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d'Alembert's other functional equation

By BRUCE EBANKS (Mississippi State) and HENRIK STETKÆR (Aarhus)

Abstract. Let G be a topological group. We find formulas for the solutions $f, g, h \in C(G)$ of the functional equation

$$f(xy) - f(y^{-1}x) = g(x)h(y), \quad x, y \in G,$$

when G is generated by its squares and its center, as for instance when G is a connected Lie group, and when G is compact. Some solutions are given by the same formulas as in the known abelian case. The new ones are expressed in terms of matrix-coefficients of irreducible, 2-dimensional representations of G and of solutions of Wilson's functional equation $\phi(xy) + \phi(xy^{-1}) = 2\phi(x)\gamma(y)$.

1. Introduction

In his series [3], [4], [5] of papers about vibrating strings d'Alembert studied not just the functional equation

$$g(x+y) + g(x-y) = 2g(x)g(y), \quad x, y \in \mathbb{R}, \tag{1.1}$$

in which $g : \mathbb{R} \to \mathbb{R}$ is the unknown function, but also

$$f(x+y) - f(x-y) = 2g(x)h(y), \quad x, y \in \mathbb{R}, \tag{1.2}$$

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We dedicate this paper to Che Tat Ng on the occasion of his retirement from the University of Waterloo. His work, particularly on Jensen's functional equation on groups, has been inspirational for us.

in which $f, g, h : \mathbb{R} \to \mathbb{R}$ are unknown functions.

Many solutions of (1.2) correspond to elementary trigonometric identities. For example, that the triple $f(x) = \cos x$, $g(x) = -\sin x$, $h(x) = \sin x$ is a solution, means that

$$\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}, \quad x, y \in \mathbb{R}. \tag{1.3}$$

d'Alembert's functional equations (1.1) and (1.2) fit into a wider context, in which $\mathbb{R}$ is replaced by a topological group G and the triple $f, g, h : \mathbb{R} \to \mathbb{R}$ by $f, g, h \in C(G)$ (the continuous, complex-valued functions on G). This is particularly simple for abelian groups $(G, +)$, where the equations are unchanged, except that G, and not $\mathbb{R}$, will be the domain of definition of the unknown functions.

Kannappan [12] solved the equation (1.1) on abelian groups. His work was extended to general groups, even monoids (where inversion is replaced by an involution), by Yang [27], Davison [9] and others, so now a satisfactory theory exists for d'Alembert's (first) functional equation.

As a contrast, the solutions $f, g, h \in C(G)$ of d'Alembert's other functional equation

$$f(x+y) - f(x-y) = g(x)h(y), \quad x, y \in G, \tag{1.4}$$

are known on abelian groups $(G, +)$ (Proposition 5), but the results have not been extended to general groups. The present paper complements and contains the existing results for (1.4) by finding the solutions $f, g, h \in C(G)$ of the extension

$$f(xy) - f(y^{-1}x) = g(x)h(y), \quad x, y \in G, \tag{1.5}$$

of it to large classes of groups G that need not be abelian. We impose no conditions like boundedness on the solutions.

(1) We get a number of results about the solutions of (1.5) on all groups, enabling us to

(2) find all its continuous solutions on compact groups (Theorem 20) and to

(3) describe its solutions on groups which are generated by their squares and their center (Theorem 21).

(4) To illustrate the theory we solve (1.5) on the symmetric group S_3, the special unitary group $SU(2)$, and the special linear group $SL(2, \mathbb{R})$ (Sections 11, 12 and 13).

(5) As a minor by-product we solve the particular instance of (1.5), in which $g = 2f$ (Proposition 26). It occurs in the literature for G abelian (see Székelyhidi [23, Theorem 12.10]).

There are often different natural ways of extending functional equations from abelian to non-abelian groups. To take an example, NG studied both Jensen's functional equation (A.7) (see [14], [15]) and the variant $f(xy)+f(y^{-1}x) = 2f(x)$ of it (see [16]). Here we focus exclusively on (1.5), and we include nothing about $f(xy) - f(xy^{-1}) = g(x)h(y)$.

Our methods are mainly algebraic. An exception is our discussion of compact groups, because we need the Haar measure. In our theory we can not resort to differential equations like d'Alembert did, since we handle also discrete groups.

Abstract harmonic analysis in the form of matrix-elements of irreducible, 2-dimensional representations enters the description of the solutions of d'Alembert's first functional equation. With that in mind it is no wonder that such matrix-elements play a crucial role also for the solutions of d'Alembert's other functional equation (1.5).

The trigonometric identity (1.3) was found by Johannes Werner about 1510. It played an important role as a precursor of the logarithm, because it gives a method to compute products by help of trigonometric tables via sums and differences. The astronomer Tycho Brahe applied the method on a big scale to speed up reductions of data from his stellar observations (see [24]).

However, let us also mention more recent contributions to the theory of functional equations related to (1.5), first for abelian groups, where (1.5) reduces to (1.4).

In 1920 WILSON [26] found on $G = \mathbb{C}$ relations between the subtractions laws for Sine and Cosine and the solutions of (1.4) in the two special cases $g = h$ and $h = f$.

SZÉKELYHIDI [23, Theorem 12.10] solved the equation

$$f(x+y) - f(x-y) = 2f(x)g(y), \quad x, y \in G,$$

for G abelian by methods from spectral analysis. It is (1.4) with $g = 2f$. Our Proposition 26 solves a generalization of it on non-abelian groups.

On $G = \mathbb{R}$ the equation (1.4) is a special case of Wilson's second generalization of d'Alembert's functional equation, the continuous solutions of which can be found in ACZÉL'S book [1, Section 3.2.2]. KANNAPPAN [13, Section 3.4.13] extends the discussion about Wilson's second generalization from $\mathbb{R}$ to abelian, 2-divisible groups.

RUKHIN [18] discussed the form of solutions of

$$f(x+y) - f(x-y) = \sum_{j=1}^{m} h_j(x)k_j(y), \quad x, y \in G,$$

when G is a 2-divisible, abelian group, and where the functions assume values in an algebraically closed field $\mathbb{F}$ of characteristic zero.

Under the same conditions as in [18] Sinopoulos [20, Theorem 1] found explicit expressions for the functions $f, g_1, g_2, h_1, h_2 : G \to \mathbb{F}$ satisfying the equation

$$f(x+y) - f(x-y) = g_1(x)h_1(y) + g_2(x)h_2(y), \quad x, y \in G.$$

Stetkær [21, Corollary III.5] derived formulas for the continuous solutions of the functional equation

$$f(x+y) - f(x+\sigma(y)) = 2g(x)h(y), \quad x, y \in G,$$

where $\sigma : G \to G$ is an involutive automorphism. Equation (1.4) is the special case of $\sigma(x) = -x$ for $x \in G$.

However, the literature has also results about the generalization (1.5) of (1.4) and equations related to it on non-abelian groups.

Chung, Ebanks, Ng, Kannappan and Sahoo [8, Theorem 4.1] solved the functional equation

$$F(xy) - F(y^{-1}x) = H(x)G(y) + K(y), \quad x, y \in G,$$

that compared with (1.5) has an extra term K on the right hand side. They did not assume that the group G was abelian, but they assumed that the function F was abelian in the sense of our Definition 1. Our Proposition 5 encompasses the results of [8] for $K = 0$.

An and Yang published in [6] a general theory of the continuous solutions $f_1, \dots, f_6 : G \to \mathbb{C}$, where G is a compact group, of

$$f_1(xy) + f_2(yx) + f_3(xy^{-1}) + f_4(y^{-1}x) = f_5(x)f_6(y), \quad x, y \in G,$$

but [6] does not provide explicit formulas for the solutions when $f_2 = f_3 = 0$ and $f_4 = -f_1$, where the functional equation reduces to (1.5). We write down the continuous solutions of (1.5) on compact groups in Section 8.

Penney and Rukhin [17, Theorem 1.2] described the square integrable solutions of functional equations similar to (1.5). We do not impose restrictions like square integrability on the solutions.

[10] treats a generalization of (1.4) of another kind than (1.5). Its functional equation is $f(xy) - f(\sigma(y)x) = g(x)h(y)$, $x, y \in S$, where $\sigma : S \to S$ is an involutive automorphism of a monoid S. It differs from (1.5), because the inversion of (1.5) is an anti-automorphism.

2. Notation and terminology

For any topological space X we let $C(X)$ denote the complex algebra of all continuous functions $f : X \to \mathbb{C}$.

If $g : X \to \mathbb{C}$ and $h : Y \to \mathbb{C}$, where X and Y are non-empty sets, we define $g \otimes h : X \times Y \to \mathbb{C}$ by $g \otimes h(x, y) := g(x)h(y)$ for $(x, y) \in X \times Y$. Note that $g \otimes h \neq 0$ iff both $g \neq 0$ and $h \neq 0$.

We let $M(n, \mathbb{C})$ denote the set of complex $n \times n$ matrices. The identity matrix is denoted I. Let $GL(n, \mathbb{C}) := \{A \in M(n, \mathbb{C}) \mid \det A \neq 0\}$, and $sl(n, \mathbb{C}) := \{A \in M(n, \mathbb{C}) \mid \operatorname{tr} A = 0\}$ with tr meaning trace. Similarly for $\mathbb{C}$ replaced by $\mathbb{R}$.

Throughout the paper G will denote a topological group with identity element e. This set up includes the purely algebraic one of a discrete group, where all complex-valued functions on G are continuous. The term compact group means a topological group, which is not just compact, but also Hausdorff, because we need the Haar measure.

