Publ. Math. Debrecen
45 / 3-4 (1994), 371–377

On the form of solutions of some iterative functional inequality

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Abstract. The paper contains two theorems of the representation type for solutions of functional, iterative inequality of the second order with functional coefficients.

I. We consider the following functional inequality of the second order:

$$\psi_1(f^2(x)) - (\lambda(f(x)) + \mu(x))\psi_1(f(x)) + \lambda(x)\mu(x)\psi_1(x) \le 0, \tag{1}$$

where ψ_1 is an unknown function, μ, λ and f are given functions, f^2 denotes the second iterate of the function f. We substitute: $\psi_2(x) = \psi_1(f(x)) - \lambda(x)\psi_1(x)$. The form of functional coefficients of the inequality (1) implies that one can study instead, the equivalent system:

$$\psi_1(f(x)) - \lambda(x)\psi_1(x) = \psi_2(x), \tag{2}$$

$$\psi_2(f(x)) - \mu(x)\psi_2(x) \le 0. \tag{3}$$

Thus, the study of the second order inequality reduces to the investigation of the first order equation and first order inequality. The inequality (3) has been considered in detail in [1] and the results obtained therein will be used in this paper. Apart from the inequality (1), we shall also consider the corresponding functional equation of the second order:

$$\phi_1(f^2(x)) - (\lambda(f(x)) + \mu(x))\phi_1(f(x)) + \lambda(x)\mu(x)\phi_1(x) = 0, \tag{4}$$

The equation (4) is equivalent to the system of two equations of the first order:

$$\phi_1(f(x)) - \lambda(x)\phi_1(x) = \phi_2(x), \tag{5}$$

$$\phi_2(f(x)) - \mu(x)\phi_2(x) = 0. \tag{6}$$

II. Let us make the following assumption on functions λ, μ and f:

(H1). $f : I \to I$, $I = [0, \alpha)$, $\alpha > 0, f$ *is continuous and strictly increasing function and* $0 < f(x) < x$, $x \in I \backslash \{0\}$.

(H2). $\mu : I \to (0, \infty)$ $\mu(0) < 1$, $\lambda : I \to \mathbf{C}$, $\mu(0) < |\lambda(0)|$, $\lambda(x) \neq 0$

Here $\mathbf{C}$ denotes the set of complex numbers.

Since complex values for functions ψ_1 and λ are allowed, we consider only functions, for which the values of expressions on the left hand side of the inequality (1) and (3) are real. Then the real part of the function $\psi_2(x) = \psi_1(f(x)) - \lambda(x)\psi_1(x)$ satisfies the inequality (3) and its imaginary part is a solution of the equation (6). The inequality (3) is thoroughly investigated in paper [1]. We recall now some results obtained there for this inequality. They will be applied many times in our considerations.

We introduce the following notation:

$$G_{1j}(x) = \prod_{k=0}^{j-1} \lambda(f^k(x)), \qquad G_{2j}(x) = \prod_{k=0}^{j-1} \mu(f^k(x)),$$

$$G_j(x) = \frac{G_{1j}(x)}{G_{2j}(x)}, \quad j \in N, \ x \in I.$$

$\mathbf{F} = \{\phi : I \to \mathbf{R}; \phi$ *is continuous and satisfies* (6) *in* I, *and* $\phi(x) > 0$ *in* $I \backslash \{0\}\}$

Definition. Function $\eta : I \to \mathbf{R}$ is called $\{f\}$-decreasing in I, if $\eta(f(x)) \leq \eta(x)$ for $x \in I$.

Lemma 1. *If* $\phi : I \to \mathbf{R}^+$ *is a solution of the equation* (6), η *is a* $\{f\}$*-decreasing function in* I, *then the function:*

$$\psi_2(x) = \eta(x)\phi(x),$$

satisfies the inequality (3).

