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On spectral variation of two-parameter matrix eigenvalue problem

By MICHAEL GIL' (Beer-Sheva)

Abstract. We consider the two-parameter eigenvalue problem $Z_j v_j - \lambda_1 A_{j1} v_j - \lambda_2 A_{j2} v_j = 0$, where $\lambda_j \in \mathbb{C}$; Z_j, A_{jk} $(j,k=1,2)$ are matrices. Bounds for the variation of the spectrum of that problem under perturbations are suggested.

1. Introduction and statement of the main result

The present paper is devoted to perturbations of the eigenvalues of a two-parameter eigenvalue problem. Multiparameter eigenvalue problems arise in numerous applications, cf. [3], [6], [15]. The classical results on that problem can be found in the books [1], [14]. For some recent presentations we refer the interested reader to the papers [11], [12], [17], [18]. In particular, the paper [11] deals with harmonic Rayleigh-Ritz extraction for the multiparameter eigenvalue problem. Linearizations of the quadratic two-parameter eigenvalue problems are investigated in [12]. Numerical methods to solving spectral problems for multi-parameter polynomial matrices are explored in [17].

In the present paper we establish bounds for the spectral variations of two-parameter eigenvalue problems. To the best of our knowledge, such bounds were not investigated in the available literature.

Let $\mathbb{C}^n$ be the complex n-dimensional Euclidean space with a scalar product $(.\,,.)$, the Euclidean norm $\|.\| = \sqrt{(.\,,.)}$ and the unit matrix I; $\mathbb{C}^{n\times n}$ is the set of $n \times n$ complex matrices.

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For an $A \in \mathbb{C}^{n\times n}$, $\|A\| = \sup_{x\in\mathbb{C}^n} \|Ax\|/\|x\|$ is the spectral (operator) norm, A^* is the adjoint operator, $\lambda_k(A)$ $(k = 1, \dots, n)$ are the eigenvalues with their multiplicities, $r_s(A)$ is the spectral radius; $\sigma(A)$ denotes the spectrum, A^{-1} is the inverse operator,

$$N_p(A) := [\text{Trace}\,(AA^*)^{p/2}]^{1/p} \quad (1 \le p < \infty)$$

is the Schatten -von Neumann norm. Besides, $N_2(A) = \|A\|_F$ is the Frobenius (Hilbert–Schmidt) norm; $\otimes$ means the tensor product.

Consider the problem

$$Z_1v_1 - \lambda_1 A_{11}v_1 - \lambda_2 A_{12}v_1 = 0, \quad Z_2v_2 - \lambda_1 A_{21}v_2 - \lambda_2 A_{22}v_2 = 0, \tag{1.1}$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$; $v_j \in \mathbb{C}^{n_j}$; Z_j, A_{j1} and $A_{j2} \in \mathbb{C}^{n_j\times n_j}$ $(j = 1, 2)$. Denote problem (1.1) by Λ. If for some λ_1, λ_2 problem Λ has a solution $v_1 \ne 0$ and $v_2 \ne 0$, then the pair $\hat\mu = (\lambda_1, \lambda_2)$ is called the eigenvalue of Λ. Besides, λ_1 is the first coordinate of $\hat\mu$ and λ_2 is called the second coordinate of $\hat\mu$. It is possible that $\lambda_1 = \infty$ or (and) $\lambda_2 = \infty$. The set of all the eigenvalues is the spectrum of Λ and is denoted by $\Sigma(\Lambda)$. Besides, the set of all the $j-th$ coordinates $(j = 1, 2)$ of the eigenvalues is denoted by $\sigma_j(\Lambda)$. We write $\Sigma(\Lambda) = (\sigma_1(\Lambda), \sigma_2(\Lambda))$.

Together with Λ consider the eigenvalue problem

$$\tilde Z_1\tilde v_1 - \tilde\lambda_1 \tilde A_{11}\tilde v_1 - \tilde\lambda_2 A_{12}\tilde v_1 = 0, \quad \tilde Z_2\tilde v_2 - \tilde\lambda_1 \tilde A_{21}\tilde v_2 - \tilde\lambda_2 \tilde A_{22}\tilde v_2 = 0, \tag{1.2}$$

where $\tilde\lambda_1, \tilde\lambda_2 \in \mathbb{C}$; $\tilde Z_j$, $\tilde A_{j1}$ and $\tilde A_{j2} \in \mathbb{C}^{n_j\times n_j}$. Problem (1.2) is denoted by $\tilde\Lambda$.