Let $\langle G^2 \rangle$ denote the subgroup of G generated by the squares $\{x^2 \mid x \in G\}$. Due to the identity $[x, y] = x^2(x^{-1}y)^2y^{-2}$ it contains the commutator subgroup $[G, G]$, so it is a normal subgroup of G. The coset space $G/\langle G^2 \rangle$ is an abelian group.

Definition 1. Let X be a set and $F : G \to X$ a function. We say that F is *abelian*, if $F(x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)}) = F(x_1x_2 \cdots x_n)$ for all $x_1, x_2, \ldots, x_n \in G$, all permutations σ and all $n = 2, 3, \ldots$. It is equivalent to $F(xyz) = F(xzy)$ for all $x, y, z \in G$. We say that F is *non-abelian*, if F is not abelian. Any abelian function F is *central*, meaning $F(xy) = F(yx)$. Central functions on groups are also called class functions in the literature.

Let H be a subgroup of G, and let f be a function on G. f is said to be a function on the coset space G/H if $f(gh) = f(g)$ for all $g \in G$ and $h \in H$. Note any function on $G/\langle G^2 \rangle$ is abelian.

For $f : G \to \mathbb{C}$ we let $\check{f}(x) := f(x^{-1})$, $x \in G$. We let $f_e := (f + \check{f})/2$ and $f_o := (f - \check{f})/2$ denote the even and odd parts of f. We say that f is *even*, if $f = f_e$, and that f is *odd*, if $f = f_o$.

An *additive function* on G is a continuous homomorphism of G into $(\mathbb{C}, +)$.

A *character* χ on G is a continuous homomorphism $\chi : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers. So characters need not be unitary in the present paper.

A *representation* of G on $\mathbb{C}^n$ is a continuous homomorphism of G into $GL(n, \mathbb{C})$.

By *solution* of a functional equation we mean a solution the components of which are continuous.

Lemma 2 lists three useful identities about *adjugation*, which is the anti-automorphism of $M(2,\mathbb{C})$ defined by

$$\operatorname{adj}(A) := \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \quad \text{for } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Lemma 2. *Let $A \in M(2,\mathbb{C})$ and $B \in sl(2,\mathbb{C})$. Then*

(a) $A \operatorname{adj}(A) = \operatorname{adj}(A) A = (\det A) I$.

(b) $A + \operatorname{adj}(A) = (\operatorname{tr} A) I$.

(c) $AB - B \operatorname{adj}(A) = \operatorname{tr}(AB) I$.

Appendix A lists the trigonometric functional equations, that we refer to in our discussion of the functional equation (1.5).

3. The trivial solutions

Definition 3. We say that the solution $f, g, h \in C(G)$ of (1.5) is *trivial*, if $g \otimes h = 0$ (that is $g = 0$ or $h = 0$), or equivalently if $f \in \mathcal{N}(G) := \{f \in C(G) \mid f(xy) = f(y^{-1}x) \text{ for all } x, y \in G\}$.

Proposition 4. *A function $\nu : G \to \mathbb{C}$ satisfies*

$$\nu(xy) - \nu(y^{-1}x) = 0 \quad \textit{for all } x, y \in G, \tag{3.1}$$

if and only if it is a function on the abelian group $G/\langle G^2 \rangle$.

The component f of any trivial solution f, g, h of (1.5) is abelian.

$\mathcal{N}(G)$ consists of the constant functions on G, if $G = \langle G^2 \rangle$.

Proof. Assume ν satisfies (3.1). Taking $x = e$ we see that ν is even, which implies that $\nu(xy) = \nu(y^{-1}x) = \nu((y^{-1}x)^{-1}) = \nu(x^{-1}y)$. In other words we may replace the first factor in the argument of ν by its inverse. Doing that in (3.1) we infer that ν is central. Then replacing x by xy in (3.1) we obtain that $\nu(xy^2) = \nu(x)$, which implies that ν is a function on $G/\langle G^2 \rangle$. And then ν is abelian, because $G/\langle G^2 \rangle$ is an abelian group as noted in Section 2.

The converse implication: Let ν be a function on $G/\langle G^2 \rangle$. Since $G/\langle G^2 \rangle$ is abelian, $\nu(y^{-1}x) = \nu(xy^{-1})$. And so $\nu(y^{-1}x) = \nu(xy^{-1}) = \nu(xy^{-1}y^2) = \nu(xy)$.

The rest of the proof is immediate, so we skip it. $\square$

For examples of groups G for which $G = \langle G^2 \rangle$ see the remarks prior to Theorem 21.

Proposition 4 is actually stronger than it appears. If we can find any particular solution (f_p, g, h) of equation (1.5), then for every solution (f, g, h) of (1.5) we have $f - f_p \in \mathcal{N}(G)$. Thus, if there is any abelian solution f_p, then all solutions f are abelian.

4. The abelian solutions

For abelian groups the set of solutions (f, g, h) of the functional equation (1.5) is described in the literature in terms of characters and additive maps. Proposition 5 takes a tiny step further. It assumes only that f is abelian, not that the group is. That is what we need later.

Proposition 5. *The set of solutions $f, g, h \in C(G)$ of (1.5) such that f is abelian, is the union of three subsets:*

(a) *The trivial solutions.*

(b) *There exist a character $\chi : G \to \mathbb{C}^*$ for which $\check{\chi} \neq \chi$, $c \in \mathbb{C} \setminus \{0\}$, $c_1, c_2 \in \mathbb{C}$ and $\nu \in \mathcal{N}(G)$ such that*

$$f = \frac{c}{2}\Big(c_1 \frac{\chi - \check{\chi}}{2} + c_2 \frac{\chi + \check{\chi}}{2}\Big) + \nu,$$
$$g = c_1 \frac{\chi + \check{\chi}}{2} + c_2 \frac{\chi - \check{\chi}}{2}, \quad h = c\frac{\chi - \check{\chi}}{2}.$$

(c) *There exist a character $\chi : G \to \mathbb{C}^*$ for which $\check{\chi} = \chi$, $c_1, c_2 \in \mathbb{C}$, an additive function $a \in C(G)$ and $\nu \in \mathcal{N}(G)$ such that*

$$f = \frac{1}{2} c_1 \chi a + \frac{1}{4} c_2 \chi a^2 + \nu, \quad g = c_1 \chi + c_2 \chi a, \quad h = \chi a.$$

Proof. (a) Any trivial solution has f abelian by Proposition 4.

(b) Aside from the trivial case (a), g and h are abelian when f is, so f, g and h are functions on the abelian group $G/[G, G]$. Thus we may assume that G is abelian. Now combine [21, Corollary III.5] and Proposition 4. Alternatively, specializing [8, Theorem 4.1] to $K = 0$ will also give you the formulas above. That [8] does not discuss continuity is easily remedied. □

If $G = \langle G^2 \rangle$ then the proposition simplifies, because in that case $\mathcal{N}(G) = \mathbb{C}$, and $\chi = 1$ is the only character such that $\check{\chi} = \chi$.

The assumption that f is abelian may sometimes be relaxed. For instance: If $G = \langle G^2 \rangle$ it suffices that f is central (Proposition 23). For other sufficient conditions consult Propositions 19 and 22.

The case of $G = \mathbb{R}$ is treated in details by [13, Section 3.4.9].

In view of our later results it seems futile to guess from Proposition 5 which parts of it persist to general groups. An example: An inspection of the formulas of the proposition reveals that the h-part of any non-trivial solution (f, g, h) is odd. That result is actually easy to derive directly and in more generality:

Lemma 6. *If $f, g, h \in C(G)$ is a solution of* (1.5) *such that g is central and $\neq 0$, then h is odd.*

Proof. The result follows from the computation

$$\begin{aligned} g(x)h(y) &= g(yxy^{-1})h(y) = f(yxy^{-1}y) - f(y^{-1}yxy^{-1}) \\ &= f(yx) - f(xy^{-1}) = -[f(xy^{-1}) - f(yx)] = -g(x)h(y^{-1}), \end{aligned}$$

which holds for all $x, y \in G$. □

However, h is not odd in general (Example 25). Luckily a simple formula describes how h transforms under inversion (Theorem 14(c)).

Another example: The formula (6.12) that tells how g transforms under inner automorphisms, is not apparent on abelian groups.

5. Certain non-abelian solutions

The following simple observation relates (1.5) and Wilson's functional equation (A.3). It will be used to find f, given g and h.

Lemma 7. *Let $h, g : G \to \mathbb{C}$ be a solution of Wilson's functional equation* (A.3), *i.e., $h(xy) + h(xy^{-1}) = 2h(x)g(y)$ for all $x, y \in G$, such that h is odd. Then*

(a) *$(h/2, g, h)$ is a solution of* (1.5).

(b) *Assuming furthermore that (f, g, h) is a solution of* (1.5), *then $f = h/2 + \nu$ for some $\nu \in \mathcal{N}(G)$.*

Proof. (a) The proof is the computation

$$\begin{aligned} h(xy) - h(y^{-1}x) &= -[h(y^{-1}x) + h(y^{-1}x^{-1})] \\ &= -2h(y^{-1})g(x) = 2g(x)h(y) \quad \text{for } x, y \in G. \end{aligned}$$

Part (b) follows from the definition (3.1) of $\mathcal{N}(G)$. □

Lemma 8 looks rather special, but it does capture a number of important solutions. They are in general non-abelian (Lemma 9).

