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Rees congruences in lattice-ordered algebras

G. BORDALO (Lisboa) and H. A. PRIESTLEY (Oxford)

Abstract. The Rees sublattices of a lattice have been characterised by J. DUDA. This characterisation is refined in case the lattice is modular of finite height or is bounded distributive. Rees congruences in mono-unary algebras are also considered, and applications are given to some important classes of distributive lattices with additional operations.

1. Introduction

Given any non-empty subset S of a set A there is a smallest equivalence relation, θ_S, with respect to which S is an equivalence class. The partition associated with this relation consists of S itself and the singleton sets $\{x\}$ for $x \notin S$. In case A is an algebra one may ask whether θ_S belongs to $\operatorname{Con} A$, the lattice of congruences of A. If this is so, then θ_S is called a *Rees congruence* and S a *Rees subset*. Rees congruences were first introduced in the context of semigroups by D. REES in the classic paper [6], and play an important role in semigroup theory. The notion was extended to arbitrary algebras by R. F. TICHY in [9], and has been further investigated in [8] (for lattices) and [3]. We observe that a Rees congruence has the property that it is determined by a single congruence class. This phenomenon is rather rare, although for certain classes of algebras — for example groups, Boolean algebras and Heyting algebras — it always holds.

A Rees subset of an arbitrary algebra need not, in general, be a subalgebra. However Rees subsets of lattices behave better. Note that the congruence classes of a lattice congruence are always convex sublattices (see, for example, [4], Ch. 5). If L is a lattice with no infinite chains, then any sublattice has universal bounds, and a convex sublattice is a closed

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interval $[a, b] := \{x \in L \mid a \leq x \leq b\}$. We shall refer to Rees subsets in this setting as Rees sublattices or Rees intervals, as appropriate. For an Rees interval $[a, b]$ the associated Rees congruence is the principal congruence $\Theta_{(a,b)}$. Note that the principal ideal of $\operatorname{Con} L$ generated by $\Theta_{(a,b)}$ is isomorphic to $\operatorname{Con}[a, b]$. In particular $\Theta_{(a,b)}$ is an atom of $\operatorname{Con} L$ if and only if the Rees sublattice $[a, b]$ is itself a simple algebra.

In [5] J. Duda gave the following elegant characterisation for Rees congruences in lattices.

Theorem 1.1. *A convex sublattice M of a lattice L is a Rees sublattice if and only if, given $a, b \in M$, $a < b$, and $x \in L \setminus M$, one of the following holds:*

(1) *$\{a, b, x\}$ is a chain;*
(2) *$\{a, b, x\}$ generates the pentagon N_5.*

Recall that a point x in a lattice L is called a *node* if for every $y \in L$, either $y \leq x$ or $x \leq y$. The zero, 0, and unit, 1, of L, when such exist, are nodes. We say a node x is non-trivial if $0 < x < 1$. Note that (1) holds in Duda's theorem whenever $M = [a, b]$ and x is a node of L. Further, if L is modular, (2) cannot occur, and $[a, b]$ is a Rees sublattice if and only if a and b are nodes. If additionally either

(i) L is of finite height (equivalently L has no infinite chains), or
(ii) L is distributive with universal bounds,

we can relate Rees intervals to natural hereditary properties of join- and meet-irreducible elements and of prime ideals and prime filters, respectively (Theorems 2.3 and 2.4). The key to these results is provided by the technical lemma 2.2.

Conditions (i) and (ii) above hold simultaneously if and only if L is finite and distributive. For this case we obtain, in Section 3, very explicit structural information relating to Rees sublattices.

Throughout our notation follows [4], where can also be found most of the basic lattice-theoretic results that we use.

Our second concern is Rees congruences in lattices with additional operations. Many of the varieties which serve as algebraic models of non-classical propositional calculi consist of algebras with an underlying distributive lattice structure. Thus we are led to consider algebras of the form $(A; \vee, \wedge, F)$, in which the reduct $(A; \vee, \wedge)$ is a lattice. An equivalence relation θ_S on A is a Rees congruence if and only if it is a Rees congruence for $(A; \vee, \wedge)$ and for $(A; F)$. In many important examples, F consists of 0 and 1 as nullary operations, plus a single unary operation. We are thus led in Section 4 to investigate Rees congruences in mono-unary algebras. Finally, with the aid of the results obtained in this case, we illustrate the ways in which Rees congruences may behave for algebras in different classes of distributive-lattice-ordered algebras.