The *spectral variation* of $\tilde\Lambda$ with respect to Λ is the pair $(sv^{(1)}_\Lambda(\tilde\Lambda), sv^{(2)}_\Lambda(\tilde\Lambda))$, where

$$sv^{(j)}_\Lambda(\tilde\Lambda) = \sup_{s\in\sigma_j(\tilde\Lambda)} \inf_{t\in\sigma_j(\Lambda)} |s - t|.$$

Introduce the $(n_1n_2) \times (n_1n_2)$-matrices

$$K_0 = A_{11}\otimes A_{22} - A_{12}\otimes A_{21}, \quad K_1 = Z_1\otimes A_{22} - A_{12}\otimes Z_2, \quad K_2 = A_{11}\otimes Z_2 - Z_1\otimes A_{21}$$

and

$$\tilde K_0 = \tilde A_{11} \otimes \tilde A_{22} - \tilde A_{12} \otimes \tilde A_{21}, \quad \tilde K_1 = \tilde Z_1 \otimes \tilde A_{22} - \tilde A_{12} \otimes \tilde Z_2,$$
$$\tilde K_2 = \tilde A_{11} \otimes \tilde Z_2 - \tilde Z_1 \otimes \tilde A_{21},$$

and put

$$q(K_l) = \|K_l - \tilde K_l\| \quad (l = 0, 1, 2).$$

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let both matrices K_0 and $\tilde{K}_0$ be invertible. Then for any $p \geq 2$ and $j = 1, 2$ we have*

$$(sv_\Lambda^{(j)}(\tilde{\Lambda}))^{n_1 n_2}$$
$$\leq \frac{q(K_j) + q(K_0) r_s(\tilde{K}_0^{-1}\tilde{K}_j)}{|\det(K_0)|(n_1 n_2 - 1)^{(n_1 n_2 - 1)/p}} (N_p(K_j) + r_s(\tilde{K}_0^{-1}\tilde{K}_j) N_p(K_0))^{n_1 n_2 - 1}. \quad (1.3)$$

The proof of this theorem is presented in the next section. In Section 3 we suggest an estimate for the spectral radius $r_s(\tilde{K}_0^{-1}\tilde{K}_j)$ of matrix $\tilde{K}_0^{-1}\tilde{K}_j$.

Letting $p \to \infty$ in (1.3), we arrive at the inequality

$$(sv_\Lambda^{(j)}(\tilde{\Lambda}))^{n_1 n_2}$$
$$\leq \frac{q(K_j) + q(K_0) r_s(\tilde{K}_0^{-1}\tilde{K}_j)}{|\det(K_0)|} (\|K_j\| + r_s(\tilde{K}_0^{-1}\tilde{K}_j)\|K_0\|)^{n_1 n_2 - 1} \quad (j = 1, 2). \quad (1.4)$$

2. Proof of Theorem 1.1

Consider the pencil $T(z) = A - zB$ $(z \in \mathbb{C})$, where A and B are $n \times n$-matrices. A number μ is a characteristic value of $T(.)$ if $\det T(\mu) = 0$, it is a regular point of $T(.)$ if $\det T(\mu) \neq 0$. The set of all characteristic values of $T(.)$ is the spectrum of $T(.)$ and is denoted by $\Sigma(T(.))$. Put $r_s(T(.)) := \sup\{|\mu| : \mu \in \Sigma(T(.))\}$.

A pencil $\tilde{T}(z) = \tilde{A} - z\tilde{B}$ with $n \times n$ matrices $\tilde{A}$ and $\tilde{B}$ will be considered as a perturbation of $T(z)$. *It is assumed that B and $\tilde{B}$ are invertible.*

The value

$$\text{var}_{T(.)}(\tilde{T}(.)) := \sup_{s \in \Sigma(\tilde{T}(.))} \inf_{t \in \Sigma(T(.))} |s - t|$$

is called the spectral variation of $\tilde{T}(.)$ with respect to $T(.)$.

Set $q_A = \|A - \tilde{A}\|$ and $q_B = \|\tilde{B} - B\|$.