Lemma 8. *Let π be a representation of G on $\mathbb{C}^2$. Let $A \in sl(2,\mathbb{C})$ and $W \in M(2,\mathbb{C})$ satisfy the following two equivalent intertwining formulas for all $x \in G$:*

$$\operatorname{adj}(\pi(x))W = W\pi(x^{-1}) \quad or \quad \pi(x)W = W\det(\pi(x))\pi(x). \tag{5.1}$$

Then the triple of functions

$$f(x) := \operatorname{tr}(\pi(x)AW), \quad g(x) := \operatorname{tr}(\pi(x)W), \quad h(x) := \operatorname{tr}(\pi(x)A), \quad x \in G,$$

satisfies (1.5) *and transforms for any $x, y \in G$ as follows:*

$$g(y^{-1}xy) = \det(\pi(y))\, g(x) \quad and \quad h(y^{-1}) = -\frac{h(y)}{\det \pi(y)}.$$

Proof. The proof of the fact that (f,g,h) satisfies (1.5) is the following computation in which we use first that the trace function is central, next the assumption (5.1) and finally Lemma 2(c):

$$\begin{aligned} f(xy) - f(y^{-1}x) &= \operatorname{tr}(\pi(x)\pi(y)AW) - \operatorname{tr}(\pi(y^{-1})\pi(x)AW) \\ &= \operatorname{tr}\{\pi(x)[\pi(y)AW - AW\pi(y^{-1})]\} = \operatorname{tr}\{\pi(x)[\pi(y)A - A\operatorname{adj}(\pi(y))]W\} \\ &= \operatorname{tr}\{\pi(x)\operatorname{tr}(\pi(y)A)W\} = \operatorname{tr}(\pi(x)W)\operatorname{tr}(\pi(y)A) = g(x)h(y). \end{aligned}$$

The next computation proves the statement about $g(y^{-1}xy)$:

$$g(y^{-1}xy) = \operatorname{tr}(\pi(y^{-1})\pi(x)\pi(y)W) = \operatorname{tr}(\pi(x)\pi(y)W\pi(y^{-1}))$$
$$= \operatorname{tr}(\pi(x)\pi(y)\det(\pi(y))\pi(y^{-1})W) = \det(\pi(y))\operatorname{tr}(\pi(x)W) = \det(\pi(y))g(x).$$

The elementary computation of $h(x^{-1})$ uses Lemma 2. □

Lemma 9. *Let π be an irreducible representation of G on $\mathbb{C}^2$, let $M \in M(2,\mathbb{C})$ and define $F(x) := \operatorname{tr}(\pi(x)M)$, $x \in G$. If F is abelian, then $M = 0$. If F is central, then $M = cI$ for some $c \in \mathbb{C}$.*

Proof. If F is abelian, then $F(xyz) = F(xzy)$ for all $x, y, z \in G$, i.e.,

$$\operatorname{tr}(\pi(x)\pi(y)\pi(z)M) = \operatorname{tr}(\pi(x)\pi(z)\pi(y)M) \quad \text{for all } x, y, z \in G.$$

By Burnside's theorem $\pi(y)\pi(z)M = \pi(z)\pi(y)M$ for each $y, z \in G$. Taking $y = e$ here Schur's lemma says that $M = cI$ for some $c \in \mathbb{C}$. We prove that $c = 0$ by contradiction. If $c \neq 0$ we get $\pi(y)\pi(z) = \pi(z)\pi(y)$, which by Schur's lemma implies that $\pi(y)$ is proportional to I for each $y \in G$. But then π is not irreducible, contradicting our hypothesis. The last statement is proved in a similar way. □

6. Properties of the solutions

We start now on our systematic study of the functional equation (1.5).

Section 6 contains the bulk of the computations of the paper. It presents some properties of the solutions $f, g, h : G \to \mathbb{C}$ of (1.5), mainly of the (g, h) part. If $g \otimes h = 0$ in (1.5) we may refer to Proposition 4, so we prove the statements under the assumption $g \otimes h \neq 0$.

Lemma 10 consists of introductory observations. Some of them will be extended and strengthened later on. For instance, the word abelian in (e) may be replaced by the word central, if $G = \langle G^2 \rangle$ (Propositions 22 and 23).

Lemma 10. *Assume that the triple $f, g, h : G \to \mathbb{C}$ satisfies (1.5) and that $g \otimes h \neq 0$.*

(a) *For all $x, y, z \in G$ we have*

$$f_o(xy) - f_o(y^{-1}x) = g_e(x)h(y), \qquad f_e(xy) - f_e(y^{-1}x) = g_o(x)h(y).$$

In particular $2f_o = g(e)h$, and f odd $\Rightarrow$ g even.

(b) *We have two useful identities, valid for all $x, y, z \in G$:*

$$g(x)h(yz) + g(x^{-1})h(yz^{-1}) + g(z)h(y^{-1}x^{-1}) + g(z^{-1})h(y^{-1}x) = 0, \tag{6.1}$$

and

$$\begin{aligned} &g(xy)h(z) + g(z^{-1}x)h(y) + g(y^{-1}z^{-1})h(x) + g(x^{-1})h(y^{-1}z^{-1}) + g(zy)h(x^{-1}) \\ &= -g(xz)h(y) - g(y^{-1}x)h(z) - g(z^{-1}y^{-1})h(x) \\ &\quad - g(x^{-1})h(z^{-1}y^{-1}) - g(yz)h(x^{-1}). \end{aligned} \tag{6.2}$$

(c) *If h is odd then $g_o = \lambda h$ for some $\lambda \in \mathbb{C}$.*

(d) *$g(e) = 0 \iff g$ is odd $\iff f$ is even.*
If $g(e) = 0$ then $h_o = cg$ for some $c \in \mathbb{C}$.

(e) *f is abelian, if and only if both g and h are abelian.*

Proof. (a) Use the definitions $f_o = (f - \check{f})/2$ and $f_e = (f + \check{f})/2$, and

$$\begin{aligned} \check{f}(xy) - \check{f}(y^{-1}x) &= -[f(x^{-1}y) - f(y^{-1}x^{-1})] \\ &= -g(x^{-1})h(y) = -\check{g}(x)h(y) \quad \text{for all } x, y \in G. \end{aligned}$$

(b) Compute each term as a difference of f-terms using (1.5).

Note for use below that (6.1) for any $x, z \in G$ and $y = e$ becomes

$$g(x)h(z) + g(x^{-1})h(z^{-1}) + g(z)h(x^{-1}) + g(z^{-1})h(x) = 0. \tag{6.3}$$

(c) When h is odd, (6.3) reduces to $g_o(x)h(z) = g_o(z)h(x)$ which implies (c).

(d) We prove first that g odd $\Rightarrow g(e) = 0 \Rightarrow f$ even $\Rightarrow g$ odd. Indeed, g odd $\Rightarrow g(e) = 0$ trivially. Next $g(e) = 0 \Rightarrow f_o = 0$ by (a), i.e., f is even. Finally f even $\Rightarrow g_e = 0$ by (a), i.e., g is odd.

When g is odd, (6.3) reduces to $g(x)[h(z) - h(z^{-1})] + g(z)[h(x^{-1}) - h(x)] = 0$, which implies the last statement.

(e) It is trivial from (1.5) that f abelian implies g and h abelian, so let us assume that g and h are abelian. We shall prove that f is abelian.

By (a) $2f_o = g(e)h$, so f_o is abelian. It is left to show that f_e is abelian. To do this we note, again by (a), that $f_e(xy) - f_e(y^{-1}x) = g_o(x)h(y)$. If $g_o = 0$ the statement follows from Proposition 4, so we may assume $g_o \neq 0$. Furthermore h is odd according to Lemma 6. Now, applying (d) to the solution (f_e, g_o, h) of (1.5) we get that $h = h_o = cg_o$ for some $c \in \mathbb{C}\setminus\{0\}$, so $f_e(xy) - f_e(y^{-1}x) = cg_o(x)g_o(y)$. By the remarks after Proposition 4 it suffices to prove that the equation

$$F(xy) - F(y^{-1}x) = g_o(x)g_o(y) \tag{6.4}$$

has a solution $F : G \to \mathbb{C}$ which is abelian. To do so we need information about g_o. That can be squeezed out of the identity (6.1) applied to (f_e, g_o, h): Since $h = cg_o$ we find that

$$g_o(x)[g_o(yz) + g_o(zy^{-1})] = g_o(z)[g_o(xy) + g_o(y^{-1}x)],$$

which implies that

$$g_o(xy) + g_o(y^{-1}x) = 2g_o(x)l(y) \quad \text{for all } x, y \in G,$$

where $l(y) := [g_o(yz_0) + g_o(z_0y^{-1})]/(2g(z_0))$, $z_0 \in G$ being chosen such that $g_o(z_0) \neq 0$. Thus g_o is a solution of Wilson's functional equation (A.3) (g is abelian). Since g_o is odd and abelian there are according to [22, Proposition 11.5] only two possibilities for g_o: The first is that there exist a character $\chi : G \to \mathbb{C}^*$ and a constant $\alpha \in \mathbb{C}$ such that $g_o = \alpha(\chi - \check{\chi})/2$. Here $F := \alpha^2(\chi + \check{\chi})/4$ is an abelian solution of (6.4). The second is that there exist a character $\chi : G \to \mathbb{C}^*$ with $\check{\chi} = \chi$ and an additive function $a : G \to \mathbb{C}$ such that $g_o = \chi a$. Here $F := \chi a^2/4$ is an abelian solution of (6.4). $\square$

We discuss the possibilities $g(e) \neq 0$ and $g(e) = 0$ in the next two lemmas. The possibilities require different treatments.