2. Rees congruences in modular and distributive lattices

Before turning to the characterisation of Rees sublattices we interpose a few elementary observations about the way in which the Rees congruences of an algebra A sit inside $\operatorname{Con} A$. Certainly the Rees congruences form a unital meet subsemilattice of $\operatorname{Con} A$, since the meet of any family of Rees subsets is again a Rees subset. (Joins clearly are much less tractable, and we do not consider them at all.) When L is modular of finite height $\operatorname{Con} L$ is Boolean and we have the following result.

Proposition 2.1. *Let L be a modular lattice of finite height and let $a < b$ in L be such that $\Theta_{(a,b)}$ is a Rees congruence. Then the following statements are equivalent:*

(i) *the complement of $\theta_{(a,b)}$ in $\operatorname{Con} L$ is a Rees congruence;*
(ii) *$a = 0$ or $b = 1$.*

Proof. We denote the least element of $\operatorname{Con} L$ by Δ.

Assume (i) holds. We have $\Theta_{(p,q)} \cap \Theta_{(s,t)} = \Delta$ if and only if $|[p \vee s, q \wedge t]| \leq 1$. Thus if $0 < a < b < 1$ then both $\Theta_{(a,b)} \cap \Theta_{(0,a)} = \Delta$ and $\Theta_{(a,b)} \cap \Theta_{(b,1)} = \Delta$. If the complement of $\Theta_{(a,b)}$ were a Rees congruence $\Theta_{(c,d)}$ then $[c,d] \subseteq [0,a]$ and $[c,d] \subseteq [b,1]$, which is impossible. Hence (ii) holds.

Now assume (ii). It will be sufficient to prove that $\Theta_{(0,a)}$ is the pseudocomplement of $\Theta_{(a,1)}$ in $\operatorname{Con} L$. Let ψ be a congruence of L such that $\psi \cap \theta_{(a,1)} = \Delta$. We wish to show that $\psi \leq \Theta_{(0,a)}$. Suppose not. Then there exist $b, c \in L$, $b \neq c$ such that $(b,c) \in \psi$ and $(b,c) \notin \Theta_{(0,a)}$. This implies that $b, c \notin [0,a]$, so that $b, c \in [a,1]$, since a is a node. Then we have $(b,c) \in \Theta_{(a,1)}$ and $(b,c) \in \psi$, which contradicts $\psi \cap \Theta_{(a,1)} = \Delta$. Thus ${\Theta_{(a,1)}}^* = \Theta_{(0,a)}$. Because $\operatorname{Con} L$ is a Boolean lattice, $\Theta_{(0,a)}$ is in fact the complement of $\Theta_{(a,1)}$. $\square$

We now take up our main theme: the characterisation of Rees sublattices in important classes of lattices. In order effectively to apply Duda's theorem to modular lattices we need the following lemma. We use the notation $x \| y$ to indicate that elements x and y in a lattice are incomparable in the underlying order.

Lemma 2.2. *Let L be a modular lattice, and suppose that a, b and x in L are such that $a < b$ and $\{a, b, x\}$ does not form a chain. Then there exists y such that $y < b$ and $a \| y$ or there exists z such that $a < z$ and $z \| b$.*

Proof. If either $x < b$ and $a \| x$ or, dually, $a < x$ and $x \| b$ then x serves as y or z. Now assume $a \| x$ and $x \| b$. We then have $a < a \vee x$ and $b \wedge x < b$. If $(a \vee x) \| b$ we take $z = a \vee x$ and if $a \| (b \wedge x)$ we take $y = b \wedge x$. Finally assume (for a contradiction) that $a \vee x$ is comparable to b and a is comparable to $b \wedge x$. Suppose it were the case that $b < a \vee x$. Then

