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Parallelism of normal Jacobi operator for real hypersurfaces in complex two-plane Grassmannians

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Abstract. In this paper, we give a partial classification of $\mathfrak{D}^{\perp}$-invariant real hypersurfaces in complex two-plane Grassmannians with Reeb parallel normal Jacobi operator.

1. Introduction

The Jacobi fields along geodesics of a given Riemannian manifold $(\tilde{M}, \tilde{g})$ satisfy a well-known differential equation. This classical differential equation naturally inspires the so-called Jacobi operator. That is, if $\tilde{R}$ is the Riemannian curvature tensor of $\tilde{M}$, and X is any tangent vector field to $\tilde{M}$, the Jacobi operator with respect to X at $p \in \tilde{M}$ is defined by

$$(\tilde{R}_X Y)(p) = (\tilde{R}(Y, X)X)(p)$$

for any $Y \in T_p\tilde{M}$, becomes a self adjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$. Clearly, each tangent vector field X to $\tilde{M}$ provides a Jacobi operator with respect to X.

In the geometry of real hypersurfaces in complex space forms or in quaternionic space forms there have been many characterizations of homogeneous hypersurfaces of type (A_1), (A_2), (B), (C), (D) and (E) in complex projective

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space $\mathbb{C}P^n$, of type (A_0), (A_1), (A_2) and (B) in complex hyperbolic space $\mathbb{C}H^n$ or of type (A_1), (A_2) and (B) in quaternionic projective space $\mathbb{H}P^n$, which are completely classified by Takagi ([14]), Cecil and Ryan ([6]), Kimura ([9]), Montial and Romero ([11]) and Martinez and Pérez ([10]), respectively.

In quaternionic space forms Berndt ([2]) has introduced the notion of normal Jacobi operator

$$\bar{R}_N = \bar{R}(X, N)N \in \text{End } T_xM, \quad x \in M$$

for real hypersurfaces M in a quaternionic projective space $\mathbb{H}P^n$ or in a quaternionic hyperbolic space $\mathbb{H}H^n$, where $\bar{R}$ denotes the Riemannian curvature tensor of $\mathbb{H}P^n$ and $\mathbb{H}H^n$ respectively. The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_iN$, $i = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ denote a canonical local basis of quaternionic Kähler structure on $\mathbb{H}P^n$ and N a unit normal vector field of M in $\mathbb{H}P^n$. He has also shown that the curvature adaptedness, that is, the normal Jacobi operator $\bar{R}_N$ commutes with the shape operator A, is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp = \text{span}\{\xi_1, \xi_2, \xi_3\}$ are invariant by the shape operator A of M, that is, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$, where $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$. And he gave a complete classification of curvature adapted real hypersurfaces in non-flat quaternionic space forms with the assumption of constant principal curvatures in the hyperbolic case (See [2]).

Now let us consider a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The ambient space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric structure. It was known that the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$. By using such kinds of two natural geometric structures, many geometers have investigated some characterizations for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. Among them, Berndt and Suh ([4], [5]) and Suh ([13]) have shown some examples of two kinds of tubes, which said to be of type (A) and of type (B), and have given a characterization of type (A) (resp. of type (B)) by the isometric Reeb flow in [5](resp. contact hypersurfaces in [13]).

As one of examples Berndt and Suh [4] considered two natural geometric conditions for hypersurfaces in $G_2(\mathbb{C}^{m+2})$ that $[\xi] = \text{span}\{\xi\}$ and $\mathfrak{D}^\perp = \text{span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator. By using such conditions and the result in Alekseevskii [1], they have proved the following

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

The structure vector field ξ, $\xi = -JN$, of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be a *Reeb* vector field. If the *Reeb* vector field ξ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, M is said to be a *Hopf hypersurface*. In such a case the integral curves of the *Reeb* vector field ξ are geodesics (See [5]).

