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Groups whose proper subgroups are (locally finite)-by-(locally nilpotent)

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Abstract. If $\mathfrak{X}$ is a class of groups, then a group G is called a *minimal non-$\mathfrak{X}$-group* if it is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$. Let π be a set of primes and let $\mathfrak{X}$ be a quotient and subgroup closed class of locally nilpotent groups such that every infinite locally graded minimal non-$\mathfrak{X}$-group is a countable p-group for some prime p. Our main result in the present paper states that G is an infinitely generated minimal non-$(L\mathfrak{F}_\pi)\mathfrak{X}$-group if and only if there exists a prime $p \notin \pi$ such that G is an infinitely generated minimal non-$\mathfrak{X}$ p-group; where $L\mathfrak{F}_\pi$ denotes the class of locally finite π-groups.

1. Introduction

If $\mathfrak{X}$ is a class of groups, then a group G is called a *minimal non-$\mathfrak{X}$-group* if it is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$. Many results have been obtained on minimal non-$\mathfrak{X}$-groups for several choices of $\mathfrak{X}$. In particular, in [10] a complete description of infinitely generated minimal non-nilpotent groups having a maximal subgroup is given. These groups are metabelian Chernikov p-groups, where p is a prime. Later in [14], infinitely generated minimal non-nilpotent groups without maximal subgroups have been studied and it was proved, among many results, that they are countable p-groups. In [15] it is proved that if G is a minimal non-$(L\mathfrak{F})\mathfrak{N}$ (respectively, non-$(L\mathfrak{F})\mathfrak{N}_c$) group, then G is a finitely generated perfect group which has no proper subgroups of finite index and $G/\operatorname{Frat}(G)$ is simple, where $L\mathfrak{F}$ (respectively, $\mathfrak{N}$, $\mathfrak{N}_c$) denotes the class of locally finite (respectively, nilpotent, nilpotent of class at most c) groups. Therefore there are

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no minimal non-$(L\mathfrak{F})\mathfrak{N}$-groups (respectively, non-$(L\mathfrak{F})\mathfrak{N}_c$-groups) which are infinitely generated (or equivalently locally graded). In the present paper, we generalize these last results by considering the classes $(L\mathfrak{F}_\pi)\mathfrak{N}$ and $(L\mathfrak{F}_\pi)\mathfrak{N}_c$, where π is a given set of primes and $L\mathfrak{F}_\pi$ denotes the class of locally finite π-groups. It turns out that infinitely generated minimal non-$(L\mathfrak{F}_\pi)\mathfrak{N}$-groups exist. For if G is an infinitely generated minimal non-nilpotent group, then it is a p-group for some prime p and hence it is an infinitely generated minimal non-$(L\mathfrak{F}_\pi)\mathfrak{N}$-group for every set π not containing p. We will prove that the converse is also true. In fact our results on $(L\mathfrak{F}_\pi)\mathfrak{N}$ and $(L\mathfrak{F}_\pi)\mathfrak{N}_c$ will be consequences of more general results on $(L\mathfrak{F}_\pi)\mathfrak{X}$ (respectively, $(L\mathfrak{F}_\pi)\mathfrak{V}$), where $\mathfrak{X}$ (respectively, $\mathfrak{V}$) denotes a quotient and subgroup closed class (respectively, a variety) of locally nilpotent groups such that infinite locally graded minimal non-$\mathfrak{X}$-groups are countable p-groups. Our main result (Theorem 4.1) states that a group G is an infinitely generated minimal non-$(L\mathfrak{F}_\pi)\mathfrak{X}$-group if and only if there exists a prime $p \notin \pi$ such that G is an infinitely generated minimal non-$\mathfrak{X}$ p-group.

Although our main result concerns the class $(L\mathfrak{F}_\pi)\mathfrak{X}$, we will introduce two more general classes, $\Omega_\pi\mathfrak{Z}$ and $\Omega_\pi\mathfrak{Y}$, in order to make our preliminary results as general as we can do, for further investigations.

Recall that a group is locally graded if every non-trivial finitely generated subgroup has a proper non-trivial finite homomorphic image.

