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Neutral stochastic differential equations driven by Brownian motion and fractional Brownian motion in a Hilbert space

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Abstract. A class of mixed neutral stochastic differential equations involving Brownian motion and fractional Brownian motion is considered. The existence, uniqueness and exponential stability for the solutions of these equations are discussed by means of semigroup of operator and the fixed point principle under some suitable assumptions. Our results extend and improve those of [2] and [11].

1. Introduction

The main object of this paper is a neutral stochastic differential equation (NSDE) in a Hilbert space X

$$\begin{cases} d[x(t)-h(t,x(t-r(t)))] = [Ax(t)+f(t,x(t-\rho(t)))]dt \\ \qquad\qquad +g(t,x(t-\eta(t)))dW(t)+\sigma(t)dB^H(t), \quad t\geq 0, \\ x(t)=\varphi(t)\in\mathcal{C}([-\tau,0],\mathcal{L}^2(\Omega,X)), \quad t\in[-\tau,0],\ \tau>0, \end{cases} \tag{1}$$

where φ is $\mathcal{F}_0$-measurable; $r,\rho,\eta:[0,\infty)\to[0,\tau]$ are continuous; $f,h:[0,\infty)\times X\to X, g:[0,\infty)\times X\to\mathcal{L}_1^0(Y_1,X)$ and $\sigma:[0,\infty)\to\mathcal{L}_2^0(Y_2,X)$ are Borel measurable functions satisfied some appropriate conditions; the integral w.r.t. $Q^{(1)}$-Brownian motion $\{W(t)\}_{t\geq 0}$ on a real separable Hilbert space Y_1 is Itô integral, and the integral w.r.t. $Q^{(2)}$-fractional Brownian motion (fBm) $\{B^H(t)\}_{t\geq 0}$

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on a real separable Hilbert space Y_2 is Wiener integral with the Hurst parameter $1/2 < H < 1$; A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ on X. For $\alpha \in (0,1]$, it is possible to define the fractional power $(-A)^\alpha : \mathcal{D}((-A)^\alpha) \to X$, which is a closed linear operator with its domain $\mathcal{D}((-A)^\alpha)$ (we refer the readers to Pazy [22] for a detailed discussion on $(-A)^\alpha$).

As we know, on one hand, it so happens that processes in hydrodynamics, telecommunications and finance demonstrate the availability of random noise that can be modeled by a Brownian motion and also a so called long memory that can be modeled with the help of fBm with Hurst index $1/2 < H < 1$. Since the seminal paper [4], mixed stochastic models containing both a standard Brownian motion and an fBm gained a lot of attention. In recent years, there has been considerable interest in studying this class of SDEs, see e.g. [8], [14], [19], [20], [24], [25].

On the other hand, the future state of many stochastic or determinate systems not only depends on the present and past states (delays) but also involves derivatives with delays. Neutral (stochastic) differential equations (N(S)DEs) are often used to describe such kind of systems. NDE is an important area of applied mathematics due to several reasons with non-instant transmission phenomena such as lossless transmission lines [3], population ecology [10], heat exchanges [9], and other engineering systems. As for NSDE, we can refer to [1], [6], [11], [12], [13], [18] only involving Brownian motions and also refer to [2], [7], [15], [26], [17], [16] only containing fBms. Luo [11] first used the fixed-point theory to consider the stability for the following stochastic partial differential equations with delays.

$$\begin{cases} dx(t) = [Ax(t) + f(t, x(t-\rho(t)))]dt + g(t, x(t-\delta(t)))dW(t), \quad t \geq 0, \\ x(t) = \varphi(t), \quad t \in [-\tau, 0], \ \tau > 0. \end{cases} \tag{2}$$

Boufoussi and Hajji [2] considered the NSDE driven by fBm as follows:

$$\begin{cases} d[x(t) - g(t, x(t-r(t)))] = [Ax(t) + f(t, x(t-\rho(t)))]dt \\ \qquad\qquad\qquad\qquad\qquad\qquad + \sigma(t)dB^H(t), \quad t \geq 0, \\ x_0(t) = \varphi(t), \quad t \in [-\tau, 0], \ \tau > 0. \end{cases} \tag{3}$$

They proved an existence and uniqueness result for mild solution of NSDE (3) and established some conditions ensuring the exponential decay to zero in mean square for the mild solution.

However, to the best of authors' knowledge, up to now, there is no paper which investigates mixed NSDEs. Thus, we will make the first attempt to research such problem in the present paper. We devote to the existence and uniqueness of mild solution for NSDE (1) and also establish the sufficient conditions to ensure the exponential stability for the solution of NSDE (1). We can see that equation 2) and equation 3) are both particular cases of NSDE (1) where $g = 0$ and $h = 0$, $\sigma = 0$ respectively, our results in the present paper extend the results of [2] and [11]. Besides, our assumptions on coefficients ensuring the exponential decay to zero in mean square for the mild solution are weaker than those of [2], so we also improve and generalize the results of [2].

The rest of this paper is organized as follows. Some elements of $Q^{(1)}$-Brownian motion and $Q^{(2)}$-fBm are given in Section 2. In Section 3, we present the assumptions and some lemmas, and we show the existence and uniqueness for mild solution of NSDE (1) as well. In Section 4, we establish the sufficient conditions which are weaker than those of [2] to ensure exponential stability for the solution.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying standard assumptions. We denote by $\mathcal{C}([-\tau, T]; \mathcal{L}^2(\Omega, X))$ the space of all continuous functions from $[-\tau, T]$ to $\mathcal{L}^2(\Omega, X)$. Moreover, for the real separable Hibert space X, we denote by $\mathcal{L}(Y_i, X)$ the space of bounded linear operators from Y_i to X, $i = 1, 2$. We assume that $\{e_n^{(i)},\ n = 1, 2, \ldots\}$ is a complete orthonormal basis in Y_i and $Q^{(i)} \in \mathcal{L}(Y_i, X)$ are two operator defined by $Q^{(i)} e_n^{(i)} = \lambda_n^{(i)} e_n^{(i)}$ with finite trace $trQ^{(i)} = \sum_{n=1}^{\infty} \lambda_n^{(i)} < \infty$, where $\lambda_n^{(i)},\ n = 1, 2, \ldots,$ are non-negative real numbers and $i = 1, 2$. Then there exits a real-valued sequence $\omega_n(t)$, $n = 1, 2, \ldots,$ of one dimensional standard Brownian motions mutually independent over $(\Omega, \mathcal{F}, \{F_t\}_{t\geq 0}, \mathcal{P})$ such that

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n^{(1)}} e_n^{(1)} \omega_n(t), \quad t \geq 0,$$

and the infinite dimensional cylindrical Y_2-valued fBm $B^H(t)$ is defined by the formal sum (see [7])

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n^{(2)}} e_n^{(2)} \beta_n^H(t), \quad t \geq 0,$$

where the sequence $\{\beta_n^H(t)\}_{n=1,2,\ldots}$ are stochastically independent scalar fBms with Hurst parameter $H \in (1/2, 1)$.