The Riemannian curvature tensor $\bar{R}(X,Y)Z$ for any tangent vector fields X, Y and Z on $G_2(\mathbb{C}^{m+2})$ is explicitly defined in [3]. In a paper [12] due to Pérez, Jeong and Suh, we have introduced a notion of normal Jacobi operator $\bar{R}_N$ for hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ in such a way that

$$\bar{R}_N X = \bar{R}(X,N)N \in \text{End } T_xM, \quad x \in M,$$

for any tangent vector field X on M, where $\bar{R}$ and N respectively denote the Riemannian curvature tensor and a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. Related to such a normal Jacobi operator $\bar{R}_N$, Jeong, Kim and Suh [7] obtained a non-existence theorem for *Hopf hypersurfaces* in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator, that is, $(\nabla_X \bar{R}_N)Y = 0$, $\forall X \in TM$, where ∇ denotes the induced Riemannian connection on M.

Motivated by this fact, in such a paper we consider more general notion of parallelism weaker than the notion of parallel normal Jacobi operator. So we consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with *Reeb parallel* normal Jacobi operator, that is, $\nabla_\xi \bar{R}_N = 0$. The normal Jacobi operator $\bar{R}_N$ is said to be *Reeb parallel* on M if the covariant derivative of the normal Jacobi operator $\bar{R}_N$ along the direction of the Reeb vector ξ identically vanishes, that is, $\nabla_\xi \bar{R}_N = 0$. Here the meaning of *Reeb parallel* normal Jacobi operator $\bar{R}_N$ gives that every eigenspaces of the normal Jacobi operator $\bar{R}_N$ are *parallel* along the integral curve γ of the Reeb vector field ξ in M. Here the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are said to be *parallel* along the curve γ if they are *invariant* under the *parallel displacement* along the curve γ in M.

Related to such a Reeb parallel normal Jacobi operator $\bar{R}_N$ in section 3 we prove an important theorem for $\mathfrak{D}^\perp$-invariant real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ as follows:

Main Theorem. *Let M be a $\mathfrak{D}^\perp$-invariant real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel normal Jacobi operator. If the distribution $\mathfrak{D}$ and $\mathfrak{D}^\perp$*

components of the Reeb vector field are eigenvectors of the shape operator at every point, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r \in (0, \pi/\sqrt{8}\,)$.

As a corollary of this theorem, together with the result in [7], we may assert the following

Corollary. *There do not exist any connected $\mathfrak{D}^{\perp}$-invariant real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator if the distribution $\mathfrak{D}$ and $\mathfrak{D}^{\perp}$ components of the Reeb vector field are eigenvectors of the shape operator at every point.*

In the sequel we will use some notations as in [4] and [5].

2. Reeb parallel normal Jacobi operator

In this section we want to derive some formulas related to the *Reeb parallel* normal Jacobi operator from the curvature tensor $\bar{R}(X,Y)Z$ of $G_2(\mathbb{C}^{m+2})$. Moreover, we will show whether the hypersurfaces of type (A) or (B) in Theorem A have Reeb parallel normal Jacobi operator, that is, result mentioned in our main theorem satisfy the assumption of *Reeb parallel*, that is, $\nabla_\xi \bar{R}_N = 0$. From now, unless otherwise stated, let us follow the notations such as η, η_ν, ϕ, and ϕ_ν in [4], [5], [7], [12] and [13].

Then first the normal Jacobi operator $\bar{R}_N$ can be defined in such a way that

$$\begin{aligned}\bar{R}_N(X) = \bar{R}(X,N)N = X + 3\eta(X)\xi + 3\sum\nolimits_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ - \sum\nolimits_{\nu=1}^{3}\big\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\big\}.\end{aligned} \tag{2.1}$$

We used these standard notations such as η, η_ν, ϕ, ϕ_ν in (2.1). These were used in [8]. Of course, we know that the normal Jacobi operator $\bar{R}_N$ is a symmetric endomorphism of T_xM, $x \in M$([8], [12]).