Our notation and terminology are standard and follow [13]. In the sequel, π denotes a given set of primes. If H is a subgroup of a group G then we denote by $T_\pi(H)$ the subgroup generated by all π-elements of H, that is elements of finite order whose prime divisors belong to π. We will write simply T_π for $T_\pi(G)$. A (not necessary periodic) group G is said to be π-free if it has no π-elements. Finally we denote by $\mathfrak{T}_\pi$ the class of π-groups, that is groups with only π-elements.

2. Minimal non-$\Omega_\pi\mathfrak{Z}$-groups

Let $\mathfrak{Z}$ be a quotient and subgroup closed class of locally nilpotent groups and let Ω_π be either the class $L\mathfrak{F}_\pi$ or $\mathfrak{T}_\pi$. In this section we will prove a result on infinitely generated minimal non-$\Omega_\pi\mathfrak{Z}$-groups which will be used many times in this note.

First we prove two useful lemmas.

Lemma 2.1. *Let G be a group generated by π-elements. If G belongs to $L(\Omega_\pi\mathfrak{N})$, then it belongs to Ω_π.*

Proof. Since $L(\Omega_\pi) = \Omega_\pi$, there is no loss of generality if we assume that G is finitely generated and hence it belongs to $\Omega_\pi\mathfrak{N}$. Let N be a normal subgroup of G such that $N \in \Omega_\pi$ and G/N is nilpotent. We have that $(G/N)/(G/N)'$ is an abelian group generated by finitely many π-elements and hence it is a finite π-group. We deduce by [13, Corollary of Theorem 2.26] that G/N is also a finite π-group as it is nilpotent. Therefore G belongs to Ω_π. □

Lemma 2.2. *Let G be an infinitely generated minimal non-$\Omega_\pi\mathfrak{Z}$-group. Then T_π belongs to Ω_π and if H is a proper subgroup of G, then so is HT_π.*

Proof. As G is not finitely generated, $G \in L(\Omega_\pi\mathfrak{Z}) \subseteq L(\Omega_\pi\mathfrak{N})$ and hence $T_\pi \in \Omega_\pi$ by Lemma 2.1. Now let H be a proper subgroup of G. If $G = HT_\pi$, then $G/T_\pi \simeq H/H \cap T_\pi$ is in $\Omega_\pi\mathfrak{Z}$ and hence it belongs to $\mathfrak{Z}$ since it is a π-free group. It follows that $G \in \Omega_\pi\mathfrak{Z}$, which is a contradiction. Therefore HT_π is a proper subgroup of G. □

Proposition 2.3. *An infinitely generated group G is a minimal non-$\Omega_\pi\mathfrak{Z}$-group if and only if the following conditions are satisfied:*

(i) *T_π belongs to Ω_π, and if H is a proper subgroup of G then so is HT_π;*

(ii) *G/T_π is a π-free infinitely generated minimal non-$\mathfrak{Z}$-group.*

Proof. Assume that G is an infinitely generated minimal non-$\Omega_\pi\mathfrak{Z}$-group. Then (i) is satisfied by Lemma 2.2. As $T_\pi \in \Omega_\pi$ by (i), we have that $G/T_\pi \notin \mathfrak{Z}$. Since G is in $L(\Omega_\pi\mathfrak{Z})$ and G/T_π is π-free, we deduce that G/T_π is a minimal non-$\mathfrak{Z}$-group which is locally in $\mathfrak{Z}$. Hence G/T_π is an infinitely generated π-free-group and (ii) is proved.