Let $\mathcal{L}_i^0(Y_i, X)$ be the space of all $Q^{(i)}$-Hilbert-Schmidt operators from Y_i to X $i = 1, 2$. Now we can show the following two definitions of norms.

Definition 2.1. ([6]) Let $g \in \mathcal{L}(Y_1, X)$ and defined

$$\|g\|^2_{\mathcal{L}_1^0} := tr\big(gQ^{(1)}g*\big) = \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n^{(1)}} g e_n^{(1)} \right\|_X^2.$$

If $\|g\|^2_{\mathcal{L}_1^0} < \infty$, then g is called a $Q^{(1)}$-Hilbert-Schmidt operator and the space $\mathcal{L}_1^0 := \mathcal{L}_1^0(Y_1, X)$ equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{L}_1^0} = \sum_{n=1}^{\infty} \langle \varphi e_n^{(1)}, \psi e_n^{(1)} \rangle$ is a separable Hilbert space.

Definition 2.2. ([2]) In order to define Wiener integrals with respect to the $Q^{(2)}$-fBm, we recall that $\sigma \in \mathcal{L}(Y_2, X)$ is called a $Q^{(2)}$-Hilbert-Schmidt operator if

$$\|\sigma\|^2_{\mathcal{L}_2^0} := tr(\sigma Q^{(2)} \sigma *) = \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n^{(2)}} \sigma e_n^{(2)} \right\|_X^2 < \infty,$$

and that the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y_2, X)$ equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n^{(2)}, \psi e_n^{(2)} \rangle$ is a separable Hilbert space.

Let $T > 0$ be an arbitrarily given number and $\{\beta^H(t)\}_{t\in[0,T]}$ be the one-dimensional fBm with Hurst parameter $H \in (1/2, 1)$. This means by definition that β^H is a centered Gaussian process with covariance function $R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. Moreover β^H has the following Wiener integral representation

$$\beta^H(t) = \int_0^t K_H(t,s) d\beta(s), \tag{4}$$

where $\beta = \{\beta(t)\}_{t\in[0,T]}$ is a Wiener process and $K_H(t,s)$ is the kernel given by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

for $t > s$, here $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}}$ and $B(\cdot,\cdot)$ denotes the Beta function. We put $K_H(t,s) = 0$ if $t \le s$.

We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. In fact $\mathcal{H}$ is the closure of set of indicator functions $\{1_{[0,t]},\ t \in [0,T]\}$ with respect to the scalar product $\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s)$. The mapping $1_{[0,t]} \to \beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos. We will denote by $\beta^H(\varphi)$ the image of φ by the previous isometry. We recall that for $\psi, \varphi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \psi(s)\varphi(t)|t-s|^{2H-2} dsdt.$$

Let us consider the operator K_H^* from $\mathcal{H}$ to $\mathcal{L}^2([0,T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(t) \frac{\partial K}{\partial t}(t,s)dt.$$

We refer to [21] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $\mathcal{L}^2([0,T])$. Moreover for any $\varphi \in \mathcal{H}$, we have

$$\beta^H(\varphi) := \int_0^T \varphi(t)d\beta^H(t) = \int_0^T (K_H^*\varphi)(t)d\beta(t).$$

It follows from [21] that the elements of $\mathcal{H}$ may be not functions but distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|^2_{|\mathcal{H}|} := \alpha_H \int_0^T \int_0^T |\psi(s)| \cdot |\psi(t)| \cdot |s-t|^{2H-2} dsdt < \infty,$$

where $\alpha_H = H(2H-1)$. The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$.

Now, let $\psi(s), s \in [0,T]$, be a function with values in $\mathcal{L}_2^0(Y_2, X)$. The Wiener integral of ψ with respect to B^H is defined by

$$\begin{aligned} \int_0^t \psi(s)dB^H(s) &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n^{(2)}} \psi(s) e_n^{(2)} d\beta_n^H(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n^{(2)}} K_H^*(\psi e_n^{(2)})(s) d\beta_n(s), \end{aligned} \tag{5}$$

where β_n is the standard Brownian motion used to present β_n^H as in (4).

The following lemma is the directly result of Proposition 2.1 of [15] and Lemma 2 of [7].

Lemma 2.1. *For any $\varphi : [0,T] \to \mathcal{L}_2^0(Y_2, X)$ and $0 \le a \le b \le T$, if $\sum_{n=1}^{\infty} \sqrt{\lambda_n^{(2)}} < \infty$ and $\sum_{k=1}^{n} \sqrt{\lambda_k^{(2)}} (\varphi e_n^{(2)})(t)$ is uniformly convergent for $t \in [0,T]$ as $n \to \infty$, then we have*

$$\mathbb{E} \left\| \int_a^b \varphi(s) dB^H(s) \right\|_X^2 \le \kappa_H (b-a)^{2H-1} \int_a^b \mathbb{E}\|\varphi(s)\|^2_{\mathcal{L}_2^0} ds, \quad H \in (1/2, 1),$$

here $\kappa_H = H(2H-1)\sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}} = c_H \alpha_H$.

3. Existence and Uniqueness

Lemma 3.1 (see Theorem 2.2 in Chapter 1 and Theorem 6.13 in Chapter 2 of [22]). *Supposed that A is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $\{S(t)\}_{t\geq 0}$ on the separable Hilbert space X. It is well known that there exist some constants $M \geq 1, \lambda \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\lambda t}$, for $t \geq 0$, and moreover, if $0 \in \rho(-A)$, where $\rho(-A)$ is the resolvent set of $-A$, then,*

(a) *for any $\alpha \geq 0$, the subspace $D((-A)^\alpha)$ is dense in X with the norm*

$$\|h\|_\alpha^2 := \sup_{t\in\mathbb{R}} \mathbb{E}\|(-A)^\alpha h(t, x(t))\|_X^2, \quad h \in D((-A)^\alpha),$$

(b) *for each $x \in \mathcal{D}((-A)^\alpha)$, we have $S(t)(-A)^\alpha x = (-A)^\alpha S(t)x$,*

