a selection of T. Suppose that T is compact. Since H_T is bounded, it follows that $PT = H_T Q_T T$ is compact. $\square$

Theorem 3.5. *Let* $T \in \mathcal{CR}(E)$ *be compact. Assume that* $T(0) \subset \mathrm{Ker}(T)$ *and that* $T(0)$ *is complemented in* E. *Then*

(i) $s((I - T)^{-1})$ *is finite.*

(ii) $I - T$ *is congruent if and only if it is surjective.*

(iii) $\sigma_s(T)$ *is countable.*

(iv) *If* $\overline{E}$ *is infinite dimensional, then* $0 \in \sigma_s(T)$.

PROOF. (i) Let P and H_T be as in Lemma 3.4. We first claim that

$$Q_T(\mathrm{Ker}(I - T)) = \mathrm{Ker}(Q_T(I - T)H_T). \tag{6}$$

Let $\overline{x} = Q_T(x)$, for some $x \in \mathrm{Ker}(I - T)$. Then $Q_T(I - T)H_T\overline{x} = Q_T(I - T)P(x)$. Since $x - P(x) \in T(0) \subset \mathrm{Ker}(T)$, it follows that $P(x) \in \mathrm{Ker}(I - T)$, which implies that $Q_T(I - T)H_T\overline{x} = \overline{0}$. We have thus proven the first inclusion $Q_T(\mathrm{Ker}(I - T)) \subset \mathrm{Ker}(Q_T(I - T)H_T)$.

Conversely, let $\overline{x} = Q_T(x)$ for some $x \in E$ be such that $Q_T(I - T)H_T\overline{x} = \overline{0}$. Then $Q_T(I - T)P(x) = \overline{0}$, so that $P(x) \in \mathrm{Ker}(I - T)$. Thus $Q_T(P(x)) = \overline{x} \in Q_T(\mathrm{Ker}(I - T))$. Hence, (6) is shown.

On the other hand,

$$Q_T(I - T)H_T = I_{\overline{E}} - Q_T T H_T. \tag{7}$$

Since $Q_T T H_T$ is a compact (singlevalued) operator from $\overline{E}$ into itself, it follows that $\mathrm{Ker}(Q_T(I - T)H_T)$ is finite dimensional. Then $\mathrm{Ker}(I - T)/T(0)$ has a finite dimension.

(ii) Since $T(0) \subset \mathrm{Ker}(T)$, it follows that $s(T) = 0$, and therefore $s(I-T) = 0$. Hence, $I - T$ is congruent if and only if $\mathrm{Ker}(I - T) = T(0)$. By this and (6) it follows that:

$$I - T \text{ is congruent if and only if } Q_T(I - T)H_T \text{ is injective.} \tag{8}$$

If $I - T$ is surjective, then so is $Q_T(I - T)$. Let $\overline{y} \in \overline{E}$, there exists $x \in E$ such that $Q_T(I - T)x = \overline{y}$. On the other hand, $H_T\overline{x} = P(x)$ and $T(0) \subset \mathrm{Ker}(I - T)$. Then $Q_T(I-T)x = Q_T(I-T)P(x)$, which shows that $Q_T(I-T)H_T$ is surjective. Conversely, if $Q_T(I - T)H_T$ is surjective, then so is $Q_T(I - T)$, which implies that $I - T$ is surjective. We have thus proven that:

$$I - T \text{ is surjective if and only if so is } Q_T(I - T)H_T. \tag{9}$$

On the other hand, since $Q_T TH_T$ is compact, according to ([12], Ch 6, Proposition 1.6), it follows by (7) that:

$$Q_T(I - T)H_T \text{ is surjective if and only if it is injective.}$$

In combination with (9) and (8), the last assertion yields (ii).

(iii) Proceeding as above, it follows that:

$$\lambda \neq 0 \in \rho_s(T) \text{ if and only if } \lambda \in \rho(Q_T TH_T).$$

Since $Q_T TH_T$ is a compact operator, the result follows from ([12], Ch 6, Theorem 1.8.).