Conversely, assume that conditions (i) and (ii) are satisfied. So G/T_π is an infinitely generated π-free group and a minimal non-$\mathfrak{Z}$-group. Hence $G \notin \Omega_\pi\mathfrak{Z}$ and is infinitely generated. Let H be a proper subgroup of G. Then $H \cap T_\pi \in \Omega_\pi$ and $H/H\cap T_\pi \simeq HT_\pi/T_\pi \neq G/T_\pi$ by (ii). It follows that $H/H\cap T_\pi \in \mathfrak{Z}$ and hence $H \in \Omega_\pi\mathfrak{Z}$. Therefore G is an infinitely generated minimal non-$\Omega_\pi\mathfrak{Z}$-group. □

Corollary 2.4. *Let G be an infinitely generated minimal non-$\Omega_\pi\mathfrak{Z}$-group. If G splits over T_π, that is if there exists a subgroup H of G such that $T_\pi \cap H = 1$ and $G = T_\pi H$, then G is a π-free infinitely generated minimal non-$\mathfrak{Z}$-group.*

Proof. Assume that G is an infinitely generated minimal non-$\Omega_\pi\mathfrak{Z}$-group such that $G = T_\pi H$. It follows by condition (i) of Proposition 2.3 that $G = H$. Since $T_\pi \cap H = 1$ we deduce that G is π-free and hence $T_\pi = 1$. It follows by condition (ii) of Proposition 2.3 that G is a π-free infinitely generated minimal non-$\mathfrak{Z}$-group. □

By Proposition 2.3, if there exists an infinitely generated minimal non-$\Omega_\pi\mathfrak{Z}$-group, then there exists an infinitely generated minimal non-$\mathfrak{Z}$-group. Since minimal non-$L\mathfrak{N}$-groups are obviously finitely generated, $L\mathfrak{N}$ being the class of locally nilpotent groups, we have the following application of Proposition 2.3.

Corollary 2.5. *If G is an infinitely generated group whose proper subgroups are in $\Omega_\pi(L\mathfrak{N})$, then so is G.*

3. Minimal non-$\Omega_\pi\mathfrak{Y}$-groups

Let $\mathfrak{Y}$ be a quotient and subgroup closed class of locally nilpotent groups such that locally graded minimal non-$\mathfrak{Y}$-groups are periodic. Our main result in this section is about finitely generated non-$(L\mathfrak{F}_\pi)\mathfrak{Y}$-groups.

Clearly infinitely generated minimal non-$\Omega_\pi\mathfrak{Y}$-groups are characterized by Proposition 2.3 which implies that they are periodic. If we take π to be the set of all primes, then $\Omega_\pi = \Omega$ is either the class of locally finite or periodic groups. In this case a non-trivial group cannot be periodic and π-free, so condition (ii) of Propositon 2.3 is not satisfied and hence Proposition 2.3 has the following consequence.

Corollary 3.1. *If G is an infinitely generated group all of whose proper subgroups are in $\Omega\mathfrak{Y}$, then so is G.*

Now we will prove a result on finitely generated minimal non-$(L\mathfrak{F}_\pi)\mathfrak{Y}$-groups. To this end we need the following lemma which implies together with the remark above that infinite locally graded minimal non-$\Omega_\pi\mathfrak{Y}$-groups are periodic.

Lemma 3.2. *Let G be a non-periodic group all of whose proper subgroups are in $\Omega_\pi\mathfrak{Y}$. If G is finitely generated and has a proper subgroup of finite index, then it belongs to $\Omega_\pi\mathfrak{Y}$.*

Proof. Let G as stated and let N be a normal proper subgroup of finite index. Then N is finitely generated and hence $N \in \Omega_\pi\mathfrak{N}$. It follows that if $H := T_\pi(N)$, then $H \in \Omega_\pi$ and $G/H \in \mathfrak{N}\mathfrak{F}$. Therefore G/H satisfies the maximal condition on subgroups and hence every proper subgroup of G/H belongs to $\mathfrak{F}_\pi\mathfrak{N}$. Consequently every proper subgroup of G is in $\Omega_\pi\mathfrak{N}$. We deduce by [15, Corollary 2.2] that G is in $\Omega\mathfrak{N}$ and hence it has a torsion part. Therefore T_π is periodic and hence it is proper in G. So that $T_\pi \in \Omega_\pi$ by Lemma 2.1. Since G/T_π is a π-free group, all its proper subgroups belong to $\mathfrak{Y}$. As G is not periodic, G/T_π is a non-periodic nilpotent-by-finite group and hence it cannot be a minimal non-$\mathfrak{Y}$-group. Consequently $G/T_\pi \in \mathfrak{Y}$ and so $G \in \Omega_\pi\mathfrak{Y}$, as claimed. □

Note that if a minimal non-$(L\mathfrak{F}_\pi)\mathfrak{Y}$-group is infinitely generated then it is an infinite locally graded group; the next result implies that the converse is true, that is an infinite locally graded minimal non-$(L\mathfrak{F}_\pi)\mathfrak{Y}$-group is infinitely generated.