$b \vee x \leq a \vee x$. But $a < b$ implies that $a \vee x \leq b \vee x$. Hence $a \vee x = b \vee x$. Dually, if $a < b \wedge x$, we would have $a \wedge x = b \wedge x$. Further x is not comparable to a or to b since $\{a, b, x\}$ is not a chain. But then $S = \{a \wedge x, a, b, x, a \vee x\}$ would form a sublattice isomorphic to N_5. Therefore we must have $a \vee x \leq b$ and $b \wedge x \geq a$. However this would imply that $a \leq x \leq b$, contrary to hypothesis. $\square$

Given a lattice L, let $\mathcal{J}(L)$ (respectively $\mathcal{M}(L)$) be the set of join-irreducible (meet-irreducible) elements of L. We note that by definition the zero element, 0, of L, if it exists, does not belong to $\mathcal{J}(L)$, and dually.

We wish to localise the notion of join- and meet-irreducible elements to intervals. For this to be profitable we need a good supply of such elements. To this end we shall assume that L has no infinite chains. This assumption ensures that every element of L majorises an element of $\mathcal{J}(L)$ and is majorised by an element of $\mathcal{M}(L)$ (see, for example, [4], Lemma 8.10). If $I = [a, b]$ is an interval in L then in general an element join-irreducible in I will not be join-irreducible in L, and dually. We remark that the modularity assumption is essential in the theorem that follows: (ii) fails to imply (iii) in N_5 for example.

Theorem 2.3. *Let L be a modular lattice of finite height and let $I = [a, b]$ in L with $a < b$. Then the following conditions are equivalent:*

(i) *I is a Rees interval;*

(ii) *$\mathcal{J}(I) = (I \setminus \{a\}) \cap \mathcal{J}(L)$ and $\mathcal{M}(I) = (I \setminus \{b\}) \cap \mathcal{M}(L)$;*

(iii) *a and b are nodes in L.*

Proof. We have already noted the equivalence of (i) and (iii). Clearly (iii) implies (ii).

We finally prove the contrapositive of (iii) implies (ii). Choose an element x such that $\{a, b, x\}$ does not form a chain. Applying Lemma 2.2 we may assume without loss of generality that there exists y such that $y < b$ and $a \| y$. Let $w = a \vee y$. This belongs to $I = [a, b]$, but not to $\mathcal{J}(L)$. It remains to prove that $w \in \mathcal{J}(I)$. Suppose not. Then there exists $a' \in \mathcal{J}(I)$ with $a' < a \vee y$. Now, by modularity, $a' = a' \wedge (a \vee y) = a \vee (a' \wedge y)$. Since $a \notin \mathcal{J}(I)$ we have $a' \neq a$. If we had $a' = a' \wedge y$ then we would have $a \leq a' \leq y$, whence a and y would be comparable, contrary to hypothesis. Thus we have shown that a' is not join-irreducible in I, and have the required contradiction. $\square$

If we remove the restriction that our modular lattice L be of finite height then we cannot expect condition (ii) above to characterise Rees intervals, since (ii) holds vacuously in L when $\mathcal{J}(I) = \mathcal{M}(I) = \emptyset$ for all intervals $I \subseteq L$ (as happens, for example, when L is the lattice of all open subsets of $\mathbb{R}$, ordered by inclusion). However we can get a parallel result for Rees intervals in bounded distributive lattices. Given a subset Y of a lattice L we denote by $\uparrow Y$ the order-filter generated by Y, *viz.* $\{c \in L \mid (\exists d \in Y)\, c \geq d\}$.

Theorem 2.4. *Let L be a distributive lattice with* 0 *and* 1 *and let* $I = [a, b]$ *in L with $a < b$. Then the following conditions are equivalent:*

(i) *I is a Rees interval;*
(ii) *$\uparrow F$ is a prime filter in L for each prime filter F in I, and dually;*
(iii) *a and b are nodes in L.*

Proof. As before (i) and (iii) are equivalent, and imply (ii).