Lemma 11. *Let $f,g,h : G\to\mathbb{C}$ be a solution of* (1.5), *such that $g\otimes h\neq 0$ and $g(e)\neq 0$. Then h is an odd solution of Wilson's functional equation corresponding to $g_e/g(e)$, i.e.,*

$$h(xy)+h(xy^{-1})=2h(x)g_e(y)/g(e),\quad \forall x,y\in G, \tag{6.5}$$

while $l:=g_e/g(e)$ satisfies d'Alembert's functional equation

$$l(xy)+l(xy^{-1})=2l(x)l(y),\quad \forall x,y\in G. \tag{6.6}$$

Proof. Putting $x=e$ in (1.5) we get that h is odd. To derive (6.5) we compute, using (1.5) and that h is odd:

$$\begin{aligned}
&g(e)[h(xy)+h(xy^{-1})]=f(xy)-f(y^{-1}x^{-1})+f(xy^{-1})-f(yx^{-1})\\
&\quad=-[f(yx^{-1})-f(xy)+f(y^{-1}x^{-1})-f(xy^{-1})]\\
&\quad=-g(y)h(x^{-1})-g(y^{-1})h(x^{-1})=g(y)h(x)+g(y^{-1})h(x)\\
&\quad=2h(x)\frac{g(y)+g(y^{-1})}{2}=2h(x)g_e(y).
\end{aligned}$$

Finally, l satisfies (6.6) according to [11, Proposition 1]. □

Corollary 12. *Let $f,g,h : G\to\mathbb{C}$ be a solution of* (1.5), *such that f is non-abelian. If $g(e)\neq 0$, then g is even. If $g(e)=0$, then g is odd.*

Proof. We claim that g is either even or odd. To prove the claim we shall by contraposition derive that f is abelian from the assumption that g is neither even nor odd.

If $g\otimes h=0$ then $f\in\mathcal{N}(G)$ and so f is abelian by Proposition 4. Thus we may from now on assume that $g\otimes h\neq 0$, so that Lemma 10 applies.

Now $g(e)\neq 0$, because g is not odd (by Lemma 10(d)). From Lemma 10(a) we infer that $h=2f_o/g(e)$, which shows that h is odd.

Lemma 10(c) tells us that $g_o=\lambda h$ for some $\lambda\in\mathbb{C}$. Actually $\lambda\neq 0$, because g would be even if $\lambda=0$.

Next we note from Lemma 10(a) that (f_e,g_o,h) is a solution of (1.5), so that (g_o,h) satisfies (6.1) by Lemma 10(b). Since h is odd and $\lambda\neq 0$ this means that $h(x)[h(yz)-h(yz^{-1})]=h(z)[h(xy)+h(y^{-1}x)]$, from which we infer that there is a function $\psi : G\to\mathbb{C}$ such that

$$h(yz)-h(yz^{-1})=2\psi(y)h(z)\quad\text{for all } y,z\in G.$$

When we to this add the identity

$$h(yz) + h(yz^{-1}) = 2h(y)\frac{g_e(z)}{g(e)} \quad \text{for all } y, z \in G,$$

from Lemma 11 we get the formula

$$h(yz) = 2h(y)\frac{g_e(z)}{g(e)} + 2\psi(y)h(z) \quad \text{for all } y, z \in G.$$

In here $\psi = g_e/g(e)$ due to Lemma 28, so the formula reduces to

$$h(yz) = 2h(y)\frac{g_e(z)}{g(e)} + 2\frac{g_e(y)}{g(e)}h(z) \quad \text{for all } y, z \in G,$$

which is the sine addition law. From Theorem 27 we know that h and g_e are abelian. Then $g_o = \lambda h$ is also abelian, and so is $g = g_o + g_e$. Now f is abelian according to Lemma 10(e). This proves the claim.

We finish the proof of Corollary 12 by noting that g is odd if and only if $g(e) = 0$ (Lemma 10(d)). □

Corollary 12 concludes that g is either even or odd, when f is non-abelian. This contrasts the abelian case, where g may be neither even nor odd, as shown by the following identity on $G = \mathbb{R}$:

$$\cos(x+y) + \sin(x+y) - (\cos(x-y) + \sin(x-y)) = 2(\cos x - \sin x)\sin y.$$

Lemma 13. *Let $f, g, h : G \to \mathbb{C}$ be a solution of* (1.5), *such that $g \otimes h \neq 0$ and $g(e) = 0$.*

Then there exists a function $l \in C(G)$ such that for all $x, y \in G$

$$g(xy) + g(y^{-1}x) = 2g(x)l(y), \quad \text{and} \tag{6.7}$$

$$h(xy) + h(yx) = 2h(x)l(y) + 2h(y)l(x), \tag{6.8}$$

i.e., (g, l) satisfies the variant (A.4) *of Wilson's functional equation, and (h, l) satisfies the symmetrized sine addition law* (A.6).

Proof. We assume first that $h_o = 0$, i.e., that h is even. Applying this and that g is odd (by Lemma 10(d)) in equation (6.2) we get

$$[g(xy) + g(y^{-1}x)]h(z) + [g(xz) + g(z^{-1}x)]h(y) = g(x)[h(yz) + h(zy)]. \tag{6.9}$$

Choosing here $x = x_0$ such that $g(x_0) \neq 0$ we get (6.8) with $l(y) := \frac{1}{2}[g(x_0 y) + g(y^{-1}x_0)]/g(x_0)$. Putting (6.8) back into (6.9) we obtain

$$[g(xy) + g(y^{-1}x) - 2g(x)l(y)]h(z) = -[g(xz) + g(z^{-1}x) - 2g(x)l(z)]h(y),$$

which is an equation of the form $\Phi_x(y)h(z) = -\Phi_x(z)h(y)$. Since $h \neq 0$ we get that $\Phi_x(y) = 0$ which is (6.7).

The remaining possibility is that $h_o \neq 0$. From Lemma 10(d) we know that $h_o = cg$ for some $c \in \mathbb{C}$. Here $c \neq 0$, since $h_o \neq 0$. We use $g = c^{-1}h_o$ in equation (6.2) and get for all $x, y, z \in G$ that

$$\begin{aligned}
&[h_o(xz) + h_o(z^{-1}x)]h(y) + [h_o(xy) + h_o(y^{-1}x)]h(z) \\
&\quad = [h_o(yz) + h_o(zy)][h(x) - h(x^{-1})] + h_o(x)[h(z^{-1}y^{-1}) + h(y^{-1}z^{-1})] \\
&\quad = [h_o(yz) + h_o(zy)]2h_o(x) + h_o(x)[h_e(yz) - h_o(yz) + h_e(zy) - h_o(zy)] \\
&\quad = [h(yz) + h(zy)]h_o(x). \qquad (6.10)
\end{aligned}$$

Choosing $x = x_0$ such that $h_o(x_0) \neq 0$ in (6.10) yields (6.8) with $l(y) := \frac{1}{2}[h_o(x_0 y) + h_o(y^{-1}x_0)]/h_o(x_0)$, $y \in G$. Also, putting $y = y_1$ such that $h(y_1) \neq 0$ in (6.10) we deduce that

$$h_o(xz) + h_o(z^{-1}x) = h_o(x)\alpha_1(z) - \alpha_2(x)h(z) \qquad (6.11)$$

for some functions $\alpha_1, \alpha_2 : G \to \mathbb{C}$.

Next, substituting (6.11) and (6.8) into (6.10), we get

$$h_o(x)[(\alpha_1(z) - 2l(z))h(y) + (\alpha_1(y) - 2l(y))h(z)] = 2\alpha_2(x)h(y)h(z),$$

from which we proceed to derive information about α_1 and α_2. Since $h \neq 0$, we get $\alpha_2 = dh_o$ for some $d \in \mathbb{C}$. And then, since h_o is assumed $\neq 0$, we get that

$$(\alpha_1(z) - 2l(z))h(y) + (\alpha_1(y) - 2l(y))h(z) = 2dh(y)h(z),$$

that we rewrite as

$$[\alpha_1(z) - 2l(z) - dh(z)]h(y) = -[\alpha_1(y) - 2l(y) - dh(y)]h(z).$$

This is an identity of the form $\phi(x)h(y) = -h(y)\psi(x)$. Such an identity holds only if either $\phi = 0$ or $h = 0$. But $h \neq 0$, so $\alpha_1 = 2l + dh$. Putting this and $\alpha_2 = dh_o$ back into (6.11), we arrive at $h_o(xz) + h_o(z^{-1}x) = 2h_o(x)l(z)$. This implies (6.7), since $g = c^{-1}h_o$ as noted above. $\square$

Theorem 14 synthesizes and extends the results above for non-trivial solutions (f, g, h) of equation (1.5). The character $\delta_g : G \to (\pm 1, \cdot)$ in (a) pervades the rest of the paper. It does not show on abelian groups, being identically 1 on such groups. Part (b) shows how g and h connect to one another through the joint δ_g-d'Alembert function l.

Theorem 14. *Let $f, g, h : G \to \mathbb{C}$ be a solution of* (1.5) *such that $g \otimes h \neq 0$.*

(a) *There exists exactly one function $\delta_g : G \to \mathbb{C}$ such that*

$$g(y^{-1}xy) = \delta_g(y)g(x) \quad \text{for all } x, y \in G. \tag{6.12}$$

δ_g is a continuous homomorphism of G into the multiplicative group $(\pm 1, \cdot)$. Furthermore $\delta_g = 1$ if and only if g is central. Finally $\delta_g = 1$ on the subgroup of G generated by the squares and the center of G.