Proposition 3.3. *Let G be an infinite finitely generated minimal non-$(L\mathfrak{F}_\pi)\mathfrak{Y}$-group. Then G is a perfect group which has no proper subgroups of finite index and $G/\operatorname{Frat}(G)$ is simple.*

Proof. Let G as stated. Assume that G has a proper normal subgroup of finite index, say N. Then N is finitely generated and hence $N \in (L\mathfrak{F}_\pi)\mathfrak{N}$. Since G is infinite, so is N and hence N is not periodic. It follows by Lemma 3.2 that G is in $(L\mathfrak{F}_\pi)\mathfrak{Y}$, which is a contradiction. So G has no proper subgroups of finite index and hence it is perfect. Since G is finitely generated, $G/\operatorname{Frat}(G)$ is non-trivial. Let $N/\operatorname{Frat}(G)$ be a non-trivial normal subgroup of $G/\operatorname{Frat}(G)$. Therefore there exists a maximal subgroup M of G such that $N \nsubseteq M$ and hence $G = MN$. Let $F := T_\pi(M)$; since $M \in (L\mathfrak{F}_\pi)\mathfrak{Y}$, $F \in L\mathfrak{F}_\pi$ by Lemma 2.1. If $g \in G$, then it is easy to see that $(NF)^g \leq NF$ and hence NF is normal in G. Moreover

$$G/NF = NM/NF \simeq M/M \cap NF \simeq (M/F)/(M \cap NF/F)$$

which is a finitely generated $\mathfrak{Y}$-group and hence it is nilpotent. But G is perfect, so we deduce that $G = NF$. It follows that G/N is a finitely generated group in $L\mathfrak{F}_\pi$ and hence it is finite. Therefore G/N is trivial, a contradiction that gives that $G/\operatorname{Frat}(G)$ is simple. □

4. Minimal non-$(L\mathfrak{F}_\pi)\mathfrak{X}$-groups

Let $\mathfrak{X}$ be a quotient and subgroup closed class of locally nilpotent groups such that infinite locally graded minimal non-$\mathfrak{X}$-groups are countable p-groups, where p is a prime. Now we are ready to prove our main result, which is an easy consequence of Proposition 2.3.

Theorem 4.1. *A group G is an infinitely generated minimal non-$(L\mathfrak{F}_\pi)\mathfrak{X}$-group if and only if there exists a prime $p \notin \pi$ such that G is an infinitely generated minimal non-$\mathfrak{X}$ p-group.*

Proof. Assume that G is an infinitely generated minimal non-$(L\mathfrak{F}_\pi)\mathfrak{X}$-group. Then by Proposition 2.3, $T_\pi \in L\mathfrak{F}_\pi$ and G/T_π is an infinite π-free locally nilpotent minimal non-$\mathfrak{X}$-group. Therefore there exists a prime $p \notin \pi$ such that G/T_π is a countable p-group. It follows that G is locally finite, so that we can deduce by [8, Lemma 1.D.4] that there exists a p-subgroup P of G such that $G = PT_\pi$. Therefore G is an infinitely generated minimal non-$\mathfrak{X}$ p-group by Corollary 2.4.

The converse is clear. □

One can deduce the following consequence of Theorem 4.1.

Corollary 4.2. *An infinitely generated group G having non-trivial π-elements and whose proper subgroups are in $(L\mathfrak{F}_\pi)\mathfrak{X}$ is itself in $(L\mathfrak{F}_\pi)\mathfrak{X}$.*

Since by [10] and [14] (respectively, by [2, Theorem 1.2]) an infinitely generated minimal non-nilpotent (respectively, non-Baer) group is a countable p-group for some prime p, then we can take $\mathfrak{X}$ to be the class $\mathfrak{N}$ (respectively, of Baer groups).