A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator, that is, $\nabla_X \bar{R}_N = 0$ for any tangent vector field X on M, satisfies the following

$$\begin{aligned}0 = (\nabla_X \bar{R}_N)Y &= 3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX \\ &+ 3\sum\nolimits_{\nu=1}^{3}\big\{g(\phi_\nu AX, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu AX\big\} \\ &- \sum\nolimits_{\nu=1}^{3}\big[2\eta_\nu(\phi AX)(\phi_\nu\phi Y - \eta(Y)\xi_\nu) - g(\phi_\nu AX, \phi Y)\phi_\nu\xi \\ &- \eta(Y)\eta_\nu(AX)\phi_\nu\xi - \eta_\nu(\phi Y)(\phi_\nu\phi AX - g(AX,\xi)\xi_\nu)\big]\end{aligned}$$

(See [8]).

From this, by putting $X=\xi$ and replacing Y by X, we have

$$\begin{aligned}0=(\nabla_\xi \bar{R}_N)X &= 3g(\phi A\xi, X)\xi + 3\eta(X)\phi A\xi \\ &+3\sum\nolimits_{\nu=1}^{3}\big\{g(\phi_\nu A\xi, X)\xi_\nu + \eta_\nu(X)\phi_\nu A\xi\big\} \\ &-\sum\nolimits_{\nu=1}^{3}\big[2\eta_\nu(\phi A\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - g(\phi_\nu A\xi, \phi X)\phi_\nu\xi \\ &-\eta(X)\eta_\nu(A\xi)\phi_\nu\xi - \eta_\nu(\phi X)(\phi_\nu\phi A\xi - g(A\xi,\xi)\xi_\nu)\big] \end{aligned} \tag{2.2}$$

for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$. And by putting $X=\xi$ into (2.2), we have

$$\begin{aligned}0=(\nabla_\xi \bar{R}_N)\xi &= 3\phi A\xi + 5\sum\nolimits_{\nu=1}^{3}\eta_\nu(\phi A\xi)\xi_\nu \\ &+3\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu A\xi + \sum\nolimits_{\nu=1}^{3}\eta_\nu(A\xi)\phi_\nu\xi. \end{aligned} \tag{2.3}$$

Now we check whether the normal Jacobi operator $\bar{R}_N$ for hypersurfaces of type (A) or of type (B) is Reeb parallel or not. By using (2.2), (2.3) and from Proposition 3 in [4], we check for a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ whether it has Reeb parallel normal Jacobi operator or not as follows:

Case I: $\xi=\xi_1\in T_\alpha$.

Then by (2.3) we have

$$(\nabla_\xi \bar{R}_N)\xi = 4\alpha\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi = 4\alpha\phi_1\xi_1 = 0.$$

Case II: $\xi_2,\xi_3\in T_\beta$.

Putting $X=\xi_2$ into (2.2) and using $\phi\xi_2=-\xi_3, \phi\xi_3=\xi_2$, we have

$$\begin{aligned}(\nabla_\xi \bar{R}_N)\xi_2 &= 3\sum\nolimits_{\nu=1}^{3}\big\{g(\phi_\nu A\xi,\xi_2)\xi_\nu + \eta_\nu(\xi_2)\phi_\nu A\xi\big\} \\ &\quad -\sum\nolimits_{\nu=1}^{3}\big\{-g(\phi_\nu A\xi,\phi\xi_2)\phi_\nu\xi + \eta_\nu(\phi\xi_2)g(A\xi,\xi)\xi_\nu\big\} \\ &= 3\alpha(g(\phi_2\xi,\xi_2)\xi_2 + g(\phi_3\xi,\xi_2)\xi_3) + 3\alpha\phi_2\xi + \alpha\phi_2\xi + \alpha\xi_3 = 0.\end{aligned}$$

Similarly, by putting $X=\xi_3$ into (2.2) we know that $(\nabla_\xi \bar{R}_N)\xi_3=0$.

Case III: $X_i\in T_\lambda$, $i=1,\dots,2(m-1)$.