(c) *the fractional power $(-A)^\alpha$ satisfies that $\|(-A)^\alpha S(t)x\|_X \leq M_\alpha e^{-\lambda t} t^{-\alpha}\|x\|_X$, $t > 0$, for any $x \in X$, where $M_\alpha \geq 1$ and $\lambda > 0$.*

Definition 3.1. A X-valued process $\{x(t), t \in [-\tau, T]\}$ is called a mild of NSDE (1) if

(1) $x(t) \in \mathcal{C}([0,T], \mathcal{L}^2(\Omega, X))$ is adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$ and $x(t) = \varphi(t)$ for $t \in [-\tau, 0]$,

(2) For arbitrary $t \in [0,T]$, $x(t)$ satisfies the integral equation

$$\begin{aligned} x(t) &= S(t)[x(0) - h(0, x(0 - r(0)))] + h(t, x(t - r(t))) \\ &+ \int_0^t AS(t-u)h(u, x(u - r(u)))du + \int_0^t S(t-u)f(u, x(u - \rho(u)))du \\ &+ \int_0^t S(t-u)g(u, x(u - \eta(u)))dW(u) + \int_0^t S(t-u)\sigma(u)dB^H(u), \text{ a.s..} \end{aligned} \tag{6}$$

In order to establish the existence and uniqueness of mild solution to NSDE (1) the following hypothesises are imposed.

(H1) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t)\}_{t\in[0,T]}$ on X and $0 \in \rho(-A)$. Note that from Lemma 3.1, if we fix some $T > 0$ then, for any $t \in [0,T]$ and $0 < \alpha < 1$, there exit some constants $M \geq 1$ and $M_{1-\alpha} > 1$ such that

$$\|S(t)\| \leq M \text{ and } \|(-A)^{1-\alpha}S(t)\| \leq \frac{M_{1-\alpha}}{t^{1-\alpha}}.$$

(H2) The functions $f : [0, +\infty) \times X \to X$ and $g : [0, +\infty) \times X \to \mathcal{L}_1^0(Y_1, X)$ satisfy the Lipschitz and linear growth conditions, i.e., there exist positive constants C_i,

$i = 1, 2, 3, 4$, such that, for all $t \geq 0$ and $x, y \in X$,

$$\|f(t,x) - f(t,y)\|_X^2 \leq C_1^2 \|x - y\|_X^2, \quad \|f(t,x)\|_X^2 \leq C_2^2(1 + \|x\|_X^2),$$
$$\|g(t,x) - g(t,y)\|_{\mathcal{L}_1^0}^2 \leq C_3^2 \|x - y\|_X^2, \quad \|g(t,x)\|_{\mathcal{L}_1^0}^2 \leq C_4^2(1 + \|x\|_X^2).$$

(H3) For the function $h : [0, +\infty) \times X \to X$, there exist positive constants C_i, $i = 5, 6$, and $\alpha \in (1/2, 1)$ such that h is $D((-A)^\alpha)$-valued and satisfies, for all $t \in [0, T]$ and $x, y \in X$,

(i) $\|(-A)^\alpha h(t,x) - (-A)^\alpha h(t,y)\|_X^2 \leq C_5^2 \|x - y\|_X^2$,

(ii) $\|(-A)^\alpha h(t,x)\|_X^2 \leq C_6^2(1 + \|x\|_X^2)$,

(iii) the constants C_5 and α satisfy the following inequality $k := C_5 \|(-A)^{-\alpha}\| < 1$.

(H4) The function $(-A)^\alpha h$ is continuous in the quadratic mean sense in time, i.e.,

$$\text{for all } s, t \in [0, T] \text{ and all } x \in X, \quad \lim_{t \to s} \mathbb{E}\|(-A)^\alpha h(t,x) - (-A)^\alpha h(s,x)\|_X^2 = 0.$$

(H5) The function $\sigma : [0, T] \to \mathcal{L}_2^0(Y_2, X)$ satisfies $\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds < \infty$, $\forall T > 0$.

Theorem 3.1. *If conditions* (H1)–(H5) *hold for some* $\alpha \in (1/2, 1)$*, then, for all* $T > 0$*, NSDE* (1) *has a unique mild solution on* $[-r, T]$ *in the mean-square sense.*

Proof. Let $\mathcal{B}_T := \mathcal{C}([-r, T], \mathcal{L}^2(\Omega, X))$ be the subspace of all continuous functions from $[-r, T]$ into $\mathcal{L}^2(\Omega, X)$ with the norm $\|\xi\|_{\mathcal{B}_T} = \sup_{u \in [-r,T]} (\mathbb{E}\|\xi(u)\|^2)^{1/2}$. Fix $T > 0$ and let us consider the set

$$\mathcal{U}_T = \{x \in \mathcal{B}_T : x(s) = \varphi(s), \quad \text{for } s \in [-r, 0]\}.$$

$\mathcal{U}_T$ is a closed subset of $\mathcal{B}_T$ provided with the norm $\|\cdot\|_{\mathcal{B}_T}$. Evidently, $\mathcal{B}_T$ and $\mathcal{U}_T$ are both Banach spaces. We define the operator ψ on $\mathcal{U}_T$ by $\psi(x)(t) = \varphi(t)$ for $t \in [-r, 0]$ and for $t \in [0, T]$

$$\begin{aligned}(\psi x)(t) = {} & S(t)[x(0) - h(0, x(0 - r(0)))] + h(t, x(t - r(t))) \\ & + \int_0^t AS(t-u)h(u, x(u - r(u)))du + \int_0^t S(t-u)f(u, x(u - \rho(u)))du \\ & + \int_0^t S(t-u)g(u, x(u - \eta(u)))dW(u) + \int_0^t S(t-u)\sigma(u)dB^H(u). \quad (7)\end{aligned}$$

To obtain the existence and uniqueness of the solution to NSDE (1), it is enough to show that the operator ψ has a unique fixed point in X. For this purpose, the proof is divided into the following three steps by using Banach fixed point theorem.