(iv) Suppose that $0 \in \rho_s(T)$, so that $Q_T TH_T$ is invertible (injective and surjective). Then $I_{\overline{E}}$ is compact, and therefore $\overline{E}$ is finite dimensional. $\square$

Proposition 8. *Let $T \in \mathcal{CR}(E)$ be such that $T(0)$ is complemented in E, and let P be the associated projection with kernel $T(0)$. Consider $\tilde{T}$ the relation defined on E by $\tilde{T}(x) = PT(x) + T(0) \cap T^{-1}(0)$. Then $\tilde{T} \in \mathcal{CR}(E)$ and satisfies $\tilde{T}(0) \subset \mathrm{Ker}(\tilde{T})$. Moreover, if T is compact and $s(T) < \infty$, then $\tilde{T}$ is a compact.*

Proof. Since T is everywhere defined and closed, it follows that $T(0)$ is closed and that T is bounded, which implies that $\tilde{T}(0) = T(0) \cap T^{-1}(0)$ is closed. Also, PT is bounded. Hence $\tilde{T}$ is closed.

Next we claim that $\mathrm{Ker}(\tilde{T}) = \mathrm{Ker}(T)$. To see this, let $x \in \mathrm{Ker}(\tilde{T})$. Then $PTx \in T(0) \cap T^{-1}(0) \subset T(0)$. This implies that $Tx = T(0)$ that is $x \in \mathrm{Ker}(T)$.

Conversely, let $x \in \mathrm{Ker}(T)$. Then $T(x) = T(0)$, so that $PT(x) = 0$, and therefore $x \in \mathrm{Ker}(\tilde{T})$. Hence our claim has been proved. Also, since $\tilde{T}(0) = T(0) \cap T^{-1}(0) \subset \mathrm{Ker}(T)$, it follows that $\tilde{T}(0) \subset \mathrm{Ker}(\tilde{T})$.

Suppose now that T is compact and that $s(T) < \infty$. Let $\widetilde{x}$ (resp $\overline{x}$) denotes the class of some $x \in E$ in $E/\tilde{T}(0)$ (resp. $E/T(0)$). Consider $S : E/\tilde{T}(0) \to E/T(0)$ defined by $S(\widetilde{x}) = \overline{x}$. Then the operator S is surjective, and $\mathrm{Ker}(S)$ is finite dimensional since $s(T) < \infty$. Thus S is a Fredholm operator. Also, $Q_T T = SQ_{\tilde{T}}\tilde{T}$. Therefore, since $Q_T T$ is compact, it follows that $Q_{\tilde{T}}\tilde{T}$ is compact, that is $\tilde{T}$ is compact. □

Theorem 3.6. *Let $T \in \mathcal{CR}(E)$ be compact. Assume that $T(0)$ is complemented in E and that $s(T) < \infty$. Then:*

(i) *The range of $I - T$ has a finite codimension.*

(ii) *$\sigma_s(T)$ is countable.*

(iii) *$s((I - T)^{-1})$ is finite.*

Proof. Since Q_T is surjective and $Q_T T$ is compact, it follows that $Q_T(I-T)$ is lower semi-Fredholm (see [18], Theoremm 5.28). Then $I - T$ is lower semi-Fredholm (see [10], V.1.1). This completes the proof of (i).

Let P and $\tilde{T}$ be as defined in Proposition 8. According to Theorem 3.5, $\sigma_s(\tilde{T})$ is countable. To prove (ii), it's enough to show that $\sigma_s(T) \subset \sigma_s(\tilde{T})$. Let $\lambda \neq 0 \in \rho_s(\tilde{T})$. If $x \in \mathrm{Ker}(\lambda I - T)$, then $(\lambda I - PT(x)) \in T(0)$. Consider $\Psi_\lambda(x)$ the class of $(\lambda I - PT)(x)$ in $\Delta(T) = T(0)/\tilde{T}(0)$. We claim that:

$$\Psi_\lambda : \mathrm{Ker}(\lambda I - T) \to \Delta(T) \text{ is surjective.} \tag{10}$$

Indeed, let $y \in T(0)$, and denote by $\bar{y}$ the class of y in $\Delta(T)$. Since $\lambda \neq 0 \in \rho_s(\tilde{T})$, it follows by Theorem 3.5 that $\lambda I - \tilde{T}$ is surjective. Then there exists $x \in E$ such that $(\lambda I - \tilde{T})x = y + \tilde{T}(0)$. This implies that $(\lambda I - PT)x \in T(0)$, and therefore $(\lambda I - T)x = T(0)$. Hence $x \in \mathrm{Ker}(\lambda I - T)$ satisfies $\Psi_\lambda(x) = \bar{y}$. We have thus proven our claim. Furthermore:

$$\begin{aligned}\mathrm{Ker}(\Psi_\lambda) &= \{x \in \mathrm{Ker}(\lambda I - T) \text{ such that } (\lambda I - PT)x \in \tilde{T}(0)\}\\ &= \{x \in \mathrm{Ker}(\lambda I - T) \text{ such that } (\lambda I - \tilde{T})x = \tilde{T}(0)\} = \mathrm{Ker}(\lambda I - \tilde{T}).\end{aligned}$$