Lemma 2. *If* $\tilde{\phi} \in \mathbf{F}$, *then the set of functions:*

$$\mathbf{F}(\tilde{\phi}) := \{\phi \in \mathbf{F} : \textit{such that } \lim_{x \to 0} \frac{\phi(x)}{\tilde{\phi}(x)} \textit{ exists}\},$$

is the one parameter family of functions:

$$\phi(x) = a\tilde{\phi}(x),$$

where $a = \lim_{x \to 0} \frac{\phi(x)}{\tilde{\phi}(x)}$.

Lemma 3. *Assume* (**H1**) *and* (**H2**). *Let* $\psi_2 : I \to \mathbf{R}^+$ *be the continuous solution of the inequality* (3) *in* I *such that the following limit exists:*

$$\lim_{x\to 0} \frac{\psi_2(x)}{\tilde{\phi}(x)} \neq 0, \quad \text{for some } \tilde{\phi} \in \mathbf{F}.$$

There exists exactly one $\phi_0 \in \mathbf{F}$ *and exactly one* $\{f\}$*-decreasing function* η, *fulfilling* $\eta(0) = 1$, *such that:*

$$\psi_2(x) = \eta(x)\phi_0(x),$$

holds. Function ϕ_0 *is given by:*

$$\phi_0(x) = \lim_{j\to\infty} \frac{\psi_2(f^j(x))}{G_{2j}(x)}, \quad x \in I.$$

Moreover

$$\psi_2(x) \geq \phi_0(x),$$

and

$$\lim_{x\to 0} \frac{\psi_2(x)}{\phi_0(x)} = 1.$$

III. In this paper, we formulate some theorems of the representation type for solutions of the inequality (1), which coefficients satisfy the assumption (**H2**). The following theorem is a generalization of Lemma 1, for the case of the inequality (1):

Theorem 1. *Assume* **(H1)** *and* **(H2)**. *Let* $\phi_2 : I \to \mathbf{R}^+$ *be a continuous solution of the equation* (6). *Then each function:*

$$\psi_1(x) = -\phi_2(x) \sum_{i=0}^{\infty} \frac{\eta(f^i(x))}{\mu(f^i(x))G_{i+1}(x)}, \qquad x \in I, \tag{7}$$

where η *is a continuous,* $\{f\}$*-decreasing function, is a continuous solution of the inequality* (1).

Proof. Since η is a continuous function and $|\lambda(0)/\mu(0)| > 1$, then the series in the formula (7) is almost uniformly convergent in I ([2] Chpt. II., pg. 53, Th. 2.7), ([3] Chpt. 3.1 C). We prove that the function ψ_1 given by (7) satisfies (1). It is sufficient to show that the function:

$$\psi_2(x) := \psi_1(f(x)) - \lambda(x)\psi_1(x),$$

where ψ_1 is given by (7), satisfies the inequality (3). Let's rewrite the function ψ_2, considering the fact, that ϕ_2 is a solution of the equation (6), and using the relation:

$$G_{i+1}(f(x)) = \frac{\mu(x)}{\lambda(x)} G_{i+2}(x)$$

$$\begin{aligned}
\psi_2(x) = &-\phi_2(f(x)) \sum_{i=0}^{\infty} \frac{\eta(f^{i+1}(x))}{\mu(f^{i+1}(x))G_{i+1}(f(x))} \\
&+ \lambda(x)\phi_2(x) \sum_{i=0}^{\infty} \frac{\eta(f^{i}(x))}{\mu(f^{i}(x))G_{i+1}(x)} \\
= &-\phi_2(x) \sum_{i=0}^{\infty} \frac{\mu(x)\eta(f^{i+1}(x))}{\mu(f^{i+1}(x))G_{i+1}(f(x))} \\
&+ \phi_2(x) \sum_{i=0}^{\infty} \frac{\lambda(x)\eta(f^{i}(x)}{\mu(f^{i}(x))G_{i+1}(x)} \\
= &-\phi_2(x) \sum_{i=0}^{\infty} \frac{\lambda(x)\mu(x)\eta(f^{i+1}(x))}{\mu(f^{i+1}(x))G_{i+2}(x)\mu(x)} \\
&+ \phi_2(x) \sum_{i=0}^{\infty} \frac{\lambda(x)\eta(f^{i}(x)}{\mu(f^{i}(x))G_{i+1}(x)} \\
= &-\phi_2(x) \sum_{i=1}^{\infty} \frac{\lambda(x)\eta(f^{i}(x))}{\mu(f^{i}(x))G_{i+1}(x)} \\
&+ \phi_2(x) \sum_{i=0}^{\infty} \frac{\lambda(x)\eta(f^{i}(x))}{\mu(f^{i}(x))G_{i+1}(x)} \\
= &\ \phi_2(x)\eta(x).
\end{aligned}$$