Step 1. We claim that $\psi(\mathcal{U}_T) \subset \mathcal{U}_T$. For this, we let $x(t) \in \mathcal{U}_T,\ t \in [-\tau, T]$. Evidently, for $t \in [-r, 0]$, $(\psi x)(t) = \varphi(t) \in \mathcal{U}_T$. Let $t \in [0, T]$, it follows from (7) that

$$\begin{aligned}
\|(\psi x)(t)\|_X^2 &\leq 6\Big\|S(t)[x(0) - h(0, x(0 - r(0)))]\Big\|_X^2 + 6\Big\|h(t, x(t - r(t)))\Big\|_X^2 \\
&+ 6\Big\|\int_0^t AS(t-u)h(u, x(u - r(u)))du\Big\|_X^2 + 6\Big\|\int_0^t S(t-u)f(u, x(u - \rho(u)))du\Big\|_X^2 \\
&+ 6\Big\|\int_0^t S(t-u)g(u, x(u - \eta(u)))dW(u)\Big\|_X^2 + 6\Big\|\int_0^t S(t-u)\sigma(u)dB^H(u)\Big\|_X^2 \\
&=: 6\sum_{i=1}^{6} \|P_i(t)\|_X^2. \qquad (8)
\end{aligned}$$

From assumption (H1), we have

$$\mathbb{E}\|P_1(t)\|_X^2 \leq M\mathbb{E}\|\varphi(0) - h(0, \varphi(-r(0)))\|_X^2. \qquad (9)$$

The bound of $(-A)^{-\alpha}$ and condition (ii) of (H3) imply that

$$\mathbb{E}\|P_2(t)\|_X^2 \leq C_6^2\|(-A)^{-\alpha}\|^2(1 + \mathbb{E}\|x(t - r(t))\|_X^2). \qquad (10)$$

From Hölder inequality, Fubini's theorem, Lemma 3.1, assumptions (H1) and (ii) of (H3), we get

$$\begin{aligned}
\mathbb{E}\|P_3(t)\|_X^2 &\leq t\int_0^t \mathbb{E}\|(-A)^{1-\alpha}S(t-u)(-A)^{\alpha}h(u, x(u - r(u)))\|_X^2 du \\
&\leq C_6^2 t M_{1-\alpha}\int_0^t (t-u)^{\alpha-1}(1 + \mathbb{E}\|x(u - r(u))\|_X^2)du. \qquad (11)
\end{aligned}$$

By Hölder inequality, Fubini's theorem, assumptions (H1) and (H2), we have

$$\mathbb{E}\|P_4(t)\|_X^2 \leq C_2^2 t M\int_0^t (1 + \mathbb{E}\|x(u - \rho(u))\|_X^2)du. \qquad (12)$$

By using BDG type of inequality for stochastic convolutions (see [23]), assumptions (H1) and (H2), we can obtain

$$\begin{aligned}
\mathbb{E}\|P_5(t)\|_X^2 &\leq M^2\int_0^t \mathbb{E}\|g(u, x(u - \eta(u)))\|_{\mathcal{L}_1^0}^2 du \\
&\leq C_4^2 M^2\int_0^t (1 + \mathbb{E}\|x(u - \eta(u))\|_X^2)du. \qquad (13)
\end{aligned}$$

Lemma 2.1, assumptions (H1) and (H5) indicate that

$$\mathbb{E}\|P_6(t)\|_X^2 \leq \kappa_H t^{2H-1} M^2 \int_0^t \|\sigma(u)\|_{\mathcal{L}_2^0}^2 du < \infty. \tag{14}$$

Plugging (9)–(14) into (8) and noting that $r(t), \rho(t), \eta(t) \in [0, \tau]$, we can obtain $(\psi x)(t) \in \mathcal{U}_T$ if $x(t) \in \mathcal{U}_T$ for $t \in [0, T]$.

Step 2. For arbitrary $x \in \mathcal{U}_T$, let us prove that $t \to \psi(x)(t)$ is continuous on the interval $[0, T]$ in the $\mathcal{L}^2$-sense. Let $0 < t < T$ and $|\delta|$ be sufficiently small then, for any fixed $x \in \mathcal{U}_T$, we have

$$\begin{aligned}
&\|\psi(x)(t+\delta) - \psi(x)(t)\|_X^2 \leq 6\|(S(t+\delta) - S(t))[x(0) - h(0, x(-r(0)))]\|_X^2 \\
&\quad + 6\|(-A)^{-\alpha}(-A)^{\alpha}[h(t+\delta, x(t+\delta-r(t+\delta))) - h(t, x(t-r(t)))]\|_X^2 \\
&\quad + 6\Big\| \int_0^t A(S(\delta) - I)S(t-u)h(u, x(u-r(u)))du \\
&\quad + \int_t^{t+\delta} AS(t+\delta-u)h(u, x(u-r(u)))du \Big\|_X^2 \\
&\quad + 6\Big\| \int_0^t (S(\delta) - I)S(t-u)f(u, x(u-\rho(u)))du \\
&\quad + \int_t^{t+\delta} S(t+\delta-u)f(u, x(u-\rho(u)))du \Big\|_X^2 \\
&\quad + 6\Big\| \int_0^t (S(\delta) - I)S(t-u)g(u, x(u-\eta(u)))dW(u) \\
&\quad + \int_t^{t+\delta} S(t+\delta-u)g(u, x(u-\eta(u)))dW(u) \Big\|_X^2 \\
&\quad + 6\Big\| \int_0^t (S(\delta) - I)S(t-u)\sigma(u)dB^H(u) + \int_t^{t+\delta} S(t+\delta-u)\sigma(u)dB^H(u) \Big\|_X^2 \\
&\quad =: 6\sum_{i=1}^{6} \|J_i(\delta)\|_X^2.
\end{aligned} \tag{15}$$

$J_1(\delta) \to 0$ can follow directly from the strong continuity of $S(t)$ and Lebesgue dominated theorem, that is,

$$\lim_{\delta \to 0} \mathbb{E}\|J_1(\delta)\|_X^2 = 0. \tag{16}$$

The bound of operator $(-A)^{-\beta}$ yields

$$\|J_2(\delta)\|_X^2 \leq 2\|(-A)^{-\alpha}\|^2 \mathbb{E}\|(-A)^{\alpha} h(t+\delta, x(t+\delta-r(t+\delta)))$$

$$- (-A)^{\alpha} h(t, x(t+\delta - r(t+\delta)))\|_X^2$$
$$+ 2\|(-A)^{-\alpha}\|^2 \mathbb{E}\|(-A)^{\alpha} h(t, x(t+\delta - r(t+\delta))) - (-A)^{\alpha} h(t, x(t - r(t)))\|_X^2.$$

Then, by conditions (i) of (H3), (H4) and the continuity of $r(t)$, we get

$$\lim_{\delta \to 0} \mathbb{E}\|J_2(\delta)\|_X^2 = 0. \tag{17}$$

For the third term $J_3(\delta)$, we suppose that $\delta > 0$ (Similar estimates hold for $\delta < 0$), by Hölder inequality, we have

$$\begin{aligned}\|J_3(\delta)\|_X^2 \le t \int_0^t \|(S(\delta) - I)(-A)^{1-\alpha} S(t-u)(-A)^{\alpha} h(u, x(u - r(u)))\|_X^2 du \\ + \Big| \int_0^{\delta} \|(-A)^{1-\alpha} S(u)(-A)^{\alpha} h(t+\delta-u, x(t+\delta-u-r(t+\delta-u)))\|_X du \Big|^2 \\ =: J_{31}(\delta) + J_{32}(\delta).\end{aligned}$$

By using $S(0) = I$, the strong continuity of $S(t)$ and the Lebesgue dominated theorem, we can conclude that $\lim_{\delta \to 0} \mathbb{E}|J_{31}(\delta)| = 0$.