Then by Proposition 3 in [4] we know that the eigenspace T_λ has the property that $\phi X=\phi_1 X$ for any $X\in T_\lambda$. Moreover, it is invariant by the structure tensor ϕ, that is $\phi T_\lambda\subset T_\lambda$. Because for any $X_i\in\mathfrak{D}$ such that $\phi X_i=\phi_1 X_i$ we have $\phi\phi X_i=$

$-X_i$ and $\phi_1\phi X_i = \phi_1^2 X_i = -X_i$. Then $\phi\phi X_i = \phi_1\phi X_i$. So it follows that $\phi X_i \in T_\lambda$. From this, together with (2.2), we have $(\nabla_\xi \bar{R}_N)X_i = 0$, $i = 1, \ldots, 2(m-1)$.

Case IV: $Y_i \in T_\mu$, $i = 1, \ldots, 2(m-1)$.

The eigenspace T_μ has the property that $\phi Y = -\phi_1 Y$ for any $Y \in T_\mu$. Moreover, such an eigenspace T_μ is ϕ-invariant, that is, $\phi T_\mu \subset T_\mu$. In fact, suppose $Y_i \in \mathfrak{D}$ such that $\phi Y_i = -\phi_1 Y_i$. Then $\phi Y_i \in \mathfrak{D}$ and satisfies $\phi\phi Y_i = -Y_i$ and $\phi_1\phi Y_i = -\phi_1^2 Y_i = Y_i$. So it follows that $\phi Y_i \in T_\mu$. Then also by using (2.2) we have $(\nabla_\xi \bar{R}_N)Y_i = 0$, $i = 1, \ldots, 2(m-1)$.

Then by these Cases I, II, III and IV we know that a real hypersurface of type (A) in Theorem A has Reeb parallel normal Jacobi operator $\bar{R}_N$ for $\xi \in \mathfrak{D}^\perp$.

Next, we check whether the normal Jacobi operator $\bar{R}_N$ for hypersurfaces of type (B) is Reeb parallel or not. Now let us consider a unit eigenvector $X \in T_\beta$ from Proposition 2 in [4]. In other words, we can substitute $X = \xi_\mu \in T_\beta$ into (2.2). Then it follows that

$$\begin{aligned} 0 = (\nabla_\xi \bar{R}_N)\xi_\mu &= 3\sum\nolimits_{\nu=1}^{3}\{g(\phi_\nu A\xi, \xi_\mu)\xi_\nu + \eta_\nu(\xi_\mu)\phi_\nu A\xi\} \\ &+ \sum\nolimits_{\nu=1}^{3} g(\phi_\nu A\xi, \phi\xi_\mu)\phi_\nu\xi = 4\alpha\phi_\mu\xi. \end{aligned}$$

Since $\alpha = -2\tan(2r)$ is non zero for some $r \in (0, \pi/4)$, we have $\phi_\mu\xi = 0$. But $g(\phi_\mu\xi, \phi_\mu\xi) = 1$, which makes a contradiction. So we know that the normal Jacobi operator $\bar{R}_N$ for hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$ can not be Reeb parallel when the Reeb vector ξ belongs to the distribution $\mathfrak{D}$.

3. Proof of Main Theorem

Now let us consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with Reeb parallel normal Jacobi operator, that is, $(\nabla_\xi \bar{R}_N)X = 0$ for any tangent vector field $X \in TM$.

We assert the following

Lemma 3.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel normal Jacobi operator. If the distribution $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field are eigenvectors of the shape operator at every point, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof. Let us assume that $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$, non-zero functions $\eta(X_0)$ and $\eta(\xi_1)$.

From this, together with formula (2.3), we have

$$
\begin{aligned}
0 &= 3(\eta(X_0)\phi AX_0 + \eta(\xi_1)\phi A\xi_1) + 5\sum\nolimits_{\nu=1}^{3} g(\xi_\nu, \eta(X_0)\phi AX_0 \\
&\quad + \eta(\xi_1)\phi A\xi_1)\xi_\nu + 3\eta(\xi_1)(\eta(X_0)\phi_1 AX_0 + \eta(\xi_1)\phi_1 A\xi_1) \\
&\quad + \sum\nolimits_{\nu=1}^{3} g(\xi_\nu, \eta(X_0)AX_0 + \eta(\xi_1)A\xi_1)\phi_\nu\xi. \qquad (3.1)
\end{aligned}
$$