It is known from Lemma 1 that the function of that kind satisfies inequality (3). This completes the proof.

Further considerations concerning the inequality (1) are performed in the class of its solutions, defined in the following way:

Definition. Denote by Ψ_2 a set of continuous solutions $\psi_1 : I \to \mathbf{C}$ of the inequality (1), satisfying the conditions:

1. *there exists:*

$$\lim_{x\to 0} \frac{\psi_1(x)}{\tilde{\phi}(x)} := d_1 + \mathbf{i}d_2 \in \mathbf{C}$$

for some $\tilde{\phi} \in \mathbf{F}$, *and*

$$(d_1 + \mathbf{i}d_2)(\mu(0) - \lambda(0)) = c_1 + \mathbf{i}c_2, \quad \text{where } c_1 \neq 0.$$

2. *the function* $\psi_2 : I \to \mathbf{C},\ \psi_2(x) := \psi_1(f(x)) - \lambda(x)\psi_1(x)$ *has the nonnegative real part.*

We formulate now the lemma, which is essential for the next theorem. Denote by $\mathcal{R}\psi_2$ the real part of the function ψ_2, and by $\mathcal{T}\psi_2$ the imaginary part of that function.

Lemma 4. *Assume* **(H1)** *and* **(H2)**. *If the function $\psi_1 \in \Psi_2$, then:*

- *There exists:*

$$\lim_{j\to\infty} \frac{\mathcal{R}\psi_2(f^j(x))}{G_{2j}(x)} = \phi_0(x) \quad \text{and } \phi_0 \in \mathbf{F} \tag{8}$$

- *There exists:*

$$\lim_{x\to 0} \frac{\psi_1(x)}{\phi_0(x)} = \frac{1+\mathbf{i}\delta}{\mu(0)-\lambda(0)} \tag{9}$$

where ϕ_0 is given by (8), *and* $\delta = \lim_{x\to 0} \frac{\mathcal{T}\psi_2(x)}{\phi_0(x)}$.

Proof. If $\psi_1 \in \Psi_2$, then there exists the limit $\lim_{x\to 0} \frac{\psi_1(x)}{\tilde{\phi}(x)}$ where $\tilde{\phi} \in \mathbf{F}$. From the relation ($\tilde{\phi}$ satisfies the equation (6)):

$$\frac{\psi_1(f(x))}{\tilde{\phi}(f(x))} - \frac{\lambda(x)}{\mu(x)} \frac{\psi_1(x)}{\tilde{\phi}(x)} = \frac{\psi_2(x)}{\mu(x)\tilde{\phi}(x)},$$

it follows, that there exists also $\lim_{x\to 0} \frac{\psi_2(x)}{\tilde{\phi}(x)}$, and its value is equal to $(d_1+\mathbf{i}d_2)(\mu(0)-\lambda(0))$. According to the notation applied in the family Ψ_2 definition, we obtain:

$$\lim_{x\to 0} \frac{\mathcal{R}\psi_2(x)}{\tilde{\phi}(x)} = c_1, \ \lim_{x\to 0} \frac{\mathcal{T}\psi_2(x)}{\tilde{\phi}(x)} = c_2, \tag{10}$$

where $c_1 \neq 0$.