It follows from assumptions (H1), (ii) of (H3) and Hölder inequality that

$$\mathbb{E}|J_{32}(\delta)| \le \frac{C_6^2 M_{1-\alpha}^2 \delta^{2\alpha-1}}{2\alpha - 1} \int_0^{\delta} (1 + \mathbb{E}\|x(t+\delta-u-r(t+\delta-u)))\|_X^2) du.$$

Due to the fact that $\frac{1}{2} < \alpha < 1$, it implies $\lim_{\delta \to 0} \mathbb{E}|J_{32}(\delta)| = 0$.

So we can obtain that

$$\lim_{\delta \to 0} \mathbb{E}\|J_3(\delta)\|_X^2 = 0. \tag{18}$$

From Hölder inequality, assumptions (H1) and (H3), a similar computation to $J_3(\delta)$ can be used to show

$$\lim_{\delta \to 0} \mathbb{E}\|J_4(\delta)\|_X^2 = 0. \tag{19}$$

Let $\delta > 0$ (Similar estimates hold for $\delta < 0$). The fifth term $J_5(\delta)$ can be estimated as

$$\|J_5(\delta)\|_X^2 \le \Big\| \int_0^t (S(\delta) - I) S(t-u) g(u, x(u - \eta(u))) dW(u) \Big\|_X^2$$
$$+ \Big\| \int_0^{\delta} S(u) g(t+\delta-u, x(t+\delta-u-\eta(t+\delta-u))) dW(u) \Big\|_X^2 =: J_{51}(\delta) + J_{52}(\delta).$$

From condition (H1) and BDG type of inequality for stochastic convolutions, we get

$$\mathbb{E}|J_{51}(\delta)| \leq M^2 \int_0^t \|(S(\delta)-I)g(u,x(u-\eta(u)))\|^2_{\mathcal{L}_1^0} du.$$

By using $S(0) = I$, the strong continuity of $S(t)$, assumption (H2) and the Lebesgue dominated theorem, we can conclude that $\lim_{\delta\to 0}\mathbb{E}|J_{51}(\delta)| = 0$.

Again by BDG type of inequality for stochastic convolutions, assumption (H2), we have

$$\mathbb{E}|J_{52}(\delta)| \leq M^2 C_4^2 \int_0^\delta (1+\mathbb{E}\|x(t+\delta-u-\eta(t+\delta-u))\|^2_X) du.$$

So, the following desired result holds,

$$\lim_{\delta\to 0}\mathbb{E}\|J_5(\delta)\|^2_X = 0. \tag{20}$$

For the term $J_6(\delta)$, assumption (H1) and Lemma 2.1 imply

$$\begin{aligned}
\mathbb{E}\|J_6(\delta)\|^2_X &\leq 2\mathbb{E}\left\|\int_0^t (S(\delta)-I)S(t-u)\sigma(u)dB^H(u)\right\|^2_X \\
&\quad + 2\mathbb{E}\left\|\int_t^{t+\delta} S(t+\delta-u)\sigma(u)dB^H(u)\right\|^2_X \\
&\leq 2\kappa_H t^{2H-1}\int_0^t \mathbb{E}\|(S(\delta)-I)S(t-u)\sigma(u)\|^2_{\mathcal{L}_2^0}du \\
&\quad + 2\mathbb{E}\left\|\int_0^\delta S(u)\sigma(t+\delta-u)dB^H(u)\right\|^2_X \\
&\leq 2\kappa_H t^{2H-1}M^2\int_0^t \mathbb{E}\|(S(\delta)-I)\sigma(u)\|^2_{\mathcal{L}_2^0}du \\
&\quad + 2\kappa_H \delta^{2H-1}M^2\int_0^\delta \mathbb{E}\|\sigma(t+\delta-u)\|^2_{\mathcal{L}_2^0}du.
\end{aligned}$$

By using $S(0) = I$, the strong continuity of $S(t)$, assumption (H5) and the Lebesgue dominated theorem, we can conclude that

$$\lim_{\delta\to 0}\mathbb{E}\|J_6(\delta)\|^2_X = 0. \tag{21}$$

Inequalities (15)–(21) together imply that $\lim_{\delta\to 0}\mathbb{E}\|\psi(x)(t+\delta)-\psi(x)(t)\|^2_X = 0$. Hence, we conclude that the function $t \to \psi(x)(t)$ is continuous on $[0,T]$ in the $\mathcal{L}^2-$sense.

Step 3: Now, we are going to show that ψ is a contraction mapping in $\mathcal{U}_{T_1}$ with some $T_1 \le T$ to be specified later. Let $x, y \in \mathcal{U}_T$ by using the inequality $(a+b+c+d)^2 \le \frac{1}{k}a^2 + \frac{3}{1-k}b^2 + \frac{3}{1-k}c^2 + \frac{3}{1-k}d^2$, where $k = C_5\|(-A)^{-\alpha}\| < 1$, we obtain for any fixed $t \in [0, T]$

$$\begin{aligned}
&\|\psi(x)(t) - \psi(y)(t)\|_X^2 \\
&\le \frac{1}{k}\|(-A)^{-\alpha}\|^2 \cdot \|(-A)^\alpha h(t, x(t-r(t))) - (-A)^\alpha h(t, y(t-r(t)))\|_X^2 \\
&\quad + \frac{3}{1-k}\Big\| \int_0^t (-A)^{1-\alpha} S(t-u)(-A)^\alpha [h(u, x(u-r(u))) - h(u, y(u-r(u)))]du \Big\|_X^2 \\
&\quad + \frac{3}{1-k}\Big\| \int_0^t S(t-u)[f(u, x(u-\rho(u))) - f(u, y(u-\rho(u)))]du \Big\|_X^2 \\
&\quad + \frac{3}{1-k}\Big\| \int_0^t S(t-u)[g(u, x(u-\eta(u))) - g(u, y(u-\eta(u)))]dW(u) \Big\|_X^2.
\end{aligned}$$