Other possibility for the class $\mathfrak{X}$ is the class ZA of hypercentral groups as by [6, Lemma 2.2] an infinitely generated minimal non-ZA-group is periodic and since ZA is a $\mathbf{N}_0$-closed class (i.e. the product of two normal hypercentral subgroups is hypercentral) [13, p. 51 of Part 1] and a countably recognizable class (i.e. a group is hypercentral whenever its countable subgroups are hypercentral) [13, Corollary 2 of Theorem 8.34].

If $G = A\langle x\rangle$ is a locally dihedral 2-group, that is the semi-direct product of a quasicyclic 2-group A by a cyclic group $\langle x\rangle$ of order 2 which acts by inversion on A, then it is easy to see that G is an infinitely generated minimal non-nilpotent group. So that if π is a set of odd primes then G is a minimal non-$(L\mathfrak{F}_\pi)\mathfrak{N}$-group.

5. Minimal non-$(L\mathfrak{F}_\pi)\mathfrak{V}$-groups

Now let $\mathfrak{V}$ be a variety of locally nilpotent groups. In this section we will prove that there are no infinite locally graded minimal non-$(L\mathfrak{F}_\pi)\mathfrak{V}$-groups.

To prove next lemma we adapt the proof of its analog on the variety $\mathfrak{N}_c$ [5, Corollary 2].

Lemma 5.1. *Let G be a locally graded minimal non-$\mathfrak{V}$-group. Then G is finite.*

Proof. Let G as stated and assume for a contradiction that G is infinite. Then G is obviously finitely generated and hence it is nilpotent-by-finite so that G satisfies the maximal condition on subgroups. It follows that every proper subgroup of G is nilpotent. If G is not nilpotent, then it is a minimal non-nilpotent group and hence $G/\operatorname{Frat}(G)$ is simple by [10, Theorem 3.3]. Therefore

$G/\operatorname{Frat}(G)$ is finite and hence so is G by [9], a contradiction. It follows that G is an infinite nilpotent group. It is known that therefore G is (torsion-free)-by-finite. Let N be a normal subgroup of G such that N is torsion-free and G/N is finite. Hence N is a residually finite p-group for every prime p. Let p and q be distinct primes and let H and K be proper normal subgroups of N such that N/H and N/K are respectively of p-power and q-power order. Then $N/H \cap K$ is a finite nilpotent group with two primary components. So $G/H_G \cap K_G$ is a finite nilpotent group with at least two primary components and hence it is a direct product of two proper subgroups, so that it belongs to $\mathfrak{V}$. It follows that $G/H_G \in \mathfrak{V}$ and hence $G \in \mathfrak{V}$ as N is a residually finite p-group, a contradiction which implies that G is finite. □

Proposition 5.2. *Let G be an infinite minimal non-$(L\mathfrak{F}_\pi)\mathfrak{V}$-group. Then G is a finitely generated perfect group which has no proper subgroups of finite index and $G/\operatorname{Frat}(G)$ is simple.*

Proof. Let G as stated and assume for a contradiction that G is not finitely generated. Then by Proposition 2.3, $T_\pi \in L\mathfrak{F}_\pi$ and G/T_π is a locally nilpotent minimal non-$\mathfrak{V}$-group and hence it is finite by Lemma 5.1. It follows that there exists a (finitely generated) proper subgroup F of G such that $G = T_\pi F$. We deduce by condition (i) of Proposition 2.3 that $G = F$, a contradiction. Therefore G is finitely generated and hence by Proposition 3.3 G is perfect, has no proper subgroups of finite index and $G/\operatorname{Frat}(G)$ is simple. □

Proposition 5.2 has the following consequence.