Lemma 2.1. *For any $p \geq 2$, one has*

$$(sv_{T(.)}(\tilde{T}(.)))^n \leq \frac{q_A + q_B r_s(\tilde{T}(.))}{(n-1)^{(n-1)/p}|\det(B)|} (N_p(A) + r_s(\tilde{T}(.)) N_p(B))^{n-1}. \quad (2.1)$$

In particular,

$$(sv_{T(.)}(\tilde{T}(.)))^n \leq \frac{q_A + q_B r_s(\tilde{T}(.))}{|\det(B)|} (\|A\| + r_s(\tilde{T}(.))\|B\|)^{n-1}. \quad (2.2)$$

Proof. Let λ be a characteristic value of $T(.)$ and $y = y(\lambda) \in \mathbb{C}^n$ the corresponding eigenvector: $(A - \lambda B)y = 0$. Then $B^{-1}Ay = \lambda y$. Hence it easily follows that $\Sigma(T(.))$ and $\sigma(B^{-1}A)$ coincide.

Now let μ be a characteristic value of $\tilde{T}(.)$ and $x = x(\mu) \in \mathbb{C}^n$ the corresponding normed eigenvalue: $\tilde{T}(\mu)x = 0$. Take an orthonormal basis $e_1, \dots, e_n$ (dependent on μ) in such a way that $e_1 = x$. Since $\tilde{T}(\mu)e_1 = 0$, by the Hadamard inequality

$$|\det(T(\mu))| \le \prod_{k=1}^{n} \|T(\mu)e_k\| = \|T(\mu)e_1 - \tilde{T}(\mu)e_1\| \prod_{k=2}^{n} \|T(\mu)e_k\|.$$

Hence,

$$|\det(T(\mu))| \le \|T(\mu) - \tilde{T}(\mu)\| \prod_{k=2}^{n} \|T(\mu)e_k\|. \tag{2.3}$$

Besides,

$$\|T(\mu) - \tilde{T}(\mu)\| \le \|A - \tilde{A}\| + \|\mu(B - \tilde{B})\| \le q_A + q_B r_s(\tilde{T}). \tag{2.4}$$

In addition, by the inequality between the arithmetic and geometric means, for any $\mu \in \Sigma(\tilde{T}(.))$ we have

$$\prod_{k=2}^{n} \|T(\mu)e_k\|^p \le \left(\frac{1}{n-1}\sum_{k=2}^{n} \|T(\mu)e_k\|^p\right)^{n-1}.$$

Due to [5, Theorem 4.7, p. 82],

$$\sum_{k=1}^{n} \|T(\mu)e_k\|^p \le N_p^p(T(\mu))$$

and therefore,

$$\prod_{k=2}^{n} \|T(\mu)e_k\|$$
$$\le \frac{(N_p(A) + |\mu| N_p(B))^n}{(n-1)^{(n-1)/p}} \le \frac{1}{(n-1)^{(n-1)/p}} (N_p(A) + r_s(\tilde{T}(.)) N_p(B))^{n-1}.$$

Now (2.3) implies

$$|\det(T(\mu))| \le \frac{1}{(n-1)^{(n-1)/p}} (q_A + q_B r_s(\tilde{T}))(N_p(A) + r_s(\tilde{T}) N_p(B))^{n-1}.$$

But

$$|\det(T(\mu))| = |\det(B)\ \det(B^{-1}A - \mu I)| \geq |\det(B)| \min_k |\lambda_k(B^{-1}A) - \mu|^n. \quad (2.5)$$

So

$$\min_k |\lambda_k(B^{-1}A) - \mu|^n \leq \frac{(q_A + q_B r_s(\tilde{T}))}{|\det(B)|(n-1)^{(n-1)/p}} (N_p(A) + r_s(\tilde{T}) N_p(B))^{n-1}. \quad (2.6)$$

Hence, taking into account that $\Sigma(\tilde{T}(.))$ and $\sigma(\tilde{B}^{-1}\tilde{A})$ coincide, we get (2.1).

Letting in (2.1) $p \to \infty$ we obtain (2.2), since $N_p(A) \to \|A\|$ as $p \to \infty$. This completes the proof. □

The literature on the linear matrix pencils is rather rich, cf. the interesting recent papers [4], [13], [16], [20] and references therein. In particular, the paper [13] deals with condition numbers of a nondefective multiple eigenvalue of a nonsymmetric matrix pencil. In the paper [20] the sensitivity of multiple eigenvalues of nonsymmetric matrix pencils is investigated. In [16] localization of the eigenvalues of positive definite matrices is investigated. To the best of our knowledge, inequalities of the type (2.1) and (2.2), which we apply below were not established in the available literature.