From assumption (H1), Lipschitz property of $(-A)^\alpha h, f$ and g combined with Hölder's inequality and BDG type of inequality for stochastic convolutions, we have

$$\begin{aligned}
\mathbb{E}\|\psi(x)(t) - \psi(y)(t)\|_X^2 &\le k\mathbb{E}\|x(t-r(t)) - y(t-r(t))\|_X^2 \\
&\quad + \frac{3C_5^2 M_{1-\alpha}^2 t^{2\alpha-1}}{(1-k)(2\alpha-1)} \int_0^t \mathbb{E}\|x(u-r(u)) - y(u-r(u))\|_X^2 ds \\
&\quad + \frac{3tM^2C_1^2}{1-k} \int_0^t \mathbb{E}\|x(u-\rho(u)) - y(u-\rho(u))\|_X^2 ds \\
&\quad + \frac{3M^2C_3^2}{1-k} \int_0^t \mathbb{E}\|x(u-\eta(u)) - y(u-\eta(u))\|_X^2 ds.
\end{aligned}$$

Theorefore

$$\sup_{u\in[-r,t]} \mathbb{E}\|\psi(x)(u) - \psi(y)(u)\|_X^2 \le \gamma(t) \sup_{u\in[-r,t]} \mathbb{E}\|x(u) - y(u)\|_X^2,$$

where $\gamma(t) = k + \frac{3C_5^2 M_{1-\alpha}^2}{(1-k)(2\alpha-1)}t^{2\alpha} + \frac{3M^2C_1^2}{1-k}t^2 + \frac{3M^2C_3^2}{1-k}t$. By condition (iii) of (H3), we have $\gamma(0) = k = C_5\|(-A)^{-\beta}\| < 1$. Then there exists $0 < T_1 \le T$ such that $0 < \gamma(T_1) < 1$ and ψ is a contraction mapping on $\mathcal{U}_{T_1}$ and therefore the operator ψ has a unique fixed point, which is a mild solution of NSDE (1) on $[-r, T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[-r, T]$ in finitely many steps. □

Remark 3.1. We mention that the existence of a unique mild solution of NSDE (1) can still be guaranteed due to the fact that (17) remains true if assumption (H4) is replace by the following hypothesis

(H4') for all $x \in \mathcal{C}([0,T], \mathcal{L}^2(\Omega, X))$,

$$\lim_{t \to s} \mathbb{E}\|(-A)^\alpha h(t, x(t)) - (-A)^\alpha h(s, x(s))\|_X^2 = 0.$$

The above result has been proved by Boufoussi and Hajji [2] in a particular case $g \equiv 0$. Therefore, our result is an improvement and extension to that of [2].

4. Exponential Stability

In this section, in order to establish some sufficient conditions ensuring the exponential decay to zero in mean square for mild solution of NSDE (1), some further assumptions are given as follows:
(H6) $\exists \lambda > 0$ and $\exists M > 0$ such that $\|S(t)\| \le Me^{-\lambda t}$, for $\forall\, t \ge 0$.
(H7) For any $t \ge 0$, functions f, g, h satisfy

$$\|f(t,0)\|_X^2 \le P_1 e^{-\lambda t}, \|g(t,0)\|_{\mathcal{L}_1^0}^2 \le P_2 e^{-\lambda t}, \|(-A)^\alpha h(t,0)\|_X^2 \le P_3 e^{-\lambda t},$$

where P_1, P_2, P_3 are three non-negative real numbers.
(H8) The function $\sigma : [0, +\infty) \to \mathcal{L}_2^0(Y_2, X)$ satisfies

$$\int_0^\infty e^{\lambda u} \|\sigma(u)\|_{\mathcal{L}_2^0}^2 du < \infty.$$

Let us recall the following lemma (see Lemma 3.1 of [5]).

Lemma 4.1. *For $\gamma > 0$, there exist three positive constants: $\lambda_i > 0$ $(i = 1, 2, 3)$ and a function $y : [-\tau, +\infty) \to [0, \infty)$. If $\lambda_2 + \frac{\lambda_3}{\gamma} < 1$ and the following inequality*

$$y(t) \le \begin{cases} \lambda_1 e^{-\gamma t} + \lambda_2 \sup_{\theta \in [-\tau, 0]} y(t + \theta) \\ \qquad\qquad\qquad + \lambda_3 \int_0^t e^{-\gamma(t-s)} \sup_{\theta \in [-\tau, 0]} y(s + \theta) ds, \quad t \ge 0, \\ \lambda_1 e^{-\gamma t}, \quad t \in [-\tau, 0],\ \tau > 0 \end{cases}$$

holds. Then one has $y(t) \le M_1 e^{-\mu t}, (t \ge -\tau)$, where μ is a positive root of the algebra equation: $\lambda_2 + \frac{\lambda_3}{\gamma - \mu} e^{\mu \tau} = 1$ and $M_1 = \max\left\{\frac{\lambda_1(\gamma - \mu)}{\lambda_3 e^{\mu\tau}}, \lambda_1\right\} > 0$.

Now we give our main result of this section.

Theorem 4.1. *Supposed that the conditions* (H1)–(H8) *hold, then the mild solution of NSDE* (1) *is exponentially stable in mean square moment if the following inequality*

$$k + \frac{4M_{1-\alpha}^2 C_5^2 \Gamma(2\alpha - 1)\lambda^{1-2\alpha}}{\lambda(1-k)} + \frac{4M^2C_1^2}{\lambda^2(1-k)} + \frac{4M^2C_3^2}{\lambda(1-k)} < 1 \tag{22}$$

holds.

Proof. By virtue of the inequality (22), we can find a number $\varepsilon > 0$ small enough such that

$$k + \frac{4M_{1-\alpha}^2 C_5^2 \Gamma(2\alpha - 1)\lambda^{1-2\alpha}}{(\lambda-\varepsilon)(1-k)} + \frac{4M^2C_1^2}{\lambda(\lambda-\varepsilon)(1-k)} + \frac{4M^2C_3^2}{(\lambda-\varepsilon)(1-k)} < 1. \tag{23}$$

Let $\mu = \lambda - \varepsilon$. If $x(t)$ is the mild solution of NSDE (1), then we have

$$\begin{aligned}
\mathbb{E}\|x(t)\|_X^2 \le & \frac{1}{k}\mathbb{E}\|h(t, x(t-r(t)))\|_X^2 \\
&+ \frac{12}{1-k}\mathbb{E}\|S(t)x(0)\|_X^2 + \frac{12}{1-k}\mathbb{E}\|S(t)h(0, x(-r(0)))\|_X^2 \\
&+ \frac{4}{1-k}\mathbb{E}\left\|\int_0^t AS(t-u)h(u, x(u-r(u)))du\right\|_X^2 \\
&+ \frac{4}{1-k}\mathbb{E}\left\|\int_0^t S(t-u)f(u, x(u-\rho(u)))du\right\|_X^2 \\
&+ \frac{4}{1-k}\mathbb{E}\left\|\int_0^t S(t-u)g(u, x(u-\eta(u)))dW(u)\right\|_X^2 \\
&+ \frac{12}{1-k}\mathbb{E}\left\|\int_0^t S(t-u)\sigma(u)dB^H(u)\right\|_X^2 =: \sum_{i=1}^{7} I_i(t).
\end{aligned} \tag{24}$$