Corollary 5.3. *Let G be an infinite locally graded group whose proper subgroups are in $(L\mathfrak{F}_\pi)\mathfrak{V}$. Then G is a $(L\mathfrak{F}_\pi)\mathfrak{V}$-group*

Clearly $\mathfrak{V}$ can stands for the variety $\mathfrak{N}_c$. According to the solution of the Restricted Burnside Problem [17], [18] the class of locally nilpotent groups of exponent a given positive integer n is a variety and hence it is another possibility for the variety $\mathfrak{V}$.

In [11] it is constructed an infinite simple torsion-free finitely generated group whose proper subgroups are cyclic. This group is both a minimal non-$(L\mathfrak{F}_\pi)\mathfrak{N}$-group and a minimal non-$(L\mathfrak{F}_\pi)\mathfrak{N}_1$-group for every set of primes π.

6. Some consequences

In this section we deduce some results on infinitely generated minimal non-$\mathfrak{C}_\pi\mathfrak{N}$ and non-$\mathfrak{F}_\pi\mathfrak{N}$-groups (respectively, non-$\mathfrak{C}_\pi\mathfrak{N}_c$ and non-$\mathfrak{F}_\pi\mathfrak{N}_c$-groups), where

$\mathfrak{C}_\pi = L\mathfrak{F}_\pi \cap \mathfrak{C}$, $\mathfrak{F}_\pi = L\mathfrak{F}_\pi \cap \mathfrak{F}$, $\mathfrak{C}$ is the class of Chernikov groups and c is a positive integer.

In [1, Theorem 2.1], it is proved that a locally graded group is a minimal non-$\mathfrak{C}\mathfrak{N}$-group if and only if it is a minimal non-$\mathfrak{N}$-group without maximal subgroups. Using Theorem 4.1 we generalise this result.

Corollary 6.1. *Let G be an infinitely generated group. Then G is a minimal non-$\mathfrak{C}_\pi\mathfrak{N}$-group if and only if G is a minimal non-$\mathfrak{N}$-group such that either G has no maximal subgroups or there exists a prime $p \notin \pi$ such that G is a p-group having maximal subgroups.*

Proof. Assume that G is a minimal non-$\mathfrak{C}_\pi\mathfrak{N}$-group. First we prove that the product of two normal $\mathfrak{C}_\pi\mathfrak{N}$-subgroups of G is likewise a $\mathfrak{C}_\pi\mathfrak{N}$-group. For if H and K are two normal $\mathfrak{C}_\pi\mathfrak{N}$-subgroups of G, then for some positive integers c and d, $\gamma_c(H)$ and $\gamma_d(K)$ are two normal $\mathfrak{C}_\pi$-subgroups of G and hence so is $\gamma_c(H)\gamma_d(K)$. Since

$$HK/\gamma_c(H)\gamma_d(K) = (H\gamma_d(K)/\gamma_c(H)\gamma_d(K))(K\gamma_c(H)/\gamma_c(H)\gamma_d(K))$$

is nilpotent, we deduce that HK is a $\mathfrak{C}_\pi\mathfrak{N}$-group. It follows by [10, Theorem 2.12] that every nilpotent image of G is abelian. Clearly G is either a $\mathfrak{C}\mathfrak{N}$-group or a minimal non-$\mathfrak{C}\mathfrak{N}$-group. Therefore G is either a $\mathfrak{C}\mathfrak{A}$-group, $\mathfrak{A}$ being the class of abelian groups, or a minimal non-$\mathfrak{N}$-group without maximal subgroups by [1, Theorem 2.1]. By Theorem 4.1 we have also that G is either a $(L\mathfrak{F}_\pi)\mathfrak{A}$-group or there exists a prime $p \notin \pi$ such that G is a minimal non-$\mathfrak{N}$-p-group. Consequently, if we assume that G is not a minimal non-$\mathfrak{N}$-group, then $G \in \mathfrak{C}\mathfrak{A} \cap (L\mathfrak{F}_\pi)\mathfrak{A}$, so that $G' \in (L\mathfrak{F}_\pi) \cap \mathfrak{C} = \mathfrak{C}_\pi$, which is a contradiction. Hence G is a minimal non-$\mathfrak{N}$-group and either G has no maximal subgroups or for some prime $p \notin \pi$ G is a p-group having maximal subgroups.