(b) *There exists exactly one function $l : G \to \mathbb{C}$ such that (g, l) satisfies the variant* (A.4) *of Wilson's functional equation, i.e.,*

$$g(xy) + g(y^{-1}x) = 2g(x)l(y) \quad \text{for all } x, y \in G. \tag{6.13}$$

This l is a δ_g-d'Alembert function, i.e., $l \in C(G)$, $l(e) = 1$ and

$$l(xy) + \delta_g(y)l(xy^{-1}) = 2l(x)l(y) \quad \text{for all } x, y \in G. \tag{6.14}$$

The pair (h, l) satisfies the symmetrized sine addition law, i.e.,

$$h(xy) + h(yx) = 2h(x)l(y) + 2h(y)l(x) \quad \text{for all } x, y \in G. \tag{6.15}$$

(c) *$h(y^{-1}) = -h(y)/\delta_g(y)$ for all $y \in G$.*

(d) *Assume $\delta_g = 1$. Then h is odd, and either*

   (i) *f is abelian, or*

   (ii) *$g(e) \neq 0$, $g/g(e)$ is a d'Alembert function, h is an odd solution of Wilson's functional equation*

$$h(xy) + h(xy^{-1}) = 2h(x)g(y)/g(e), \quad x, y \in G,$$

   and $f = g(e)h/2 + \nu$ for some $\nu \in \mathcal{N}(G)$.

(e) *Assume $\delta_g \neq 1$. Then g is odd.*

Proof. We start the proof with (b), not (a), because we need equation (6.13) of (b) to derive (a).

(b) The uniqueness of l is immediate from equation (6.13). So are $l \in C(G)$ and $l(e) = 1$. We derive the existence of l and the formulas (6.13) and (6.15) case by case.

Suppose $g(e) \neq 0$ and g is not even. The function f is abelian by Corollary 12. The explicit formulas in (b) and (c) of Proposition 5 give (6.13) and (6.15) with $l = g_e/g(e) = (\chi + \check{\chi})/2$.

Suppose $g(e) \neq 0$ and g is even. We take $l = g_e/g(e)$ as in Lemma 11. Here $l = g/g(e)$, g being even. That h is an odd solution of (6.5) implies that (6.15) holds (see [22, Lemma 11.3(d)]). The identity (6.6) can be written in the form

$$l(xy) + l(y^{-1}x) = 2l(x)l(y), \quad \forall x, y \in G,$$

because any solution of d'Alembert's functional equation is central [22, Corollary 9.18(a)]. And in this form it is (6.13).

Suppose $g(e) = 0$. Here (b) follows from Lemma 13.

Under (b) it remains to derive (6.14). We do this below after having proved (a), because δ_g enters the formulation of (6.14).

(a) Except for the last one, the statements are general facts about solutions of the functional equation (6.13). See [11, Theorem 3]. Since the homomorphism δ_g assumes only the values ± 1, we have $\delta_g(x^2) = (\delta_g(x))^2 = (\pm 1)^2 = 1$ for all $x \in G$, so $\delta_g = 1$ on any square. It is an immediate consequence of the defining equation (6.12) that $\delta_g(y) = 1$ for all central elements y. It follows that $\delta_g = 1$ on the subgroup of G generated by the squares of G and the center of G.

(b) (remaining statement) It is known that (6.14) follows from (6.13) ([11, Theorem 3(b)] or [22, Lemma 11.16(d)]).

(c) We derive the formula from (1.5) as follows:

$$\begin{gathered} g(x)h(y) = f(xy) - f(y^{-1}x) = f(y(y^{-1}xy)) - f((y^{-1}xy)y^{-1}) \\ = -[f((y^{-1}xy)y^{-1}) - f(y(y^{-1}xy))] = -g(y^{-1}xy)h(y^{-1}) = -\delta_g(y)g(x)h(y^{-1}). \end{gathered}$$

(d) We get from (c) that h is odd. During the rest of the proof of (d) we may assume that f is non-abelian.

We prove by contradiction that $g(e) \neq 0$. Indeed, assume $g(e) = 0$. From Lemma 10(d) we get that $h_o = cg$ and so $h = h_o = cg$ for some $c \in \mathbb{C}$. Note that $c \neq 0$, because $h \neq 0$. Since g is central (by (a)), so is h. From the equation (6.15) we see that (h, l) satisfies the sine addition law. But then h is abelian (Theorem 27), and hence so is $g = c^{-1}h$. Finally, Lemma 10(e) tells us that f is abelian. This contradicts our assumption that f is non-abelian.

Since $g(e) \neq 0$ we obtain from Corollary 12 that g is even. We get the remaining statements from Lemma 11 and Lemma 7.

(e) Taking $x = e$ in (6.12) shows that $g(e) \neq 0 \Rightarrow \delta_g = 1$. Hence $g(e) = 0$, which means that g is odd (Lemma 10(d)). □

The following result about g extends Corollary 12 by adding a relation to the character δ_g from Theorem 14.

Corollary 15. *Let $f, g, h : G \to \mathbb{C}$ be a non-abelian solution of* (1.5) *such that $g \otimes h \neq 0$. Then*

$$\delta_g = 1 \iff g \text{ is even} \iff g(e) \neq 0,$$

$$\delta_g \neq 1 \iff g \text{ is odd} \iff g(e) = 0.$$

Proof. When $\delta_g = 1$, then $g/g(e)$ is by Theorem 14(d) a d'Alembert function. In particular g is even. When $\delta_g \neq 1$, then g is odd according to Theorem 14(e). The corollary follows from Corollary 12. □

7. General groups

This section presents formulas for the solutions of (1.5) on general groups. The function l of Theorem 14 plays an important role, because it is a pre-d'Alembert function, so that we can use Davison's results in [9]. When l is non-abelian we get a complete picture of the set of solutions. When l is abelian we obtain some information about it. The discussion enables us to describe all the solutions on compact groups (Section 8), and to express the solutions explicitly or in terms of solutions of Wilson's functional equation on groups that are generated by their squares (Section 9).

Proposition 16. *Let $f, g, h : G \to \mathbb{C}$ be a solution of* (1.5) *such that $g \otimes h \neq 0$ and such that the function l from Theorem 14 is non-abelian. In that case:*

$$f(x) = \operatorname{tr}(\pi(x)AW) + \nu(x),$$

$$g(x) = \operatorname{tr}(\pi(x)W), \;\; h(x) = \operatorname{tr}(\pi(x)A) \quad \text{for all } x \in G, \tag{7.1}$$

where $\nu \in \mathcal{N}(G)$, π is an irreducible representation of G on $\mathbb{C}^2$, $A \in sl(2, \mathbb{C}) \setminus \{0\}$ and W is an invertible 2×2 matrix such that

$$W\pi(x) = \operatorname{adj}(\pi(x^{-1}))W = \det(\pi(x))^{-1}\pi(x)W \quad \text{for all } x \in G. \tag{7.2}$$

Furthermore

(a) $\det \pi(x) = \delta_g(x)$ *for all $x \in G$. In particular* $\det \pi(x) \in \{-1, 1\}$ *for all $x \in G$.*

(b) *If* $\det \pi(x) = 1$ *for all* $x \in G$, *then* $W = cI$ *for some* $c \in \mathbb{C} \setminus \{0\}$. *Also,* g *is central and even and* $g(e) \neq 0$, *while* h *is odd.*

(c) *If* $\det \pi(x_0) = -1$ *for some* $x_0 \in G$, *then* g *is odd, and* $W \in sl(2, \mathbb{C})$. *Finally* $\dim\{W \in M(2, \mathbb{C}) \mid (7.2)$ *holds*$\} = 1$.

Conversely, any triple (f, g, h) *of the form* (7.1) *is a solution of* (1.5).

Proof. Our point of departure is the functional equation (6.13) which is treated in [11]. Compressing the two cases of [11, Corollary 6] into one we see that there exists an irreducible representation π of G on $\mathbb{C}^2$ such that

$$l(x) = \frac{1}{2} \operatorname{tr} \pi(x) \text{ and } g(x) = \operatorname{tr}(\pi(x)W) \quad \text{for } x \in G,$$

where W is a complex 2×2 matrix satisfying $\pi(x)W = \det(\pi(x))W\pi(x)$ for all $x \in G$, i.e., (7.2).

l is a pre-d'Alembert function by [22, Example 8.4], which by assumption is non-abelian. Since (h, l) satisfies (6.15), we get from [22, Theorem 8.24] that there exists an irreducible, 2-dimensional representation ρ of G such that

$$l(x) = \frac{1}{2} \operatorname{tr} \rho(x) \text{ and } h(x) = \operatorname{tr}(\rho(x)B) \quad \text{for } x \in G,$$

where $\operatorname{tr} B = 0$. Since matrix-elements of inequivalent representations form direct sum (see [7, Proposition 2 of Chap. VIII, §13, no. 3]) we get from $l(x) = \frac{1}{2} \operatorname{tr} \pi(x) = \frac{1}{2} \operatorname{tr} \rho(x)$ that π and ρ are equivalent. Hence $h(x) = \operatorname{tr}(\pi(x)A)$ for some $A \in sl(2, \mathbb{C})$.