Furthermore, if $B = I$, then $\det(B) = 1$, $r_s(\tilde{T}(.)) = r_s(\tilde{A})$ and (2.2) implies

$$(sv_A(\tilde{A}))^n \leq \|A - \tilde{A}\|(\|A\| + r_s(\tilde{A}))^{n-1}. \quad (2.7)$$

Here naturally, $sv_A(\tilde{A})$ is the spectral variation of $\tilde{A}$ with respect to A:

$$sv_A(\tilde{A}) := \sup_{s \in \sigma(\tilde{A})} \inf_{t \in \sigma(A)} |s - t|.$$

Since, $r_s(\tilde{A}) \leq \|\tilde{A}\|$, inequality (2.7) slightly refines the well-known (Elsner) inequality

$$(sv_A(\tilde{A}))^n \leq \|A - \tilde{A}\|(\|A\| + \|\tilde{A}\|)^{n-1},$$

cf. [19]. In addition, (2.1) implies

$$(sv_{A)}(\tilde{A})^n \leq \frac{\|A - \tilde{A}\|}{(n-1)^{(n-1)/p}} (N_p(A) + r_s(\tilde{A}))^{n-1}. \quad (2.8)$$

Proof of Theorem 1.1. Following Atkinson [1], multiply the equation $Z_1 v_1 - \lambda_1 A_{11} v_1 - \lambda_2 A_{12} v_1 = 0$ in the Kronecker sense on the right by $A_{22} v_2$:

$$(Z_1 - \lambda_1 A_{11} - \lambda_2 A_{12}) v_1 \otimes A_{22} v_2 = 0.$$

Rewrite this equation as

$$(Z_1 \otimes A_{22} - \lambda_1 A_{11} \otimes A_{22} - \lambda_2 A_{12} \otimes A_{22})\hat{v} = 0, \tag{2.9}$$

where $\hat{v} = v_1 \otimes v_2$. Now multiply the equation $Z_2 v_2 - \lambda_2 A_{21} v_2 - \lambda_2 A_{22} v_2 = 0$ in the Kronecker sense on the left by $A_{12} v_1$. Then

$$(A_{12} \otimes Z_2 - \lambda_1 A_{12} \otimes A_{21} - \lambda_2 A_{12} \otimes A_{22})\hat{v} = 0.$$

Subtracting with equation from (2.9) we arrive at the relation

$$(Z_1 \otimes A_{22} - A_{12} \otimes Z_2 - \lambda_1 (A_{11} \otimes A_{22} - A_{12} \otimes A_{21}))\hat{v} = 0.$$

Or $(K_1 - \lambda_1 K_0)\hat{v} = 0$. So λ_1 belongs to the spectrum of the pencil $K_1 - zK_0$.

Similarly one can check that any $\lambda_2 \in \sigma_2(\Lambda)$ belongs to the spectrum of the pencil $K_2 - zK_0$.

Now the assertion of Theorem 1.1 directly follows from Lemma 2.1. □

3. A bound for $r_s(\tilde{K}_0^{-1} \tilde{K}_j)$

We need the following result.

Lemma 3.1. *Let A be an invertible $n \times n$-matrix. Then*

$$\|A^{-1} \det A\| = \prod_{k=1}^{n-1} s_k(A),$$

where $s_k(A)$ $(k = 1, \ldots, n)$ are the singular numbers of A counted with their multiplicities in the decreasing order.

Proof. First, let A be a positive definite Hermitian matrix and $\lambda_n(A) = \min_{k=1,\ldots,n} \lambda_k(A)$. Then $\|A^{-1}\| = \lambda_n^{-1}(A)$ and

$$\|A^{-1} \det A\| = \prod_{k=1}^{n-1} \lambda_k(A). \tag{3.1}$$

Now let A be arbitrary invertible. Then the matrix $B^* = AA$ is positive definite, $\|B^{-1}\| = \|A^{-1}\|^2$, $\lambda_k(B) = s_k^2(A)$ and $\det B = \det AA^* = |\det A|^2$. Hence, replacing in (3.1) A for B we obtain the required result. □

Corollary 3.2. *Let A be an invertible $n \times n$-matrix. Then*

$$\|A^{-1} \det A\| \le \frac{N_p^{n-1}(A)}{(n-1)^{(n-1)/p}} \quad (1 \le p < \infty) \tag{3.2}$$

and

$$\|A^{-1} \det A\| \le \|A\|^{n-1}. \tag{3.3}$$

Indeed, due the latter lemma and the inequality between the arithmetic and geometric mean values we get

$$\|A^{-1} \det A\|^p = \prod_{k=1}^{n-1} s_k^p(A) \le \left[(n-1)^{-1} \sum_{k=1}^{n-1} s_k^p(A)\right]^{n-1} \le [(n-1)^{-1} N_p^p(A)]^{n-1},$$

proving (3.2). Taking into account that $s_j(A) \le \|A\|$, we obtain (3.3).