And we consider the distribution $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field are eigenvectors of the shape operator at every point. Then this gives the following

$$
AX_0 = g(AX_0, X_0)X_0, \quad A\xi_1 = g(A\xi_1, \xi_1)\xi_1.
$$

By using this in (3.1), we have

$$
\begin{aligned}
0 &= 3\{\eta(X_0)g(AX_0, X_0)\phi X_0 + \eta(\xi_1)g(A\xi_1, \xi_1)\phi\xi_1\} \\
&\quad + 5\sum\nolimits_{\nu=1}^{3} g(\xi_\nu, \eta(X_0)g(AX_0, X_0)\phi X_0 + \eta(\xi_1)g(A\xi_1, \xi_1)\phi\xi_1)\xi_\nu \\
&\quad + 3\eta(\xi_1)\eta(X_0)g(AX_0, X_0)\phi_1 X_0 + \eta(\xi_1)g(A\xi_1, \xi_1)\phi_1\xi.
\end{aligned}
$$

From this, by the assumption of $\eta(X_0)\eta(\xi_1) \neq 0$, we have

$$
0 = g(A\xi_1, \xi_1)\phi_1 X_0,
$$

where we have used the following

$$
\phi\xi_1 = \eta(X_0)\phi_1 X_0, \quad \phi X_0 = -\eta(\xi_1)\phi_1 X_0, \quad \eta_\nu(\phi_1 X_0) = 0 \ (\nu = 1, 2, 3).
$$

So we assert that $g(A\xi_1, \xi_1) = 0$.

From this, we obtain the following

$$
A\xi = \eta(X_0)g(AX_0, X_0)X_0.
$$

So we put $A\xi = \sigma X_0$, where $\sigma = \eta(X_0)g(AX_0, X_0)$.

On the other hand, we may put $X = X_0 \in \mathfrak{D}$ in (2.2). Then we have the following

$$
0 = \sigma\eta(X_0)\eta(\xi_1)\phi_1 X_0.
$$

From this, it follows that $\sigma = 0$, where we have used the assumption of $\eta(X_0)\eta(\xi_1) \neq 0$. Then we say that the geodesic Reeb flow is vanishing, that is, $A\xi = 0$.

Since M is Hopf, we can use the result due to Bernt and Suh (see [5], p. 92). Then we have the following

$$
0 = \sum\nolimits_{\nu=1}^{3} \eta_\nu(\xi)\phi\xi_\nu = \eta_1(\xi)\phi\xi_1 = \eta(\xi_1)\eta(X_0)\phi_1 X_0.
$$

We have $\phi_1 X_0 = 0$, where we have used the assumption of $\eta(X_0)\eta(\xi_1) \neq 0$. This makes a contradiction, so the result follows. □

According to Lemma 3.1 we divide two cases such that $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^{\perp}$. First, we consider the case that ξ belongs to the distribution $\mathfrak{D}^{\perp}$.

Lemma 3.2. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel normal Jacobi operator. If the Reeb vector ξ belongs to the distribution $\mathfrak{D}^{\perp}$, then M becomes a Hopf hypersurface.*

Proof. Now let us consider the case $\xi \in \mathfrak{D}^{\perp}$. So we may put $\xi = \xi_1$. Then we know that

$$\phi_2\xi = -\xi_3, \quad \phi_3\xi = \xi_2, \eta_2(\phi A\xi) = \eta_3(A\xi), \quad \eta_3(\phi A\xi) = -\eta_2(A\xi).$$

By using these formulas and together with (2.3) we get the following

$$0 = 3\phi A\xi + 6\eta_3(A\xi)\xi_2 - 6\eta_2(A\xi)\xi_3 + 3\phi_1 A\xi. \tag{3.2}$$