Then, it follows from (10) that:

$$\dim \mathrm{Ker}(\lambda I - T)/Ker(\lambda I - \tilde{T}) = \dim \ \Delta(T) = s(T). \tag{11}$$

On the other hand, $\tilde{T}(0) \subset \mathrm{Ker}(\tilde{T})$, and therefore $s(\tilde{T}) = 0$, which implies by Corollary 2.1 that $s(\lambda I - \tilde{T}) = 0$. Hence, since $\lambda \in \rho_s(\tilde{T})$ it follows that $\mathrm{Ker}(\lambda I -$

$\tilde{T}) = \tilde{T}(0)$. Observing that $\tilde{T}(0) = T(0) \cap T^{-1}(0) = T(0) \cap (\lambda I - T)^{-1}(0)$, the identity (11) shows that $\lambda I - T$ is congruent. On the other hand, it follows from (Theorem 3.5, (ii)) that $\lambda I - \tilde{T}$ is surjective, and then so is $\lambda I - T$. Thus $\lambda \in \rho_s(T)$, and therefore $\rho_s(\tilde{T}) \subset \rho_s(T)$. Hence (ii) has been shown.

Turning to (iii), by equation (11), $\dim(\operatorname{Ker}(I - T)/\operatorname{Ker}(I - \tilde{T})) < \infty$. On the other hand, according to (Theorem 3.5, (i)), it follows that $\dim(\operatorname{Ker}(I - \tilde{T})/\tilde{T}(0)) < \infty$. Then $s((I - T)^{-1}) = \dim(\operatorname{Ker}(I - T)/\tilde{T}(0))$ is finite. □

Corollary 3.7. *Let $T \in \mathcal{CR}(E)$ be compact. Assume that* $\dim(T(0)) < \infty$. *Then:*

(i) $0 \in \sigma_s(T)$.

(ii) $\operatorname{Ker}(I - T)$ *is finite dimensional.*

(iii) $R(I - T)$ *has finite codimension.*

(iv) $\sigma_s(T)$ *is countable.*

4. Stability results

Observe that, by Proposition 1, if $s(T)$ is finite then so is $s(T + K)$ for all linear relation K satisfying $K(T(0) \cap T^{-1}(0)) \subset T(0)$. In the following theorem we get further important stability results.

Theorem 4.1. *Let T and K be in $\mathcal{CR}(E)$. Assume that K is compact and that $K(T(0) \cap T^{-1}(0)) \subset T(0)$. Then, if T is closed range and $s(T^{-1})$ is finite, $T + K$ is closed range and $s((T + K)^{-1})$ is finite.*

Proof. Let P_T be the canonical projection on $\hat{E} := E/(T(0) \cap T^{-1}(0))$, and consider the operator $\hat{T} = Q_T T P_T^{-1} : \hat{E} \to \overline{E}$. Then $\operatorname{Ker}(\hat{T}) = P_T(\operatorname{Ker}(T))$, which implies that $n(\hat{T}) = s(T^{-1})$ is finite. Also, $R(\hat{T}) = R(Q_T T)$ is closed since T is closed range. Hence $\hat{T}$ is an upper semi-Fredholm operator. On the other hand, since $K(T(0) \cap T^{-1}(0)) \subset T(0)$, the operator $\hat{K} = Q_T K P_T^{-1} : \hat{E} \to \overline{E}$ is singlevalued. Also $\hat{K}(B_{\hat{E}}(0, 1)) = Q_T K(B_E(0, 1))$, where $B_{\hat{E}}(0, 1)$ (resp. $B_E(0, 1)$) denotes the unit ball of $\hat{E}$ (resp. E). According to Proposition 6, $Q_T K$ is compact, which implies that $\hat{K}$ is compact. Since T and K are everywhere defined, it follows from ([10], I.4.2) that $Q_T(T+K)P_T^{-1} = \hat{T}+\hat{K}$. Now, since $\hat{K}$ is compact and $\hat{T}$ is upper semi-Fredholm, $\hat{T}+\hat{K}$ is also upper semi-Fredholm. This implies that $R(Q_T(T+K)) = R(\hat{T}+\hat{K})$ is closed, and then so is $R(T+K)$. Also, since $T(0)\cap T^{-1}(0) \subset \operatorname{Ker}(T+K)$, it follows that $P_T(\operatorname{Ker}(T+K)) = \operatorname{Ker}(\hat{T}+\hat{K})$, which is finite dimensional. Hence $s((T + K)^{-1})$ is finite. □