Since $r_s(\tilde{K}_0^{-1}\tilde{K}_j) \le \|\tilde{K}_0^{-1}\tilde{K}_j\| \le \|\tilde{K}_0^{-1}\|\|\tilde{K}_j\|$, (3.2) and (3.3) give us a bound for the quantity $r_s(\tilde{K}_0^{-1}\tilde{K}_j)$, provided we have a lower bound for $\det(\tilde{K}_0)$. Such a bound is discussed below.

Let $\|A\|_{C^n}$ mean an arbitrary operator norm of $n \times n$ matrix A. Recall that $\|A\|$ is the spectral norm. Let A and $\tilde{A}$ be $n \times n$-matrices. The following inequality is well-known [2, p. 107]:

$$|\det A - \det \tilde{A}| \le n M_2^{n-1} \|A - \tilde{A}\|, \tag{3.4}$$

where $M_2 := \max\{\|A\|, \|\tilde{A}\|\}$. The spectral norm is unitarily invariant, but often is not easy to compute that norm. To the best of our knowledge, for non-spectral norms, the similar inequalities were not published in available literature. It is supposed that for a given matrix norm, there is a constant α_n independent of A, such that

$$|\det A| \le \alpha_n \|A\|_{C^n}^n. \tag{3.5}$$

Lemma 3.3. *Let condition* (3.5) *hold. Then*

$$|\det A - \det \tilde{A}| \le \gamma_n \, \|A - \tilde{A}\|_{C^n} \, (\|A - \tilde{A}\|_{C^n} + \|A + \tilde{A}\|_{C^n})^{n-1}, \tag{3.6}$$

where

$$\gamma_n := \frac{\alpha_n n^n}{2^{n-1}(n-1)^{n-1}}.$$

Proof. Let X and Y be complex normed spaces with norms $\|.\|_X$ and $\|.\|_Y$, respectively, and F be a Y-valued function defined on X. Assume that $F(C+\lambda\tilde{C})$ $(\lambda \in \mathbb{C})$ is an entire function for all $C, \tilde{C} \in X$. That is, for any $\phi \in Y^*$, the functional $\langle \phi, F(C+\lambda\tilde{C})\rangle$ defined on Y is an entire scalar valued function of λ. In [7] (see also Lemma 2.14.1 from [9]), the following result has been proved:

Let $F(C+\lambda\tilde{C})$ ($\lambda \in \mathbb{C}$) be an entire function for all $C, \tilde{C} \in X$ and there be a monotone non-decreasing function $G : [0,\infty) \to [0,\infty)$, such that

$$\|F(C)\|_Y \le G(\|C\|_X) \quad (C \in X). \tag{3.7}$$

Then for all $C, \tilde{C} \in X$ we have

$$\|F(C) - F(\tilde{C})\|_Y \le \|C-\tilde{C}\|_X \; G\left(1 + \frac{1}{2}\|C+\tilde{C}\|_X + \frac{1}{2}\|C-\tilde{C}\|_X\right). \tag{3.8}$$

Furthermore, take $C = A$, $\tilde{C} = \tilde{A}$ and $F(A+\lambda\tilde{A}) = \det(A+\lambda\tilde{A})$. Then due to (3.5) we have (3.7) with $G(\|A\|_{C^n}) = \alpha_n\|A\|^n_{C^n}$. Now (3.8) implies the inequality

$$|\det A - \det\tilde{A}| \le \alpha_n\|A-\tilde{A}\|_{C^n}\left(1 + \frac{1}{2}\|A-\tilde{A}\|_{C^n} + \frac{1}{2}\|A+\tilde{A}\|_{C^n}\right)^n. \tag{3.9}$$

For a constant $c > 0$ put $A_1 = cA$ and $\tilde{A}_1 = c\tilde{A}$. Then by (3.9)

$$|\det A_1 - \det\tilde{A}_1| \le \|A_1-\tilde{A}_1\|_{C^n}\alpha_n\left(1 + \frac{1}{2}\|A_1-\tilde{A}_1\|_{C^n} + \frac{1}{2}\|A_1+\tilde{A}_1\|_{C^n}\right)^n.$$