The function $x \mapsto \operatorname{tr}(\pi(x)AW)$ is by Lemma 8 a solution of (1.5), corresponding to the pair (g, h), so $f(x) = \operatorname{tr}(\pi(x)AW) + \nu(x)$, where $\nu \in \mathcal{N}(G)$.

$W \neq 0$, because $g \neq 0$. By (7.2) W intertwines two irreducible representations, so by Schur's lemma W is an isomorphism.

(a) Combining Lemma 8 and the definition of δ_g (Theorem 14(a)) we get that $\det \pi(x) = \delta_g(x)$ for all $x \in G$. Theorem 14(a) also says that $\delta_g(x) \in \{-1, 1\}$ for all $x \in G$.

(b) When $\det \pi(x) = 1$ for all $x \in G$, then (7.2) becomes $W\pi(x) = \pi(x)W$, which says that W is an intertwining operator for the irreducible representation π. By Schur's lemma $W = cI$ for some $c \in \mathbb{C}$. Here $c \neq 0$, because W is invertible.

We note from Lemma 9 that f is not abelian. The rest of (b) follows from Theorem 14(d), because any d'Alembert function is even and central ([22, Corollary 9.18]).

(c) That g is odd is stated in Theorem 14(d). Taking trace on both sides of (7.2) we get that $\operatorname{tr} W = \det(\pi(x_0)) \operatorname{tr} W = -\operatorname{tr} W$, which implies that $\operatorname{tr} W = 0$. The statement about the dimension comes from Schur's lemma.

The formulas above define solutions of (1.5) by Lemma 8. □

Due to the statement about the dimension in Proposition 16(c) it suffices for given π to find or guess just one solution $W \neq 0$ of (7.2) to have them all.

Remark 17. We get the same set of solutions (f, g, h) in (7.1), if the representation π is replaced by an equivalent representation $S^{-1}\pi(\cdot)S$. In other words, it suffices to pick one and only one representative π of the equivalence class $[\pi]$, and we may choose any one we wish, because all choices produce the same solution set.

Proposition 16 treated the case of l non-abelian. Proposition 18 complements it by discussing the possibilities for abelian l. We exclude the abelian solutions from the proposition, as they have been written down earlier (Proposition 5).

Proposition 18. *Let $f, g, h : G \to \mathbb{C}$ be a non-abelian solution of* (1.5) *such that $g \otimes h \neq 0$ and such that the function l from Theorem 14 is abelian. In that case we have in addition to the formulas* (6.13) *and* (6.15) *that*

$$l = \frac{\chi + \delta_g \check{\chi}}{2} \tag{7.3}$$

for some character χ of G, and the following:

(a) *Assume $\delta_g(x) = 1$ for all $x \in G$. Then $g = c(\chi + \check{\chi})/2$, where $c \in \mathbb{C} \setminus \{0\}$. Furthermore, h is an odd, non-central solution of Wilson's functional equation*

$$h(xy) + h(xy^{-1}) = 2h(x)\frac{g(y)}{g(e)}, \quad x, y \in G, \tag{7.4}$$

and $f = g(e)h/2 + \nu$ for some $\nu \in \mathcal{N}(G)$.

(b) *Assume there is an $x_0 \in G$ such that $\delta_g(x_0) = -1$. Then $\check{\chi} = \chi$, and g is odd. Furthermore there exists an additive function $a \in C(\ker \delta_g)$ such that $a(x_0^{-1} x x_0) = -a(x)$ for all $x \in \ker \delta_g$ and*

$$g(x) = \begin{cases} \chi(x)a(x) & \text{for } x \in \ker \delta_g \\ 0 & \text{for } x \in G \setminus \ker \delta_g \end{cases}$$

Proof. Since l is abelian, we get from Theorem 14(b) and [22, Proposition 9.31] that there exists a character χ of G such that (7.3) holds.

(a) From Theorem 14(d) we note that $g/g(e)$ is a d'Alembert function. In particular g is even. Taking $x = e$ in (6.13) we see that $g = g(e)l$, and so $g = c(\chi + \check{\chi})/2$, where $c = g(e) \in \mathbb{C} \setminus \{0\}$. The rest of (a) is contained in

Theorem 14(d), except for the claim that h is non-central. We prove the claim by contradiction. If h is central, then (6.15) shows that h is a solution of the sine addition law. Hence it is abelian by Theorem 27. Now f would be abelian according to Lemma 10(e), contradicting our assumption.

(b) is [11, Corollary 9 and Proposition 10]. □

Although l is written down explicitly in (7.3), Proposition 18 only describes the solutions of (1.5) in terms of the solutions of Wilson's functional equation (7.4) = (A.3), and (implicitly in (b) to find h) of the symmetrized sine addition law (6.15) = (A.6). Neither (A.3) nor (A.6) have been completely resolved on general groups.

8. Compact groups

In this section we apply the above results to compact groups. Theorem 20 gives a complete description of the form of the solutions of (1.5) on compact groups. We use them to find the solutions on two very different examples, the symmetric group S_3 and the special unitary group $SU(2)$.

Proposition 19 and Lemma 9 are tools to decide whether solutions are abelian.

Proposition 19. *Let G be a compact group. Let the triple $f, g, h \in C(G)$ be a solution of* (1.5) *such that $g \otimes h \neq 0$. Then g is abelian $\iff$ h is abelian $\iff$ f is abelian.*

Proof. Assuming g abelian, we will prove that so is h. We multiply (1.5) by $\overline{g(x)}$ and integrate the result with respect to a Haar measure dx on G. After a change of variables we obtain

$$\int_G f(x)\overline{g(xy^{-1})}\,dx - \int_G f(x)\overline{g(yx)}\,dx = \left(\int_G |g(x)|^2 dx\right) h(y).$$

Now $\int_G |g(x)|^2 dx \in \,]0,\infty[$, because g is continuous and $\neq 0$, so it suffices to prove that each term on the left hand side is an abelian function of y. But that is immediate since g is abelian. That h abelian implies g abelian is proved along the same lines. Lemma 10(e) connects g and h to f. □

Theorem 20. *The solutions $f, g, h \in C(G)$ of* (1.5) *on the compact group G are the following, where $c \in \mathbb{C} \setminus \{0\}$, $\nu \in \mathcal{N}(G)$ and π is an irreducible representation of G on $\mathbb{C}^2$.*

(a) *The trivial solutions.*

(b) *The abelian solutions for which* $g \otimes h \neq 0$. *They are the triples of the form*

$$f = \frac{c}{2}\Big(c_1 \frac{\chi - \check{\chi}}{2} + c_2 \frac{\chi + \check{\chi}}{2}\Big) + \nu,$$
$$g = c_1 \frac{\chi + \check{\chi}}{2} + c_2 \frac{\chi - \check{\chi}}{2}, \quad h = c\frac{\chi - \check{\chi}}{2},$$

where $\chi : G \to \mathbb{C}^*$ *is a character such that* $\check{\chi} \neq \chi$, *and* $(c_1, c_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$.

(c) *The triples* (f, g, h) *of the form*

$$f(x) = c\,\mathrm{tr}(\pi(x)A) + \nu(x),$$
$$g(x) = c\,\mathrm{tr}(\pi(x)) \text{ and } h(x) = \mathrm{tr}(\pi(x)A) \quad \text{for } x \in G,$$

where $\pi(G) \subseteq SL(2, \mathbb{C})$, *and* $A \in sl(2, \mathbb{C}) \setminus \{0\}$. *Here* g *is even and central, and* h *is odd.*

(d) *The triples* (f, g, h) *of the form*

$$f(x) = \mathrm{tr}(\pi(x)AW) + \nu(x),$$
$$g(x) = \mathrm{tr}(\pi(x)W) \text{ and } h(x) = \mathrm{tr}(\pi(x)A) \quad \text{for } x \in G,$$

where $\det \pi(G) = \{\pm 1\}$, $A \in sl(2, \mathbb{C}) \setminus \{0\}$, *while* $W \in sl(2, \mathbb{C}) \setminus \{0\}$ *satisfies*

$$\pi(x)W = W \det(\pi(x))\pi(x), \quad x \in G. \tag{8.1}$$

The vector space of matrices W *satisfying* (8.1) *has dimension* 1. *Furthermore* g *is odd.*

In both (c) *and* (d): $\delta_g(x) = \det \pi(x)$ *for all* $x \in G$, *and* π *may be assumed unitary.*

Proof. It is straightforward (use Proposition 5) to check that the formulas in (a) and (b) describe solutions, and it follows from Lemma 8 that the formulas in (c) and (d) do it, too. It is thus left to show that any solution (f, g, h) of (1.5) falls into at least one af the cases (a)-(d).

If $g \otimes h = 0$ we are in case (a), so we may from now on assume that $g \otimes h \neq 0$. If g is abelian, then f, g and h are all three abelian by Proposition 19, so we are in case (b). Here the solutions are described in Proposition 5. However, the formulas of the proposition reduce to those of (b), because continuous, additive functions on a compact group vanish.

If g is not abelian, we look at the function l of Theorem 14(b). Multiplying (6.13) by $\overline{l(y)}$ and integrating over G with respect to the normalized Haar measure dy we find after changes of variables that

$$2g(x)\int_G |l(y)|^2 dy = \int_G g(y)\overline{l(x^{-1}y)}dy + \int_G g(y^{-1})\overline{l(xy)}dy.$$

We see from this that l abelian $\Rightarrow$ g abelian. But g is not abelian, so we conclude that l is non-abelian. Now Proposition 16 gives that the solution (f, g, h) falls into case (c) or (d).