By assumptions (H7) and (i) of (H3), and noting that $0 < \mu < \lambda$, we get

$$\begin{aligned}
I_1(t) \le & \frac{\|(-A)^{-\alpha}\|^2}{k}\Big(\mathbb{E}\|(-A)^\alpha h(t, x(t-r(t))) - (-A)^\alpha h(t,0)\|_X^2 \\
& + \mathbb{E}\|(-A)^\alpha h(t,0)\|_X^2\Big) \le k\mathbb{E}\|x(t-r(t))\|_X^2 + K_1 e^{-\mu t},
\end{aligned} \tag{25}$$

where $K_1 = \frac{\|(-A)^{-\alpha}\|^2 P_3}{k}$.

Assumption (H6) yields

$$I_2(t) \le K_2 e^{-2\lambda t} \le K_2 e^{-\mu t}, \tag{26}$$

where $K_2 = \frac{12M^2}{1-k}\mathbb{E}\|\varphi(0)\|_X^2$.

It follows from assumptions (H6), (H7) and (i) of (H3) that

$$I_3(t) \le K_3 e^{-2\lambda t} \le K_3 e^{-\mu t}, \tag{27}$$

where $K_3 = \frac{12\|(-A)^{-\alpha}\|^2 M^2}{1-k}(C_5^2\mathbb{E}\|\varphi(-r(0))\|_X^2 + P_3)$.

From (c) of Lemma 3.1, Hölder inequality, assumptions (i) of (H3) and (H7), we derive

$$\begin{aligned} I_4(t) &\le \frac{4}{1-k}\mathbb{E}\left|\int_0^t \|(-A)^{1-\alpha}S(t-u)(-A)^\alpha h(u,x(u-r(u)))\|_X du\right|^2 \\ &\le \frac{4M_{1-\alpha}^2\Gamma(2\alpha-1)\lambda^{1-2\alpha}}{1-k}\int_0^t e^{-\lambda(t-u)}\mathbb{E}\|(-A)^\alpha h(u,x(u-r(u)))\|_X^2 du \\ &\le \frac{4M_{1-\alpha}^2 C_5^2\Gamma(2\alpha-1)\lambda^{1-2\alpha}}{1-k}\int_0^t e^{-\lambda(t-u)}\mathbb{E}\|x(u-r(u))\|_X^2 du + K_4 e^{-\mu t}, \end{aligned} \tag{28}$$

where $K_4 = \frac{4M_{1-\alpha}^2\Gamma(2\alpha-1)\lambda^{1-2\alpha}P_3}{(1-k)(\lambda-\mu)}$.

Using Hölder inequality, assumptions (H2), (H6) and (H7), we obtain

$$\begin{aligned} I_5(t) &\le \frac{4}{1-k}\mathbb{E}\left|\int_0^t \|S(t-u)f(u,x(u-\rho(u)))\|_X du\right|^2 \\ &\le \frac{4M^2}{\lambda(1-k)}\int_0^t e^{-\lambda(t-u)}\mathbb{E}\left\|f(u,x(u-\rho(u))) - f(u,0) + f(u,0)\right\|_X^2 du \\ &\le \frac{4M^2C_1^2}{\lambda(1-k)}\int_0^t e^{-\lambda(t-u)}\mathbb{E}\|x(u-\rho(u))\|_X^2 du + K_5 e^{-\mu t}, \end{aligned} \tag{29}$$

where $K_5 = \frac{4M^2P_1}{\lambda(1-k)(\lambda-\mu)}$.

By BDG type of inequality for stochastic convolutions, assumptions (H2), (H6) and (H7) imply that

$$\begin{aligned} I_6(t) &\le \frac{4}{1-k}\int_0^t \mathbb{E}\|S(t-u)g(u,x(u-\eta(u)))\|_{\mathcal{L}_1^0}^2 du \\ &\le \frac{4M^2}{1-k}\int_0^t e^{-\lambda(t-u)}\mathbb{E}\|g(u,x(u-\eta(u))) - g(u,0) + g(u,0)\|_{\mathcal{L}_1^0}^2 du \\ &\le \frac{4M^2C_3^2}{1-k}\int_0^t e^{-\lambda(t-u)}\mathbb{E}\|x(u-\eta(u))\|_X^2 du + K_6 e^{-\mu t}, \end{aligned} \tag{30}$$

where $K_6 = \frac{4M^2P_2}{(1-k)(\lambda-\mu)}$. From assumption (H6), (H8) and Lemma 2.1, we have

$$I_7(t) \le \frac{12\kappa_H}{1-k}t^{2H-1}\int_0^t \mathbb{E}\|S(t-u)\sigma(u)\|_{\mathcal{L}_2^0}^2 du$$

$$\leq \frac{12M^2\kappa_H}{1-k} t^{2H-1} e^{-\lambda t} \int_0^t e^{\lambda u} \|\sigma(u)\|^2_{\mathcal{L}_2^0} du. \tag{31}$$

Condition (H8) ensures the existence of a positive constant K_7 such that

$$\frac{12M^2\kappa_H}{1-k} t^{2H-1} e^{-(\lambda-\mu)t} \int_0^t e^{\lambda u} \|\sigma(u)\|^2_{\mathcal{L}_2^0} du \leq K_7 \quad \text{for all } t \geq 0.$$

Then $I_7(t) \leq K_7 e^{-\mu t}$.