Conversely, if G is a minimal non-$\mathfrak{N}$-group having maximal subgroups, then G is π-free and hence $G \notin \mathfrak{C}_\pi\mathfrak{N}$ which gives that G is a minimal non-$\mathfrak{C}_\pi\mathfrak{N}$-group. Now if G is a minimal non-$\mathfrak{N}$-group without maximal subgroups, then it is a minimal non-$\mathfrak{C}\mathfrak{N}$-group by [1, Theorem 2.1] and hence $G \notin \mathfrak{C}_\pi\mathfrak{N}$ which again gives that G is a minimal non-$\mathfrak{C}_\pi\mathfrak{N}$-group. □

In [12, Theorem 1], it is proved that a locally graded group whose proper subgroups are in $\mathfrak{C}\mathfrak{N}_c$ is itself a $\mathfrak{C}\mathfrak{N}_c$-group. We generalise this result using Corollary 5.3.

Corollary 6.2. *Let G be an infinite locally graded group whose proper subgroups are in $\mathfrak{C}_\pi\mathfrak{N}_c$. Then G is a $\mathfrak{C}_\pi\mathfrak{N}_c$-group.*

Proof. Let G as stated; then G belongs to $\mathfrak{C}\mathfrak{N}_c$ by [12, Theorem 1]. Since G is a $(L\mathfrak{F}_\pi)\mathfrak{N}_c$-group by Corollary 5.3, we deduce that G belongs to $\mathfrak{C}_\pi\mathfrak{N}_c$. □

Corollary 6.2 has the following consequence which will be used in next results.

Lemma 6.3. *If G is a minimal non-$\mathfrak{F}_\pi\mathfrak{N}_c$-group, then G is a minimal non-$\mathfrak{F}\mathfrak{N}_c$-group.*

Proof. Let G be a minimal non-$\mathfrak{F}_\pi\mathfrak{N}_c$-group. Then by Corollary 6.2, G belongs to $\mathfrak{C}_\pi\mathfrak{N}_c$ and hence $G \notin \mathfrak{F}\mathfrak{N}_c$. Therefore G is a minimal non-$\mathfrak{F}\mathfrak{N}_c$-group. □

Combining [16, Theorem 3.5] and the fact that infinitely generated minimal non-$\mathfrak{N}$-groups are soluble by [3, Theorem 1.3] we have that a group is an infinitely generated minimal non-$\mathfrak{F}\mathfrak{N}$-group if and only if it is an infinitely generated minimal either non-$\mathfrak{N}$-group or non-$\mathfrak{F}\mathfrak{A}$-group. We generalise this result using Theorem 4.1.

Corollary 6.4. *Let G be an infinitely generated group. Then G is a minimal non-$\mathfrak{F}_\pi\mathfrak{N}$-group if and only if G is either a minimal non-$\mathfrak{F}_\pi\mathfrak{A}$-group, or a minimal non-$\mathfrak{N}$-group which is either without maximal subgroups, or there exists a prime $p \notin \pi$ such that G is a p-group having maximal subgroups.*

Proof. Assume that G is a minimal non-$\mathfrak{F}_\pi\mathfrak{N}$-group. As in the proof of Corollary 6.1, one can prove that every nilpotent image of G is abelian. Clearly G is either a $\mathfrak{F}\mathfrak{N}$-group or a minimal non-$\mathfrak{F}\mathfrak{N}$-group. So G is either a $\mathfrak{F}\mathfrak{A}$-group, or a minimal non-$\mathfrak{F}\mathfrak{A}$-group or a minimal non-$\mathfrak{N}$-group without maximal subgroups. On the other hand G is either a $L(\mathfrak{F}_\pi)\mathfrak{N}$-group or a minimal non-$L(\mathfrak{F}_\pi)\mathfrak{N}$-group, which implies that G is either a $L(\mathfrak{F}_\pi)\mathfrak{A}$-group or a minimal non-$\mathfrak{N}$-$p$-group for some prime $p \notin \pi$. Therefore if G is not a minimal non-$\mathfrak{N}$, then G is in $L(\mathfrak{F}_\pi)\mathfrak{A}$ and is either a $\mathfrak{F}\mathfrak{A}$-group or a minimal non-$\mathfrak{F}\mathfrak{A}$-group. Since clearly $G \notin \mathfrak{F}\mathfrak{A}$, we have that G is a minimal non-$\mathfrak{F}\mathfrak{A}$-group which is a $(L\mathfrak{F}_\pi)\mathfrak{A}$ and hence it is a minimal non-$\mathfrak{F}_\pi\mathfrak{A}$.