That the representations in (c) and (d) may be chosen unitary is a general fact from the theory of group representations: Any representation of a compact group on a Hilbert space (here $\mathbb{C}^2$) is similar to a unitary representation on the Hilbert space.

The statement about the dimension is a consequence of Schur's lemma. □

Remark 17 is pertinent also here: Equivalent representations give the same set of solutions in Theorem 20(c) and (d).

9. Groups generated by their squares and their center

In this section we write down formulas for the solutions (f, g, h) of (1.5) on groups G that are generated by their squares and their center. Replacing this algebraic condition by the topological one of G being connected would also work, because each of the conditions ensures the crucial property that $\delta_g = 1$ (solutions are continuous by convention).

Connected Lie groups (see [22, Lemma A.11]) and simple groups are generated by their squares. So is $(\mathbb{R}^n, +)$. Thus the condition holds on large classes of groups. The symmetric group S_3 is a counter-example, because it is not generated by its squares and its center. Actually, S_3 harbours a solution of (1.5) with $\delta_g \neq 1$ (Example 25).

Theorem 21. *Let G be a group which is generated by its squares together with its center. The solutions $f, g, h : G \to \mathbb{C}$ of (1.5) are the following, where $\nu \in \mathcal{N}(G)$ and $c \in \mathbb{C} \setminus \{0\}$.*

(a) *The trivial and the abelian solutions described by Propositions 4 and 5.*

(b) *With π an irreducible representation of G on $\mathbb{C}^2$ such that $\pi(G) \subseteq SL(2, \mathbb{C})$, $A \in sl(2, \mathbb{C}) \setminus \{0\}$, and $x \in G$:*

$$f(x) = ch(x) + \nu, \quad g(x) = c\,\mathrm{tr}\,\pi(x), \quad h(x) = \mathrm{tr}(\pi(x)A).$$

Replacing here π by an equivalent representation leaves the set of solutions unchanged.

(c) *With $\chi : G \to \mathbb{C}^*$ a character:*

$$f = \frac{c}{2}h + \nu, \quad g = c\frac{\chi + \check{\chi}}{2}$$

while h is an odd, non-central solution of Wilson's functional equation corresponding to $g/g(e)$, i.e.,

$$h(xy) + h(xy^{-1}) = 2h(x)g(y)/c, \quad x, y \in G.$$

In (b) *and* (c)*: g is even and central, and h is odd, but f is not central.*

Proof. We note first that the formulas of the theorem define solutions of (1.5). Case (a) is trivial, (b) is Lemma 8, and (c) follows from Lemma 7.

Thus it is left to show that any solution (f, g, h) which does not fall into the category (a), lies in (b) or (c). In particular $g \otimes h \neq 0$. We use the notation of Theorem 14(b). Note that $\delta_g = 1$ by our assumption on G (Theorem 14(a)).

If l is non-abelian then only case (b) of Proposition 16 applies. It gives the formulas of (b). If l is abelian then only case (a) of Proposition 18 applies. It gives the formulas of (c).

Consider the cases (b) and (c). That f is not central in (b) can be inferred from Lemma 9, while in (c) it follows from h being non-central. In both cases we see by inspection that g is central. That g is even follows from Corollary 15. Finally h is odd by Lemma 6. □

Theorem 21 simplifies a little, when G is generated by its squares alone, because in that case $\mathcal{N}(G) = \mathbb{C}$ (by Proposition 4).

10. Sufficient conditions for solutions to be abelian

The solutions (f, g, h) of (1.5) for which f is abelian, are known (Proposition 5), so it is of interest to find sufficient conditions ensuring that f is abelian. That we do in the present section.

Proposition 22. *Let (f, g, h) be a solution of* (1.5) *on G such that h is central. Then f is abelian,*

(a) *if g is also central, or*

(b) *if G is generated by its squares and its center.*

PROOF. By Proposition 4 we may assume that $g \otimes h \neq 0$.

In the notation of Theorem 14(b) there exists a function $l \neq 0$, such that (h, l) satisfies (6.15). Since h is central, the pair (h, l) satisfies the sine addition law (A.5). Both components of any solution (h, l) with $h \neq 0$ of the sine addition law are abelian (see Theorem 27), so h and l are abelian.

(a) According to Theorem 14(b) the pair (g, l) satisfies (6.13). Here g is assumed central, so (g, l) is a solution of Wilson's functional equation (A.3), such that g is central and l abelian. Now g is abelian by the formulas of [22, Proposition 11.5]. And then f is abelian according to Lemma 10(e).

(b) Due to (a) it suffices to show that g is central. But that is contained in Theorem 14(a). □

Proposition 23. *Let $f, g, h : G \to \mathbb{C}$ be a solution of* (1.5) *such that f is central. If $g(e) \neq 0$, or if G is generated by its squares and its center, then f is abelian.*

PROOF. By Proposition 4 we may assume that $g \otimes h \neq 0$.

Let us first assume that $g(e) \neq 0$. If g is not even, we get from Corollary 12 that f is abelian, so we may assume that g is even. The formula $2f_o = g(e)h$ from Lemma 10(a) combined with the assumption that f is central tells us that h is central. Now (6.15) reduces to (h, l) being a solution of the sine addition law (A.5). Then h is abelian (see Theorem 27). From Lemma 11 we read that

$$h(xy) + h(xy^{-1}) = 2h(x)\frac{g_e(y)}{g(e)} = 2h(x)\frac{g(y)}{g(e)}, \quad x, y \in G,$$

which implies g is also abelian. Now f is abelian by Lemma 10(e).

Let us next assume that G is generated by its squares and its center. The case of $g(e) \neq 0$ has been treated, so we may assume that $g(e) = 0$. Due to our assumption on G we get $\delta_g = 1$ in Theorem 14(a), so that g is central. Then h is odd according to Lemma 6, and so we get via Lemma 10(d) that $h = h_o = cg$ for some $c \in \mathbb{C}$. Hence h is central, g being so. We are through by Proposition 22(a). □

11. Example: The symmetric group S_3

In this section we solve (1.5) on the symmetric group on three objects

$$G = S_3 := \{e, a, a^2, b, ba, ba^2 \mid ab = ba^2,\ a^2b = ba,\ a^3 = b^2 = e\}.$$

S_3 is intricate, partly because it is not abelian, partly because it is not generated by its squares. S_3 is compact, because it is finite, so according to Theorem 20 the solutions of (1.5) can be expressed in terms of matrix-coefficients of irreducible representations of S_3 on $\mathbb{C}^2$ and of characters of S_3.

The (equivalence classes of the) irreducible representations of S_3 are known from the theory of finite groups: There are one 2-dimensional representation and two characters. Let us describe the representation. Geometrically S_3 is the group of symmetries of (= the rigid motions leaving invariant) the equilateral triangle in the complex plane with vertices $1, \omega, \omega^2$, where $\omega := \exp 2\pi i/3$. The symmetries are formed by compositions of a = the rotation around the origin by an angle of $120^0 = 2\pi/3$ in the positive direction, and b = the reflection in the x-axis. Each symmetry $x \in S_3$ is a linear operator of $\mathbb{R}^2$. We get a homomorphism π of S_3 into $GL(2, \mathbb{R})$ by associating to $x \in S_3$ its matrix, denoted $\pi(x)$. For the two generators a and b we find

$$\pi(a) = \begin{pmatrix} -\dfrac{1}{2} & -\dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \end{pmatrix} \quad \text{and} \quad \pi(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrices $\pi(x)$, $x \in S_3$, are orthogonal, being rigid motions, so $\pi(S_3) \subseteq O(2) \subseteq U(2)$. The 2-dimensional, irreducible representation is π. One of the characters is the trivial one $\pi_1(x) = 1$ for all $x \in S_3$. The other character is the sign of the permutation, sgn, given by $\operatorname{sgn}(x) = \det \pi(x)$ for $x \in S_3$, or equivalently by $\operatorname{sgn}(a) = 1$, $\operatorname{sgn}(b) = -1$ on the generators a and b of S_3.

It follows from Proposition 4 and $\langle G^2 \rangle = \{e, a, a^2\}$ that $\mathcal{N}(G)$ consists of the functions $\nu : S_3 \to \mathbb{C}$ such that

$$\nu(e) = \nu(a) = \nu(a^2) = c_0 \quad \text{and} \quad \nu(b) = \nu(ba) = \nu(ba^2) = c_1, \tag{11.1}$$

where $c_0, c_1 \in \mathbb{C}$.

Proposition 24. *On S_3 the non-trivial solutions (f, g, h) of (1.5) are*

$$f(x) = \alpha \operatorname{tr}(\pi(x)AJ) + \nu,$$
$$g(x) = \alpha \operatorname{tr}(\pi(x)J) \text{ and } h(x) = \operatorname{tr}(\pi(x)A) \quad \text{for } x \in S_3,$$

where $\alpha \in \mathbb{C} \setminus \{0\}$, $A \in sl(2, \mathbb{C}) \setminus \{0\}, \nu \in \mathcal{N}(G)$, and $J := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. It might be added that $\delta_g = \operatorname{sgn}$.

PROOF. According to Theorem 20 there are four kinds of solutions.

(a) The trivial solutions.

(b) The abelian solutions are described in terms of characters χ such that $\check{\chi} \neq \chi$. But 1 and sgn have $\check{\chi} = \chi$, so this case is void.

(c) This case is also void: Up to equivalence π is the only irreducible, 2-dimensional representation of S_3. Theorem 20(c) requires $\det \pi(x) = 1$ for all $x \in G$, but $\det \pi(b) = -1$.

