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A Coincidence point theorem for densifying mappings

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Abstract. A common fixed point theorem for a new class of densifying mappings is obtained. Our result generalizes many previously known theorems and can be regarded as an extension of Jungck's fixed point theorem for densifying mappings.

1. Introduction

Using the fact that a fixed point of any mapping can be regarded as a common fixed point of the mapping and the identity mapping, JUNGCK [3] obtained a generalization of the celebrated Banach Contraction Principle by replacing the identity mapping by a continuous mapping. In the past few years, Jungck Contraction Principle has been extensively studied by many mathematicians for single-valued as well as for multi-valued mappings in metric, 2-metric, Banach, uniform and probabilistic metric spaces.

In this note, we intend to prove a generalization of Jungck's fixed point theorem for a class of densifying mappings, a notion introduced and studied by FURI and VIGNOLI [2]. It is well-known that a contraction mapping, completely continuous mappings and a number of others are densifying. Also the results due to FURI and VIGNOLI [2] are more general than a number of known results.

We remark that we are not aware of any research paper dealing with the ideas presented here.

2. Preliminaries

Let (X, d) denote a metric space, and f be a mapping of X into itself.

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Definition 2.1. (KURATOWSKI [4]). Let A be a bounded subset of X. Then $\alpha(A)$, the measure of non-compactness of A, is the infimum of all $\varepsilon > 0$ such that A admits a finite covering consisting of subsets with diameters less than ε.

The following properties of α are well-known:

(i) $0 \le \alpha(A) \le \delta(A)$, where $\delta(A)$ stands for the diameter of A.

(ii) $\alpha(A) = 0 \implies A$ is pre-compact.

(iii) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$ for bounded subsets A and B of X,

(iv) $A \subset B \implies \alpha(A) \le \alpha(B)$.

Definition 2.2. (FURI and VIGNOLI [2]). A continuous mapping f of a metric space X into itself is said to be densifying, if for every bounded subset A of X with $\alpha(A) > 0$, we have $\alpha(f(A)) < \alpha(A)$.

Definiton 2.3. (SASTRY and NAIDU [8]). A self-mapping f on a metric space X is said to be nearly-densifying if $\alpha(f(A)) < \alpha(A)$ for every f-invariant and bounded subset A of X with $\alpha(A) > 0$.

Definition 2.4. (SASTRY and NAIDU [8]). Let f, g and s be three self-mappings on a metric space X, and S be the subsemigroup generated by f, g and s in the semigroup of all self-mappings on X with composition operation. Then for any $x \in X$, the orbit $\theta(x)$ at x is defined by

$$\theta(x) = \{y \in X : y = x \quad \text{or} \quad y = hx \quad \text{for some} \quad h \in S\}.$$

3. Results

Throughout this section, X stands for a complete metric space, and for some $x_0 \in X$ the orbit $\theta(x_0)$ is assumed to be bounded.

Let $F_1, F_2 : X \times X \to [0, \infty)$ be such that either F_1 or F_2 is lower semi-continuous, and further $F_1(x, x) = F_2(x, x) = 0$ for all $x \in X$.

The following is our main result.

Theorem 3.1. *Let f, g and s be three continuous and nearly densifying self-mappings on X such that s commutes with f and g. Suppose that*

(i) $\ldots F_1(fx, gy) < \max\{F_2(sx, sy),\ F_2(sx, fx), F_1(sy, gy),$ $\{\min\{F_2(sx, gy),\ F_1(fx, sy)\}\}$ *for* $sx \ne sy$ *and* $fx \ne gy$, *and also*

(ii) $\ldots F_2(gx, fy) < \max\{F_1(sx, sy),\ F_1(sx, gx), F_2(sy, fy),$ $\{\min\{F_1(gx, sy),\ F_2(sx, fy)\}\}$ *for* $sx \ne sy$ *and* $gx \ne fy$. *Then f and s or g and s have a coincidence point provided that $\theta(x_0)$ is bounded for some $x_0 \in X$.*

PROOF. Let $x_0 \in X$ such that $\theta(x_0)$ is bounded. Put $A = \theta(x_0)$. Then

$$A = \{x_0\} \cup f(A) \cup g(A) \cup s(A).$$

So

$$\alpha(A) = \max\{\alpha(f(A)),\ \alpha(g(A)),\ \alpha(s(A))\}.$$

As f, g and s are nearly densifying mappings and X is complete, it follows that $\bar{A}$ is compact. Let

$$B = \bigcap_{n=1}^{\infty} S^n(\bar{A}).$$

Then as proved in Theorem 2 of Shih and Yeh [9], we can show that B is a non-empty compact subset of $\bar{A}$ and $s(B) = B.So$ $s^2(B) = B$. Further, it is clear that $f(B) \subset B$ and $g(B) \subset B$. Now, assume that F_1 is lower semi-continuous. Define $\phi : B \to [0, \infty)$ by putting $\phi(x) = F_1(sx, gx)$. Then ϕ is a lower semi-continuous function on a compact set B and hence attains its minimum value $p \in B$. Clearly, $p \in s^2(B)$. So there is a $\omega \in B$ such that $p = s^2(\omega)$. Suppose that neither f and s nor g and s have a coincidence point. Then