Suppose (ii) holds but (iii) fails. We may, as in the proof of Theorem 2.3, assume without loss of generality that there exists y such that $y < b$ and $a \| y$. Let F be the principal filter F in L generated by $a \vee y$ and define $G = F \cap I$. Certainly $\emptyset \neq G$ (since $b \in G$), $a \notin G$ (since $a \vee y > a$), and G is a filter. By the Prime Filter Theorem applied to the bounded distributive lattice I we can find a prime filter G' of I such that $G \subseteq G'$. By hypothesis $\uparrow G'$ is a prime filter in L. But $a \vee y \in G \subseteq \uparrow G'$, so that either $a \in \uparrow G'$ or $y \in \uparrow G'$. The former would imply $a \in G'$, which is impossible. Thus there exists $x \in G'$ such that $y \geq x$. But then $y \geq a$, contradicting the incomparability of a and y. $\square$

In Theorem 2.4 we have only considered Rees intervals, although not every Rees sublattice need be an interval. The reason for this restriction is our need to be able to invoke the Prime Filter Theorem and the Prime Ideal Theorem in the sublattice. This would not be possible if it lacked bounds.

Note that if L is finite and distributive then the map $x \mapsto \uparrow x$ is an order-isomorphism from $\mathcal{J}(L)$ onto the prime filters of L, and dually. Thus Theorems 2.3 and 2.4 coincide in this case.

3. The finite distributive case

Throughout this section we consider only lattices which belong to the class $\boldsymbol{D}_f$ of finite distributive lattices. The Birkhoff-Priestley duality between $\boldsymbol{D}_f$ and the class $\boldsymbol{P}_f$ of finite posets is then available. It allows us to identify a finite distributive lattice L with the lattice $\mathcal{O}(\mathcal{J}(L))$ of order-ideals (*alias* down-sets) of $\mathcal{J}(L)$ (its *dual space*). As usual in a poset P, we denote by $\downarrow U$ the order-ideal

$$\{y \in P \mid (\exists x \in U)\, y \leq x\}$$

generated by the subset U of P (and dually). Any set of join-irreducible elements of a lattice will be assumed to carry the induced order.

The criteria for a Rees interval obtained in Section 3 translate nicely into dual space terms, as follows.

Proposition 3.1. *Let L be a finite distributive lattice and $P = \mathcal{J}(L)$ be its dual space. Let $I = [a, b]$ in L, with $a < b$. Then the following are equivalent:*

(i) *I is a Rees interval;*
(ii) *P can be constructed as a linear sum $P = P_1 \oplus P_2 \oplus P_3$ of convex subposets P_1, P_2 and P_3, where $P_2 \cong \mathcal{J}(I)$.*

Proof. We identify L with $\mathcal{O}(P)$. We invoke Lemma 3.2 of [1], which establishes that, under duality, the correspondence $J \mapsto \mathcal{J}(J)$ sets up a bijection between intervals J in L and convex subposets of P. Given a convex subposet Q of P, the associated interval $[u, v]$ has $v = {\downarrow}Q$ and $u = {\downarrow}Q \setminus Q$.

We first show that if P has the structure described in (ii) then a and b are nodes (and therefore $I = [a, b]$ is a Rees interval). Note first that $P_1 = {\downarrow}P_2 \setminus P_2$. Thus $a = P_1$ and $b = P_1 \cup P_2$. Further, a is the order-ideal generated by the maximal elements of P_1, and b is the order-ideal generated by the maximal elements of P_2. Now take any order-ideal $c \in \mathcal{O}(P)$. If $c \cap P_3 = \emptyset$ then $c \subseteq b$. If there exists $x \in c \cap P_3$, then x is above every element of P_2, so $b \subseteq c$. Hence b is a node. In a similar way, every order-ideal U is either contained in P_1 or meets P_2, in which case it contains P_2. We deduce that a is a node.

Consider now $I = [a, b]$, where a and b are nodes. In P the associated convex subposet is $b \setminus a$. Take $x \in P$, $x \notin b \setminus a$. The order-ideal ${\downarrow}x$ generated by x must be such that ${\downarrow}x \subseteq b$ or ${\downarrow}x \supseteq b$. If ${\downarrow}x \subseteq b$ then $x \in {\downarrow}a$ (because $x \notin b \setminus a$). If on the other hand ${\downarrow}x \supseteq b$, then every maximal element of b must lie below x. We conclude that $P = {\downarrow}a \oplus (b \setminus a) \oplus (P \setminus b)$ and is the linear sum of convex subposets. $\square$

Certainly directly decomposable lattices have no non-trivial nodes, and therefore have no proper Rees sublattices. The converse fails. Consider for example the 4-element fence $\mathbb{N}$ in Figure 1 below which is not of the form described in Proposition 3.1. Thus the lattice $\overline{N} := \mathcal{O}(\mathbb{N})$ is directly indecomposable and has no non-trivial proper Rees sublattices.