(d) It it easy to check that $W = \alpha J$ satisfies (8.1) for the two generators $x = a$ and $x = b$ of S_3 and consequently for all $x \in S_3$. We may now refer to the formulas of Theorem 20(d). There we also find that $\delta_g(x) = \det \pi(x)$, which implies that $\delta_g = \mathrm{sgn}$. □

Example 25. S_3 harbours a non-trivial solution (f, g, h) of (1.5) such that $\delta_g \neq 1$ and h is even. Take $\alpha = 1$ and $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in Proposition 24. That $\delta_g \neq 1$ is contained in the proposition.

Theorem 14(c) shows that h is even if $h = 0$ on $\ker \delta_g$, which means if $h(e) = h(a) = h(a^2) = 0$. Putting $y = e$ in (1.5) we get $h(e) = 0$. Since $h(a^2) = 2h(a)l(a)$ by (6.15), it suffices to prove that $h(a) = 0$. And that is easily verified from the explicit formulas for h and π.

12. The special unitary group $SU(2)$

In this section we solve (1.5) on the special unitary group

$$G = SU(2) := \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

$SU(2)$ is a connected Lie group, so it is generated by its squares. It follows that $\mathcal{N}(SU(2)) = \mathbb{C}$. Up to equivalence $SU(2)$ has exactly one irreducible representation in each dimension $n = 1, 2 \ldots$. For $n = 1$ it is the character 1, and for $n = 2$ it is the identity representation

$$\rho(x) = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \quad \text{for } x = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2).$$

This means that (b) and (d) of Theorem 20 are void for $SU(2)$, and so we find that the solutions (f, g, h) of (1.5) are

(1) The trivial solutions. Here f is a constant function.

(2) The non-trivial solutions. They are parametrised by $c \in \mathbb{C}\backslash\{0\}$, $A \in sl(2,\mathbb{C})\backslash\{0\}$ and $c_1 \in \mathbb{C}$ as follows:

$$f(x) = c\,\mathrm{tr}(xA) + c_1,\ g(x) = c\,\mathrm{tr}\,x \text{ and } h(x) = \mathrm{tr}(xA), \quad x \in SU(2).$$

13. The special linear group $SL(2,\mathbb{R})$

In this section we solve (1.5) on $SL(2,\mathbb{R}) := \{A \in M(2,\mathbb{R}) \mid \det A = 1\}$. It is a connected Lie group, so it is generated by its squares. Hence $\mathcal{N}(G) = \mathbb{C}$.

Since furthermore $SL(2,\mathbb{R})$ is semisimple, we read from for example [25, Corollary 3.18.10] that $[G,G] = G$. It follows that 1 is the only character of G, and 0 the only additive function on G.

It is known that there is up to equivalence only one irreducible representation π of G on $\mathbb{C}^2$, viz. the identity representation

$$\pi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

To find the solutions of (1.5) on $SL(2,\mathbb{R})$ we apply Theorem 21. Point (c) of the theorem does not contribute, because Wilson's functional equation becomes Jensen's in (c), and the only odd solution of Jensen's functional equation on $SL(2,\mathbb{R})$ is 0 (see [22, Example 12.24]). Also all the abelian solutions are trivial: Indeed $h = 0$ in Proposition 5 for any abelian, non-trivial solution (f,g,h) of (1.5). Thus Theorem 21 tells us that the solutions $f,g,h \in C(G)$ of (1.5) are the following.

(1) The trivial solutions. Here f is a constant.

(2) The non-trivial solutions. They are parametrised by $c \in \mathbb{C}\backslash\{0\}$, $A \in sl(2,\mathbb{C})\backslash\{0\}$ and $c_1 \in \mathbb{C}$ as follows:

$$f(x) = c\,\mathrm{tr}(xA) + c_1,\ g(x) = c\,\mathrm{tr}(x) \text{ and } h(x) = \mathrm{tr}(xA), \quad x \in G.$$

14. A particular instance

In this short section we find by help of the results in Sections 3, 4 and 6 on any group G the solutions $f,h : G \to \mathbb{C}$ of the functional equation

$$f(xy) - f(y^{-1}x) = 2f(x)h(y), \quad x,y \in G, \tag{14.1}$$

mentioned in the introduction. It is the particular case of (1.5) for which $g = 2f$. Székelyhidi solved (14.1) on abelian groups in [23, Theorem 12.10]. Proposition 26, which extends his result to any group, states that the solution formulas are the same as on abelian groups.

Proposition 26. *Let G be a group. The solutions $f, h : G \to \mathbb{C}$ of (14.1) are the following:*

(a) *$f = 0$ and h is arbitrary in $C(G)$.*

(b) *$h = 0$ and $f \in C(G)$ is a function on $G/\langle G^2\rangle$.*

(c) *$f = c\chi$ and $h = (\chi - \check{\chi})/2$, where χ is a character of G and $c \in \mathbb{C}$.*

Proof. We leave out the simple verifications that the formulas of (a), (b) and (c) define solutions. It is thus left to prove that any solution $f, h : G \to \mathbb{C}$ of (14.1) falls into one of these three categories.

If $f = 0$, then (a) is the case. If $h = 0$, then (b) is the case according to Proposition 4. From now on we assume that $f \neq 0$ and $h \neq 0$. We view (14.1) as (1.5) with $g = 2f$. By Lemma 10(d) we have that $g = 2f$ is neither even nor odd, because $f \neq 0$. Then f is abelian by Corollary 12. Finally, computations based on the formulas of Proposition 5 give (c). $\square$

Appendix A. Functional equations that we refer to

Our investigations show that the components g and h of any solution (f, g, h) of (1.5) satisfy functional equations like *d'Alembert's functional equation*

$$g(xy) + g(xy^{-1}) = 2g(x)g(y), \quad x, y \in G, \tag{A.1}$$

which can (by [22, Theorem 7.1(b)]) equivalently be expressed as

$$g(xy) + g(y^{-1}x) = 2g(x)g(y), \quad x, y \in G. \tag{A.2}$$

A *d'Alembert function* is a solution g of (A.1) such that $g(e) = 1$.

We encounter also *Wilson's functional equation*

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G, \tag{A.3}$$

the following *variant of Wilson's functional equation*

$$f(xy) + f(y^{-1}x) = 2f(x)g(y), \quad x, y \in G, \tag{A.4}$$

the *sine addition law*

$$f(xy) = f(x)g(y) + f(y)g(x), \quad x, y \in G, \tag{A.5}$$

and the *symmetrized sine addition law*

$$f(xy) + f(yx) = 2f(x)g(y) + 2f(y)g(x), \quad x, y \in G, \tag{A.6}$$

where $f, g : G \to \mathbb{C}$ denote the unknown functions. In examples we meet *Jensen's functional equation* which is Wilson's functional equation with $g = 1$, i.e.,

$$f(xy) + f(xy^{-1}) = 2f(x), \quad x, y \in G. \tag{A.7}$$

A function $g : G \to \mathbb{C}$ satisfying the *pre-d'Alembert functional equation*

$$\begin{aligned} g(xyz) + g(xzy) = {} & 2g(x)g(yz) + 2g(y)g(xz) + 2g(z)g(xy) \\ & - 4g(x)g(y)g(z), \quad x, y, z \in G, \end{aligned} \tag{A.8}$$

is said to be a *pre-d'Alembert function* if $g(e) = 1$.

The above basic functional equations have been studied extensively. The present paper is based on existing knowledge of their solutions. For example we use Theorem 27 several times. It can be found as [22, Theorem 4.1(e)].

Theorem 27. *If the pair $f, g : G \to \mathbb{C}$ satisfies the* sine addition law (A.5) *and $f \neq 0$, then f and g are abelian functions.*

Lemma 28, which concerns an extension of (A.5), is new.

Lemma 28. *Let $h, g_1, g_2 : G \to \mathbb{C}$. If h is odd and $\neq 0$ and*

$$h(xy) = h(x)g_1(y) + h(y)g_2(x) \quad \text{for all } x, y \in G,$$

then $g_2 = \breve{g}_1$.

Proof. Comparing the left and right hand sides of the computation

$$\begin{gathered} h(x)g_1(y) + h(y)g_2(x) = h(xy) = -h(y^{-1}x^{-1}) \\ = -[h(y^{-1})g_1(x^{-1}) + h(x^{-1})g_2(y^{-1})] = h(y)g_1(x^{-1}) + h(x)g_2(y^{-1}) \end{gathered}$$

we infer that

$$h(x)[g_1(y) - g_2(y^{-1})] = h(y)[g_1(x^{-1}) - g_2(x)], \tag{A.9}$$

from which we see that $\breve{g}_1 - g_2 = ch$ for some $c \in \mathbb{C}$. Plugging that back into (A.9) we get that $h(x)[ch(y^{-1})] = h(y)[ch(x)]$ or equivalently $-ch(x)h(y) = ch(y)h(x)$. Thus $c = 0$, and so $\breve{g}_1 - g_2 = ch = 0$. $\square$

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BRUCE EBANKS
DEPARTMENT OF MATHEMATICS & STATISTICS
MISSISSIPPI STATE UNIVERSITY
P.O. DRAWER MA
MISSISSIPPI STATE, MS 39762
USA

E-mail: ebanks@math.msstate.edu

HENRIK STETKR
DEPARTMENT OF MATHEMATICS
AARHUS UNIVERSITY
NY MUNKEGADE 118
DK-8000 AARHUS C
DENMARK

E-mail: stetkaer@imf.au.dk

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