From this, by taking an inner product with ξ_3 and ξ_2, respectively, we obtain the following

$$\eta_2(A\xi) = 0, \quad \eta_3(A\xi) = 0. \tag{3.3}$$

Then substituting (3.3) into (3.2) implies

$$0 = \phi A\xi + \phi_1 A\xi.$$

From this, if we apply the structure tensor ϕ, we have

$$0 = -A\xi + \eta(A\xi)\xi + \phi\phi_1 A\xi. \tag{3.4}$$

And we have

$$\phi\phi_1 A\xi = \phi_1\phi A\xi = \phi_1\nabla_\xi\xi = -(\nabla_\xi\phi_1)\xi = q_2(\xi)\xi_2 + q_3(\xi)\xi_3 - A\xi + \eta(A\xi)\xi. \tag{3.5}$$

Now substituting (3.5) into (3.4), we have

$$0 = -2A\xi + 2\eta(A\xi)\xi + q_2(\xi)\xi_2 + q_3(\xi)\xi_3. \tag{3.6}$$

On the other hand, by using assumption we have

$$\nabla_\xi\xi = \nabla_\xi\xi_1.$$

So we have

$$\phi A\xi = q_3(\xi)\xi_2 - q_2(\xi)\xi_3 + \phi_1 A\xi.$$

From this, by taking an inner product with ξ_2 and ξ_3, respectively, we obtain the following

$$q_3(\xi) = 2\eta_3(A\xi), \quad q_2(\xi) = 2\eta_2(A\xi).$$

And by using this, together with formula (3.3) we have

$$q_3(\xi) = 0, \quad q_2(\xi) = 0.$$

Then substituting these formulas into (3.6) gives

$$A\xi = \eta(A\xi)\xi.$$

This means that a real hypersurface M satisfying Reeb parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$ becomes a Hopf hypersurface in this case. □

Next in the latter case we consider that ξ belongs to the distribution $\mathfrak{D}$.

Lemma 3.3. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel normal Jacobi operator. If the Reeb vector ξ belongs to the distribution $\mathfrak{D}$, then M becomes a Hopf hypersurface.*

Proof. Now let us consider the case $\xi \in \mathfrak{D}$. Then by using (2.3) we have

$$0 = 3\phi A\xi + 5\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi)\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(A\xi)\phi_\nu\xi. \tag{3.7}$$

From this, by taking an inner product with ξ_μ, $\mu = 1, 2, 3$, we have

$$\eta_\mu(\phi A\xi) = 0, \quad \mu = 1, 2, 3. \tag{3.8}$$

By taking an inner product with $\phi_\mu\xi$, $\mu = 1, 2, 3$, into (3.8), we have

$$\eta_\mu(A\xi) = 0, \ \mu = 1, 2, 3. \tag{3.9}$$

Applying (3.8) and (3.9) into (3.7) gives

$$0 = \phi A\xi.$$

From this, if we apply the structure tensor ϕ, we have

$$A\xi = \eta(A\xi)\xi.$$

This means that a real hypersurface M with Reeb parallel normal Jacobi operator and $\xi \in \mathfrak{D}$ becomes also a Hopf hypersurface. □

Remark. If M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ and the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, the normal Jacobi operator $\bar{R}_N$ becomes Reeb parallel. Even in such a case the condition of Hopf does not give us any meaning when we consider the Reeb parallel normal Jacobi operator. From such a point of a view, instead of Hopf we have considered the notion of $\mathfrak{D}^\perp$-invariant real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ in our Main Theorem.

Then summing up Lemmas 3.1, 3.2, 3.3 and together with Theorem A ([4]), we know that a $\mathfrak{D}^{\perp}$-invariant real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel normal Jacobi operator is locally congruent to of type (A) or of type (B) if the distribution $\mathfrak{D}$ and $\mathfrak{D}^{\perp}$ components of the Reeb vector field are eigenvectors of the shape operator at every point.

The converse part of our main result is checked in section 2, in which a real hypersurface of type (A) in Theorem A satisfies Reeb parallel normal Jacobi operator for $\xi \in \mathfrak{D}^{\perp}$. But we can easily verify that the normal Jacobi operator $\bar{R}_N$ for hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$ can not be Reeb parallel for $\xi \in \mathfrak{D}$. From this, we have completed the proof of our Main Theorem in the introduction.

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