The following lemma shall be used in the proof of next theorem.

Lemma 4.2. *Let T be an upper semi-Fredholm operator between normed vector spaces E and F, with negative index. There exists a finite rank operator K such that $T+K$ is injective closed range.*

PROOF. Write $\mathrm{Ker}(T) = \mathrm{vect}\{e_1, \ldots, e_{n(T)}\}$, and $E = \mathrm{Ker}(T) \oplus E_1$. Since $d(T) \geq n(T)$, there exits $F_1 = \mathrm{vect}\{f_1, \ldots, f_{n(T)}\} \subset F$, with $\dim(F_1) = n(T)$, such that $F_1 \cap R(T) = \{0\}$. The finite rank operator $K : E \to F$ defined by $K(x) = 0$ for all $x \in E_1$ and $K(e_i) = f_i$, for $i = 1, \ldots, n(T)$ provides an answer for the lemma. $\square$

Theorem 4.3. *Let $T \in \mathcal{CR}(E)$.*

(i) *If $s(T)$ is finite, then there exists some compact linear operator K_1 such that $s(T+K_1) = 0$.*

(ii) *Assume that T is closed range, $s(T^{-1})$ is finite and that $s(T^{-1}) \leq d(T)$. Then there exists some compact linear relation $K_2 \in \mathcal{CR}(E)$ such that $T+K_2$ is closed range and $s((T+K_2)^{-1}) = 0$.*

(iii) *Assume that T is closed range and that it is not lower semi-Fredholm. Then if $s(T)$ and $s(T^{-1})$ are finite, there exists some compact linear relation $K \in \mathcal{CR}(E)$ such that $T+K$ is closed range and $s(T+K) = s((T+K)^{-1}) = 0$.*

PROOF. Since $s(T)$ is finite, $T(0) = T(0) \cap T^{-1}(0) \oplus E_1$ for some subspace E_1 of E, with $\dim(E_1) = s(T) := s$. Write $E_1 = \mathrm{vect}\{e_1, \ldots, e_s\}$, and consider $u_i \in T(e_i)$, $i = 1, \ldots, s$. Let a finite rank operator K_1 be such that, $R(K_1) = \mathrm{vect}\{u_1, \ldots, u_s\}$, $K_1(e_i) = -u_i$ for each i, and $K_1(T(0) \cap T^{-1}(0)) = \{0\}$. Then $(T+K_1)(0) = T(0) \subset \mathrm{Ker}(T+K_1)$. This completes the proof of (i).

As to (ii), an argument similar to that given in the proof of Theorem 4.1, shows that the operator $\hat{T} = Q_T T P_T^{-1} : \hat{E} \to \overline{E}$ is singlevalued upper semi-Fredholm, with $n(\hat{T}) = s(T^{-1})$. Moreover, since $R(\hat{T}) = R(Q_T T)$, it follows immediately that $d(\hat{T}) = d(T) \leq s(T^{-1}) = n(\hat{T})$. Hence, by Lemma 4.2, there exists a finite rank operator $\hat{K}_2 : \hat{E} \to \overline{E}$ such that $\hat{T} + \hat{K}_2$ is injective. Consider the compact linear relation $K_2 = Q_T^{-1} \hat{K}_2 P_T$. Observe that K_2 is bounded and that $K_2(0) = T(0)$ is closed, showing that K_2 is closed. Also

$$K_2(T(0) \cap T^{-1}(0)) = T(0). \tag{12}$$

Since $\hat{K}_2 = Q_T K_2 P_T^{-1}$, it follows that $P_T(\mathrm{Ker}(T+K_2)) = \mathrm{Ker}(\hat{T} + \hat{K}_2) = \{0\}$. Thus $s((T+K_2)^{-1}) = \dim(P_T(\mathrm{Ker}(T+K_2))) = 0$. Finally, it follows from Theorem 4.1 together with the identity (12), that $T+K_2$ is closed range. Hence (ii) has been proved.