Conversely, if G is a minimal non-$\mathfrak{F}_\pi\mathfrak{A}$-group, then G is a minimal non-$\mathfrak{F}\mathfrak{A}$-group by Lemma 6.3 and hence G is a minimal non-$\mathfrak{F}\mathfrak{N}$-group by [16, Theorem 3.5]. We deduce that $G \notin \mathfrak{F}_\pi\mathfrak{N}$ and hence G is a minimal non-$\mathfrak{F}_\pi\mathfrak{N}$-group. If G is a minimal non-$\mathfrak{N}$-p-group for some prime $p \notin \pi$, then G is π-free and hence $G \notin \mathfrak{F}_\pi\mathfrak{N}$, so that G is a minimal non-$\mathfrak{F}_\pi\mathfrak{N}$-group. Now if G is a minimal non-$\mathfrak{N}$-p-group without maximal subgroups, for some prime $p \in \pi$, then G is a minimal non-$\mathfrak{F}\mathfrak{N}$ by [16, Theorem 3.5] and a $L(\mathfrak{F}_\pi)\mathfrak{A}$-group by Theorem 4.1 and hence G is a minimal non-$\mathfrak{F}_\pi\mathfrak{N}$-group. □

In [4, Theorem 2], it is proved that a locally graded group is a minimal non-$\mathfrak{F}\mathfrak{N}_c$-group if and only if it is a minimal non-$\mathfrak{F}\mathfrak{A}$-group. We generalise this result using Corollary 5.3.

Corollary 6.5. *An infinitely generated group G is a minimal non-$\mathfrak{F}_\pi\mathfrak{N}_c$-group if and only if G is a minimal non-$\mathfrak{F}_\pi\mathfrak{A}$-group.*

Proof. Let G be a minimal non-$\mathfrak{F}_\pi\mathfrak{N}_c$-group. Then G is a minimal non-$\mathfrak{F}\mathfrak{N}_c$-group by Lemma 6.3. It follows that G is a minimal non-$\mathfrak{F}\mathfrak{A}$ by [4, Theorem 2] and hence G is a minimal non-$\mathfrak{F}\mathfrak{N}$-group by [16, Theorem 3.5]. Since the product of two normal $\mathfrak{F}\mathfrak{N}$-subgroups of G is a $\mathfrak{F}\mathfrak{N}$-group and as G is a $(L\mathfrak{F}_\pi)\mathfrak{N}_c$-group by Corollary 5.3, we deduce that G belongs to $(L\mathfrak{F}_\pi)\mathfrak{A}$. Consequently G is a minimal non-$\mathfrak{F}_\pi\mathfrak{A}$-group.

Conversely assume that G is a minimal non-$\mathfrak{F}_\pi\mathfrak{A}$-group. Then clearly every proper subgroup of G is in $\mathfrak{F}_\pi\mathfrak{N}_c$. Since G cannot be a $\mathfrak{F}_\pi\mathfrak{N}_c$-group by Lemma 6.3, it is a minimal non-$\mathfrak{F}_\pi\mathfrak{N}_c$-group. □

It is easy to see that the well known example of Heineken-Mohamed [7], which is a minimal non-$\mathfrak{N}$-group without maximal subgroups, is both a minimal non-$\mathfrak{F}_\pi\mathfrak{N}$ and non-$\mathfrak{C}_\pi\mathfrak{N}$-group for all sets π of primes.

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