Figure 1

Consider now a directly indecomposable lattice $L \in \boldsymbol{D}_f$ with no proper Rees intervals and $|L| > 2$. The dual space P of L cannot be obtained as a linear sum of subspaces, since under duality linear sums in

$\boldsymbol{P}_f$ correspond to reduced (*alias* vertical) linear sums in $\boldsymbol{D}_f$ ([4], Exercise 8.9), and any non-trivial reduced linear sum in $\boldsymbol{D}_f$ has a non-trivial node. On the other hand P must be connected (because L is indecomposable). We deduce that P cannot be series-parallel (that is, constructed from singletons using only disjoint unions and linear sums). Series-parallel orders have been studied in particular by I. Rival [7] and, in the context of distributive lattice duality by G. Bordalo and H. A. Priestley [1], [2]. A finite poset is series-parallel if and only if it does not contain a subposet isomorphic to $\mathbb{N}$. A proof of this characterisation can be found in [7]. We conclude that an indecomposable lattice in $\boldsymbol{D}_f$ with no non-trivial nodes always admits $\overline{N}$ as a homomorphic image.

Proposition 3.2. *Let L be a finite distributive lattice and assume that $L \cong L_1 \times \cdots \times L_n$, where each L_i is directly indecomposable. For each i such that $|L_i| > 2$ either L_i has a proper Rees sublattice or has $\overline{N}$ as a homomorphic image.*

Corollary 3.3. *Every directly indecomposable finite distributive lattice can be written as a reduced linear sum of sublattices each of which is a 2-element chain, is directly decomposable, or has $\overline{N}$ as a homomorphic image.*

Proof. Let L be a finite directly indecomposable distributive lattice. If L has no node x such that $0 < x < 1$ then L itself has $\overline{N}$ as a homomorphic image. If L has non-trivial nodes, then the nodes must form a chain $0 < x_1 < \cdots < x_n < 1$. We now invoke Proposition 3.1 and the fact that the dual space of a reduced linear sum of sublattices is the linear sum of their dual spaces. $\square$

Corollary 3.4. *Let L be a directly indecomposable finite distributive lattice with $|L| > 2$. Then the congruence lattice $\operatorname{Con} L$ either contains a congruence ϕ with $L/\phi = \{[a, b]\} \cup \{\{x\} \mid x \notin [a, b]\}$ for some $a < b$ or a congruence θ such that $L/\theta \cong \overline{N}$.*

4. Rees congruences in distributive lattices with additional operations

In this section we consider Rees subsets in algebras with a lattice reduct. This is quite a simple task, thanks to the following elementary observation. Let $(A; F)$ be any algebra, let F' be a subset of the fundamental operations F. Then θ is a Rees congruence of $(A; F)$ if and only if it is a Rees congruence of its reduct $(A; F')$ and is a congruence of $(A; F \setminus F')$.

Some interesting algebras arise from distributive lattices endowed with an additional unary operation, for example pseudocomplemented distributive lattices and Ockham algebras (which include de Morgan algebras and

Stone algebras). It is therefore of some interest to consider the mono-unary algebra reduct of a distributive-lattice-ordered algebra of this type and to determine its Rees subsets. We note that Proposition 4.1 and its corollaries generalise easily to an arbitrary unary algebra.