Turning to the proof of (iii). By (i), there exists a compact operator K_1 such that $s(T + K_1) = 0$. Since $K_1(T(0) \cap T^{-1}(0)) = \{0\} \subset T(0)$, and T is closed range, it follows from Theorem 4.1 that $T + K_1$ is closed range and that $s((T + K_1)^{-1})$ is finite. Moreover, T is not lower semi-Fredholm, so neither is $T + K_1$. Hence, by (ii), there is a compact linear relation $K_2 \in \mathcal{CR}(E)$ such that $T + K_1 + K_2$ is closed range, and $s((T + K_1 + K_2)^{-1}) = 0$. Since $s(T + K_1) = 0$, it follows immediately that $(T + K_1)(0) \subset \mathrm{Ker}(T + K_1)$. Hence, $K_2((T + K_1)(0)) = (T + K_1)(0)$. It follows from Proposition 1 that $s(T + K_1 + K_2) = 0$. It is clear that $K := K_1 + K_2 \in \mathcal{CR}(E)$. This completes the proof. □

References

[1] El H. Alaarabiou, Calcul fonctionnel et puissance fractionnaire d'opérateurs linéaires multivoques non négatifs, *C. R. A. S. Paris* **313** (1991), 163–166.

[2] T. Alvarez, On the Browder essential spectrum of linear relation, *Publ. Math. Debrecen* **73** (2008), 145–154.

[3] T. Alvarez, Small perturbation of normally solvable relations, *Publ. Math. Debrecen* **80** (2012), 155–168.

[4] T. Alvarez, A. Ammar and A. Jeribi, On the essential spectra of some matrix of linear relations, *Math. Methods Appl. Sci.* **37** (2014), 620-644.

[5] T. Alvarez, R. W. Cross and D. Wilcox, Multivalued Fredholm type operators with abstract generalised inverses, *J. Math. Anal. App.* **261** (2001), 403–417.

[6] T. Alvarez, R. W. Cross and D. Wilcox, Perturbation theorems for upper and lower semi-Fredholm linear relations, *Publ. Math. Debrecen.* **65** (2004), 179–191.

[7] T. Alvarez and D. Wilcox, Perturbation theory of multivalued Atkinson operators in normed spaces, *Bull. Austral. Math. Soc.* **76** (2007), 195–204.

[8] R. Arens, Operational calculus of linear relations, *Pacific J. Math.* **11** (1961), 9-23.

[9] Y. Chamkha and M. Mnif, Browder spectra of upper triangular matrix linear relations, *Publ. Math. Debrecen* **82** (2013), 569–590.

[10] R. W. Cross, Multivalued Linear Operators, Monographs and Textbooks in Pure and Applied Mathematics 213, *Marcel Dekker*, 1998.

[11] F. Fakhfakh and M. Mnif, Perturbation theory of lower semi-Browder multivalued linear operator, *Publ. Math. Debrecen* **78** (2011), 595-606.

[12] F. Hirsch and G. Lacombe, Elements of Functional Analysis, *Springer*, *New York*, 1999.

[13] J. PH. Labrousse, A. Sandovici and H. De Snoo, Quasi-Fredholm relations in Hilbert spaces, *Stad. Cercet. Stiint. Ser. Mat. Univ. Bacau* **16** (2006), 93–105.

[14] J. PH. Labrousse, A. Sandovici, H. De Snoo and H. Winkler, The Kato decomposition of quasi-Fredholm relations, *Oper. Matrices* **4** (2010).

[15] A. Sandovici, Some basic properties of polynomials in a linear relation in linear spaces, *Oper. Theory Adv. Appl.* **175** (2007), 231–240.

[16] A. Sandovici and H. De Snoo, The structure of linear relations in Euclidean spaces, *Lin. Alg. and its App.* **397** (2007), 2160–2171.

[17] A. Sandovici and H. De Snoo, An index formula for the product of linear relations, *Lin. Alg. and its App.* **431** (2009), 141–169.

[18] M. Schechter, Principal of Functional Analysis, *Academic Press*, *New York*, 1971.

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