But $\|A_1\|_{C^n} = c\|A\|_{C^n}$, $\|\tilde{A}_1\|_{C^n} = c\|\tilde{A}\|_{C^n}$ and

$$|\det A_1 - \det\tilde{A}_1| = c^n|\det A - \det\tilde{A}|.$$

Thus,

$$c^n|\det A - \det\tilde{A}| \le c\alpha_n\|A-\tilde{A}\|_{C^n}(1+cb)^n, \tag{3.10}$$

where

$$b = \frac{1}{2}\|A-\tilde{A}\|_{C^n} + \frac{1}{2}\|A+\tilde{A}\|_{C^n}.$$

Denote $x = bc$. Then from inequality (3.10) we obtain

$$|\det A - \det\tilde{A}| \le \alpha_n b^{n-1}\|A-\tilde{A}\|_{C^n}\frac{(1+x)^n}{x^{n-1}}.$$

Minimize the function

$$f(x) = \frac{(1+x)^n}{x^{n-1}}, \quad x > 0.$$

Simple calculation show that

$$\min_{x\ge 0} f(x) = \frac{n^n}{(n-1)^{n-1}}.$$

So

$$|\det A - \det\tilde{A}| \le \frac{\alpha_n n^n}{(n-1)^{n-1}} b^{n-1}\|A-\tilde{A}\|_{C^n}.$$

This is the assertion of the lemma. □

For the spectral norm inequality (3.6), can be worse than (3.4). Indeed, for the spectral norm we have $\alpha_n = 1$. If we take $A = a\tilde{A}$ with a positive constant $a < 1$, then $\|A - \tilde{A}\| + \|A + \tilde{A}\| = 2\|\tilde{A}\| = 2M_2$, but

$$\gamma_n 2^{n-1} = n\left(1 + \frac{1}{n-1}\right)^{n-1} \geq n.$$

Furthermore, due to the inequality between the arithmetic and geometric mean values,

$$|\det A|^p = \prod_{k=1}^{n} |\lambda_k(A)|^p \leq \left(\frac{1}{n}\sum_{k=1}^{n} |\lambda_k(A)|^p\right)^n.$$

Thus,

$$|\det A| \leq \frac{1}{n^{n/p}} N_p^n(A).$$

So in this case $\alpha_n = \frac{1}{n^{n/p}}$ and $\gamma_n = \zeta_{n,p}$, where

$$\zeta_{n,p} := \frac{n^n}{2^{n-1} n^{n/p} (n-1)^{n-1}}.$$

Now (3.6) implies

Corollary 3.4. *One has*

$$|\det A - \det \tilde{A}| \leq \Delta_p(A),$$

where

$$\Delta_p(A) := \zeta_{n,p} N_p(A - \tilde{A}) \left(N_p(A - \tilde{A}) + N_p(A + \tilde{A})\right)^{n-1}.$$

Note that in [9, Corollary 3.3] a result weaker than the previous corollary proved for the norm $N_1(.)$. The latter corollary gives us the invertibility conditions for $\tilde{K}_0$. Moreover, Corollaries 3.2 and 3.4 imply

$$\|\tilde{K}_0^{-1}\| \leq \frac{\|\tilde{K}_0\|^{n-1}}{|\det \tilde{K}_0|} \leq \frac{\|\tilde{K}_0\|^{n-1}}{|\det(K_0)| - \Delta_p(K_0)} \quad (|\det(K_0)| > \Delta_p(K_0)).$$

We thus have proved

Corollary 3.5. *On has*

$$r_s(\tilde{K}_0^{-1}\tilde{K}_j) \leq \frac{\|\tilde{K}_j\|\|\tilde{K}_0\|^{n-1}}{|\det(K_0)| - \Delta_p(K_0)}, \text{ provided } |\det(K_0)| > \Delta_p(K_0).$$

Other bounds for other bounds for $r_s(\tilde{K}_0^{-1}\tilde{K}_j)$ and therefore for the spectral radius of the two parameter eigenvalue problem can be found in the paper [10] and references given therein.

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MICHAEL GIL'
DEPARTMENT OF MATHEMATICS
BEN GURION UNIVERSITY OF THE NEGEV
P.0. BOX 653, BEER-SHEVA 84105
ISRAEL

E-mail: gilmi@bezeqint.net

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