$$\begin{aligned}
&\phi(fg(w)) = F_1(sfg(w), gfg(w)) = F_1(fsg(w), gfg(w))\\
&\ < \max\{F_2(s^2g(w), sfg(w)), F_2(s^2g(w), fsg(w)),\\
&F_1(sfg(w), gfg(w)), \min\{F_2(s^2g(w), gfg(w)), F_1(fsg(w), sfg(w))\}\}\\
&\ = F_2(s^2g(w), sfg(w)) \qquad \text{(By (i))}\\
&\ = F_2(gs^2(w), fsg(w)) < \max\{F_1(s^3(w), s^2g(w)), F_1(s^3(w), gs^2(w)),\\
&F_2(s^2g(w), fsg(w)), \min\{F_1(gs^2(w), s^2g(w)), F_2(s^3(w), fsg(w))\}\}\\
&\ = F_1(s^3(w), s^2g(w)) \qquad \text{(By (ii))}\\
&\ = F_1(s(s^2(w)), g(s^2(w))) = F_1(s(p), g(p)) = \phi(p),
\end{aligned}$$

a contradiction to the choice of p. Hence f and s or g and s must have a coincidence point. Similarly, when F_2 is lower semi-continuous, we can prove the existence of a coincidence point of f and s or g and s.

Theorem 3.2. *Let f, g, s, F_1 and F_2 be as in the statement of Theorem 3.1. If z is a common coincidence point of f, g and s, then sz is a unique common fixed point of f, g and s.*

Proof. Given that z is a common coincidence point of f, g and s. The $fz = gz = sz$. Using commutativity of s with f and g, we see that $f(sz) = s(fz) = s(sz) = s(gz) = g(sz)$. Now suppose that $s^2z \neq sz$.

Then

$$\begin{aligned}
F_1(s^2z, sz) &= F_1(fsz, gz) \\
&< \max\big\{F_2(s^2z, sz), F_2(s^2z, fsz), F_1(sz, gz), \\
&\qquad \min\{F_2(s^2z, gz), F_1(fsz, sz)\}\big\} \\
&= F_2(s^2z, sz) = F_2(gsz, fz) \\
&< \max\big\{F_1(s^2z, sz), F_1(s^2z, gsz), F_2(sz, fz), \\
&\qquad \min\{F_1(gsz, sz), F_2(s^2z, fz)\}\big\} \\
&= F_1(s^2z, sz),
\end{aligned}$$

which is a contradiction. Hence $s^2z = sz$. Thus sz is a common fixed point of f, g and s.

The unicity of a common fixed point follows from (i) and (ii). This completes the proof.

Corollary 3.3. *Let f, g and s be three continuous and nearly densifying self-mappings on X such that s commutes with f and g. Suppose that*

(iii) $\ldots F_1(fx, gy) < \max\{F_2(sx, gy), F_2(sx, fx), F_1(sy, gy)\}$. *for $sx \neq sy$ and $fx \neq gy$, and also*

(iv) $\ldots F_2(gx, fy) < \max\{F_1(sx, sy), F_1(sx, gy), F_2(sy, fy)\}$. *for $sx \neq sy$ and $gx \neq fy$. Then f, g and s have a unique common fixed point.*

Remark. Corollary 3.3 extends results due to Ray-Fisher [5], Fisher-Khan [1], Ray-Chatterjee [6] and Singh [10].

Corollary 3.4. *Let f, g and s be three continuous and nearly densifying self-mappings on X such that s commutes with f and g. Suppose that*

$$F_1(fx, gy) < F_2(sx, sy),$$

for $sx \neq sy$ and $fx \neq gy$, and also

$$F_2(gx, fy) < F_1(sx, sy),$$

for $sx \neq sy$ and $gx \neq fy$. Then f, g and s have a unique common fixed point.

Remark. For $F_1 = F_2$ and $f = g$, Corollary 3.4 can be regarded as an extension of Jungck's theorem [3] for denisfying mappings.

Finally, we state the following result which is motivated by the contraction condition given in Rhoades [7]. It can be proved using techniques of Theorem 3.1.

Theorem 3.5. *Let f, g and s be three continuous and nearly densifying self-mappings on X such that s commutes with f and g. Suppose that the inequality*

$$F(fx, gy) < \max\big\{F(sx, sy), F(sx, fy), F(sy, gy), \\ \tfrac{1}{2}[F(sx, gy) + F(sy, fx)]\big\},$$

holds for $sx \neq sy$ and $fx \neq gy$, where $F : X \times X \to [0, \infty)$ is a lower semi-continuous symmetric function satisfying triangle inequality and $F(x, x) = 0$ for all $x \in X$. Then f, g and s have a unique common fixed point.

Example. Consider $X = \{0, 1\}$ with the usual metric. Define $f, g : X \to X$ as

$$f(0) = 0, g(0) = 1,$$
$$f(1) = 1, g(1) = 0.$$

Then $d(fx, gy) < d(x, y)$, for $x \neq y$, $fx \neq gy$, because $0 \neq 1 \implies d(f0, g1) = 0 < d(0, 1) = 1$, and

$$1 \neq 0 \implies d(f1, g0) = 0 < d(1, 0) = 1.$$

Clearly, $fg = gf$. But f and g have no common fixed point.

Thus in all results (except Theorem 3.2), we can just conclude that either f and s or g and s have a coincidence point. In the above example s is taken as the identity map on X.

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