Substituting (25)–(31) into (24), we get

$$\mathbb{E}\|x(t)\|_X^2 \leq \begin{cases} \gamma e^{-\mu t} + k \sup_{-\tau \leq u \leq 0} \mathbb{E}\|x(t+u)\|_X^2 \\ \qquad + k' \int_0^t e^{-\mu(t-s)} \sup_{-\tau \leq u \leq 0} \mathbb{E}\|x(s+u)\|_X^2 ds, \quad t \geq 0, \\ \gamma e^{-\mu t}, \quad t \in [-\tau, 0], \end{cases}$$

where

$$\gamma = \max\left\{ \sum_{i=1}^{7} K_i, \sup_{-\tau \leq u \leq 0} \mathbb{E}\|\varphi(u)\|_X^2 \right\}$$

and

$$k' = \frac{4M_{1-\alpha}^2 C_5^2 \Gamma(2\alpha-1)\lambda^{1-2\alpha}}{1-k} + \frac{4M^2C_1^2}{\lambda(1-k)} + \frac{4M^2C_3^2}{1-k}.$$

Thus, according to Lemma 4.1, we can complete the proof. □

Remark 4.1. Note that (H7) implies conditions: $f(t,0) \equiv 0$, $g(t,0) \equiv 0$, $(-A)^\alpha h(t,0) \equiv 0$, which are imposed in [11]. Moreover, if $h \equiv 0$ and $\sigma \equiv 0$ of NSDE (1), then, the inequality (22) can be written as $3M^2\left(\frac{C_1^2}{\lambda} + C_2^2\right) < \lambda$, which is the same as (3.2) of [11] in the case that $p = 2$. So, as a particular case of our paper, the result of [11] is improved and generalized.

Remark 4.2. We also remark that if (H7) is replaced by the following condition

(H7') $\|f(t,x)\|_X^2 \leq Q_1\|x\|_X^2 + P_1 e^{-\lambda t}$, $\|g(t,x)\|^2_{\mathcal{L}_1^0} \leq Q_2\|x\|_X^2 + P_2 e^{-\lambda t}$,
$\|(-A)^\alpha h(t,x)\|_X^2 \leq Q_3\|x\|_X^2 + P_3 e^{-\lambda t}$,

where Q_1, Q_2, Q_3, P_1, P_2, P_3 are some non-negative real numbers, then, Theorem 4.1 remains true. This result was proved by Boufoussi and Hajji [2] in the particular case $g \equiv 0$. At this point, the inequality (22) can be replaced by

$$k + \frac{4M_{1-\alpha}^2 Q_3 \Gamma(2\alpha-1)\lambda^{1-2\alpha}}{\lambda(1-k)} + \frac{4M^2Q_1}{\lambda^2(1-k)} < 1$$

accordingly. It is easy to see that (H7) in our paper is weaker than (H7') in [2], therefore, our results improve and extend those of [2].

5. Applications

Let us consider the following initial-boundary value problem-neutral stochastic heat differential equation-with finite variable delays driven by both a Brownian motion and an fBm:

$$\begin{cases} d[u(t,\zeta)-H(t,u(t-r(t),\zeta))]=\left[\dfrac{\partial^2}{\partial^2\zeta}u(t,\zeta)+F(t,u(t-\rho(t),\zeta))\right]dt \\ \qquad +G(t,u(t-\eta(t),\zeta))dW(t)+\Sigma(t)dB^H(t), \quad t\geq 0, \\ u(t,0)=u(t,\pi)=0, \quad t\geq 0, \\ u(t,\zeta)=\varphi(t,\zeta)\in\mathcal{C}([-\tau,0],\mathcal{L}^2(\Omega,X)), \quad t\in[-\tau,0],\ \tau>0,\ 0\leq\zeta\leq\pi, \end{cases} \tag{32}$$

where W is a Brownian motion on a real separable Hilbert space Y_1, B^H is an fBm on a real separable Hilbert space Y_2, $F,H:\mathbb{R}^+\times X\to X$ $G:\mathbb{R}^+\times X\to\mathcal{L}_1^0(Y_1,X)$, $\Sigma:\mathbb{R}^+\to\mathcal{L}_2^0(Y_2,X)$.

To study this system, we consider the space $X=\mathcal{L}^2([0,\pi],\mathbb{R})$ and the operator $A:D(A)\subset X\to X$ given by $Ay=y''$ with

$$D(A)=\{y\in X: y',y''\in X,\ y(0)=y(\pi)=0\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ on X. Furthermore, A has discrete spectrum with eigenvalues $-n^2, n\in\mathbb{N}$ and the corresponding normalized eigenfunctions given by

$$e_n:=\sqrt{\frac{2}{1}}\sin nx, \quad n=1,2\,.$$

In addition $(e_n)_{n\in\mathbb{N}}$ is a complete orthonormal basis in X and

$$S(t)x=\sum_{n=1}^{\infty}e^{-n^2t}\langle x,e_n\rangle e_n,$$

for $x\in X$ and $t\geq 0$. It follows from this representation that $S(t)$ is compact for every $t>0$ and that $\|S(t)\|\leq e^{-t}$ for every $t\geq 0$.

By using Theorem 3.1, if we impose suitable conditions on functions F, G, H, Σ to verify assumptions (H2)-(H5) and on the delay functions $r(\cdot),\rho(\cdot),\eta(\cdot)$ to verify assumptions on Theorem 3.1, we can conclude that the system (32) has a unique mild solution on $[-r,T]$ for all $T>0$ in the mean-square sense.

Furthermore, by using Theorem 4.1, if we suppose that the assumptions (H2)–(H8) on functions F, G, H, Σ and inequality (22) hold as well, then the mild solution of system (32) is exponentially stable in mean square moment.

Example 5.1. Consider the semilinear neutral stochastic heat equation with finite variable delays

$$\begin{cases} d[u(t,\zeta) - \frac{1}{2}u(t-\cos t,\zeta)] = \left[\dfrac{\partial^2}{\partial^2\zeta}u(t,\zeta) - \frac{\epsilon u(t-\sin t,\zeta)}{1+|u(t-\sin t,\zeta)|} - e^{-t}\right]dt \\ \qquad + \left[\dfrac{\epsilon u(t-\cos t,\zeta)}{1+|u(t-\cos t,\zeta)|} + e^{-t}\right]dW(t) + e^{-t}dB^H(t), \quad t \geq 0, \\ u(t,0) = u(t,\pi) = 0, \quad t \geq 0, \\ u(t,\zeta) = \varphi(t,\zeta) \in \mathcal{C}([-1,0],\mathcal{L}^2(\Omega,X)), \quad t \in [-1,0],\ 0 \leq \zeta \leq \pi, \end{cases} \tag{33}$$

Obviously, by Theorem 3.1, for any given ϵ, (33) has a unique mild solution. Furthermore, by Theorem 4.1, we can also deduce that if

$$\frac{1}{2} + \frac{2M_{1-\alpha}^2\Gamma(2\alpha-1)}{\|(-A)^{-\alpha}\|^2} + 16\epsilon^2 < 1,$$

then the mild solution of this equation is exponentially stable in mean square moment, where $A = \frac{\partial^2}{\partial^2\zeta}$, one can refer [22] for the expression of $M_{1-\alpha}$ and $\|(-A)^{-\alpha}\|$.

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