Proposition 4.1. *Let $\mathcal{U} = (U; f)$ be a mono-unary algebra. Let $\emptyset \neq U' \subseteq U$ be a subset. Then U' is a Rees subset of $\mathcal{U}$ if and only if either U' is a subalgebra of U or the restriction $f{\restriction}U'$ is a constant function.*

Proof. Suppose that U' is a Rees subset of $\mathcal{U}$. Then $\theta = (U' \times U') \cup \Delta$ is a congruence of $\mathcal{U}$. Suppose we can find $x \in U'$ such that $f(x) \notin U'$. However, for all $y \in U'$, we have $y \ \theta \ x$, so that $f(y) \ \theta \ f(x)$. But $f(x) \notin U'$ then implies that $f(y) = f(x)$ for all $y \neq x$. The converse result is obvious. $\square$

Corollary 4.2. *The subalgebra lattice* $\operatorname{Sub}\mathcal{U}$ *of a mono-unary algebra* $\mathcal{U} = (U; f)$ *is isomorphic to a distributive sublattice of the congruence lattice* $\operatorname{Con}\mathcal{U}$ *consisting of Rees congruences.*

Proof. Consider the mapping $\phi\colon \operatorname{Sub}\mathcal{U} \to \operatorname{Con}\mathcal{U}$ defined by $\phi(\mathcal{U}') = \Theta_{U'}$, the Rees congruence generated by the subalgebra $\mathcal{U}'$. Certainly ϕ is one-to-one and preserves meets and joins. The subalgebra lattice Sub $\mathcal{U}$ is a ring of sets and so is distributive. To complete the proof it suffices to note that $\Theta_{(\mathcal{U}_1 \cup \mathcal{U}_2)}$ is the (Rees) congruence relation having as classes $\{\mathcal{U}_1 \cup \mathcal{U}_2\} \cup \{\{x\} \mid x \notin \mathcal{U}_1 \cup \mathcal{U}_2\}$, which is certainly the smallest congruence containing $\Theta_{\mathcal{U}_1}$ and $\Theta_{\mathcal{U}_2}$. $\square$

Corollary 4.3. *Let S be a Rees subset of a mono-unary algebra $\mathcal{U} = (U; f)$ such that S is not a subalgebra, and let $\mathcal{F}$ be the family of all subalgebras $\mathcal{U}'$ of $\mathcal{U}$ such that $S \cup \mathcal{U}'$ is again a subalgebra. Then $\mathcal{F}$ is non-empty and has a minimum element.*

Proof. If S is a Rees subset which is not a subalgebra, then $f{\restriction}S$ is a constant function, with image a, say. Consider the subalgebra $\langle a \rangle$ generated by a. Every subalgebra $\mathcal{U}'$ which contains a has the property that $S \cup \mathcal{U}'$ is a subalgebra of $\mathcal{U}$. Thus $\mathcal{F}$ is non-empty, with $\langle a \rangle$ as its minimum element. Moreover, the subalgebra generated by S is $S \cup \langle a \rangle$. $\square$

Note that if the unary operation f of a mono-unary algebra $(A; f)$ is a permutation (that is, if $(A; f)$ is a union of cycles) then Proposition 4.1 yields that every Rees subset must be a subalgebra.

We can derive some direct consequences for Ockham algebras, a rich class of distributive-lattice-ordered algebras. We recall that an Ockham algebra is an algebra $(A; \vee, \wedge, \sim, 0, 1)$ for which $(A; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\sim$ is a negation operation satisfying de Morgan's laws and interchanging 0 and 1. From Proposition 4.1 we deduce the following fact.

Proposition 4.4. *Let $(A; \vee, \wedge, \sim, 0, 1)$ be an Ockham algebra and let $I = [a, b] \subseteq A$ be a Rees interval with $a < b$. The lattice congruence $\Theta_{(a,b)}$ is an Ockham congruence if and only if $\sim I := \{\sim x \mid x \in I\}$ is either contained in I or is a singleton.*

If the Ockham algebra $(A; \vee, \wedge, \sim, 0, 1)$ is a de Morgan algebra (that is, $\sim$ satisfies $\sim^2 a = a$), then $I = [a, b]$ with $a < b$ is a Rees subset S of the algebra $(A; \vee, \wedge, \sim, 0, 1)$ if and only if it is a Rees subset of the reduct $(A; \vee, \wedge)$ and is closed under $\sim$. (The same conclusion holds if the law $\sim^2 a = a$ is replaced by $\sim^{2k} a = a$ for any $k = 1, 2, 3, \dots$). By contrast it is possible to find Ockham algebras whose Rees subsets are less closely tied to those of the lattice reduct. Consider for example the 4-element chain $0 < a < b < 1$ with the negation specified by

(i) $\sim a = b$, $\sim b = 0$,
(ii) $\sim a = 1$, $\sim b = 1$,
(iii) $\sim a = b$, $\sim b = a$.

Then $[a, b]$ is a Rees interval in cases (ii) and (iii), but not in case (i), while $[0, a]$ is a Rees interval only in case (ii). In a similar manner it is possible to contrive examples exhibiting a variety of other behaviours.

We now turn to pseudocomplemented lattices (not necessarily distributive) and to Heyting algebras. Members of these classes differ from the Ockham algebras in that their additional operations, respectively * and $\rightarrow$, are determined by the underlying lattice structure. Accordingly our proofs are purely lattice-theoretic.

Proposition 4.5. *Let $(L; \vee, \wedge, {}^*, 0)$ be a pseudocomplemented lattice and let $[a, b]$ be a Rees interval in L with $a < b$. Then the following are equivalent:*

(i) *$\Theta_{(a,b)}$ is a *-congruence;*
(ii) *$a \neq 0$.*

Proof. We only need to prove (ii) $\Longrightarrow$ (i) in the non-modular case. Assume that L is any lattice with zero such that for each a in L there exists $a^* = \max\{x \in L \mid x \wedge a = 0\}$. Suppose that $\Theta_{(a,b)}$ is a Rees congruence and that $a \neq 0$. We show that $\Theta_{(a,b)}$ is a *-congruence. Either (1) or (2) in Duda's Theorem must hold, for any $x \notin [a, b]$. Note that we cannot have $a^* \in [a, b]$ since $a \wedge a^* = 0$. If $\{a, b, a^*\}$ is a chain we must have $y^* = 0$ for any $y \geq a$ so (i) holds. Otherwise $\{a, b, a^*\}$ generates a sublattice isomorphic to N_5. Note that $a \wedge a^* = 0$ and $b^* \leq a^*$. If $b^* < a^*$ we have $b \wedge a^* > 0$ which contradicts the fact that $\{a, a^*, b\}$ generates N_5. Thus we must have $b^* = x^* = a^*$ for all $x \in [a, b]$, so (i) holds. $\square$

We conclude by examining, analogously, Heyting algebras viewed as bounded distributive lattices carrying an additional binary operation $\rightarrow$.

Proposition 4.6. *Let* $(H; \vee, \wedge, \rightarrow, 0, 1)$ *be a Heyting algebra with a node* $a \neq 0$*. Then* $[a, 1]$ *is a Rees sublattice which is a Heyting subalgebra, and the lattice congruence* $\Theta_{(a,1)}$ *is a Heyting congruence. In particular if* $L \oplus \mathbf{1}$ *is a subdirectly irreducible Heyting algebra, then the unique atom of* Con $(L \oplus \mathbf{1})$ *is a Rees congruence.*

PROOF. For the first part simply note that for $x, y \geq a$ the implication $x \rightarrow y$ is given by $\max\{z \mid z \wedge x \leq y\}$ and this lies above a since $a \wedge x = a \leq y$.

The final assertion relies on the fact that 1_L is a node in $L \oplus \mathbf{1}$, so that $\Theta_{(1_L,1)}$ is a Rees congruence. □

Note that the Rees congruences of a Heyting algebra H are precisely the lattice congruences of the form $\Theta_{(a,1)}$ with $a < 1$, $a \neq 0$, and a a node, plus the zero and unit of Con$H$. Therefore in this case the meet subsemilattice of Con$H$ formed by the Rees congruences is a sublattice, in fact the Rees congruences form a chain in Con$H$.

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G. BORDALO
DEPARTMENTO DE MATEMATICA,
BLOCO C1, PISO 3°,
RUA ERNESTO VASCONCELOS,
1700 LISBOA,
PORTUGAL.

H. A. PRIESTLEY
MATHEMATICAL INSTITUTE,
24/29 ST GILES,
OXFORD 0X1 3LB,
ENGLAND.

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