Applying Lemma 2 to the function $\mathcal{R}\psi_2$, we get (8). Moreover

$$\mathcal{R}\psi_2(x) = \eta(x)\phi_0(x),$$

where η is a uniquely determined, continuous function, $\{f\}$-decreasing ($\eta(0) = 1$) and $\lim_{x\to 0} \frac{\mathcal{R}\psi_2(x)}{\phi_0(x)} = 1$. From the latter and from (10) we conclude that $\lim_{x\to 0} \frac{\tilde{\phi}(x)}{\phi_0(x)}$ exists and is different from zero. From Lemma 2 we have the existence of the limit: $\lim_{x\to 0} \frac{\psi_1(x)}{\phi_0(x)}$. Dividing the equation (2) by $\phi_0(f(x))$ in $I\backslash\{0\}$ and using the fact, that ϕ_0 is a solution of the (6), we obtain:

$$\frac{\psi_1(f(x))}{\phi_0(f(x))} - \frac{\lambda(x)}{\mu(x)} \frac{\psi_1(x)}{\phi_0(x)} = \frac{\eta(x)\phi_0(x)+\mathbf{i}\mathcal{T}\psi_2(x)}{\mu(x)\phi_0(x)}.$$

Denoting by g the value of $\lim_{x\to 0}\frac{\psi_1(x)}{\phi_0(x)}$ and going to the limit with x tending to zero, we have:

$$g-\frac{\lambda(0)}{\mu(0)}g=\frac{1}{\mu(0)}(1+\mathbf{i}\delta),$$

from where (9) follows. This completes the proof.

Using the properties of functions from the class Ψ_2, shown in Lemma 4, we are going to prove the following theorem:

Theorem 2. *Assume* **(H1)** *and* **(H2)***. If the function $\psi_1\in\Psi_2$, then there exists exactly one function $\eta,\{f\}$-decreasing, continuous and satisfying $\eta(0)=1$ such that ψ_1 is of the form:*

$$\psi_1(x)=-\phi_0(x)\sum_{j=0}^{\infty}\frac{\eta(f^j(x))}{\mu(f^i(x))G_{j+1}(x)}$$

$$-\mathbf{i}\mathcal{T}\psi_2(x)\sum_{j=0}^{\infty}\frac{1}{\mu(f^i(x))G_{j+1}(x)},\qquad x\in I, \tag{11}$$

where ϕ_0 is the function defined by (8)*.*

Proof. It follows from Lemma 4 that ϕ_0 exists and $\phi_0\in\mathbf{F}$ and (9) holds. Moreover the function:

$$\psi_2(x):=\psi_1(f(x))-\lambda(x)\psi_1(x),$$

can be represented in a following, unique way:

$$\psi_2(x)=\eta(x)\phi_0(x)+\mathbf{i}\mathcal{T}\phi_2(x),\qquad x\in I,$$

where η is a continuous $\{f\}$-decreasing function and $\eta(0)=1$. Thus the function:

$$\bar{\psi}_1(x)=\begin{cases}\frac{\psi_1(x)}{\phi_0(x)}, & x\in I\backslash\{0\},\\ \frac{1+\mathbf{i}\delta}{\mu(0)-\lambda(0)}, & x=0,\end{cases} \tag{12}$$

is a continuous solution of the equation:

$$\bar{\psi}_1 f(x))-\frac{\lambda(x)}{\mu(x)}\bar{\psi}_1(x)=\frac{\eta(x)\phi_0(x)+\mathbf{i}\mathcal{T}\psi_2(x)}{\mu(x)\phi_0(x)}.$$

Since $|\lambda(0)/\mu(0)|>1$, the equation above has exactly one continuous solution ([2] Chpt. II. pg. 53, Th. 2.7), ([3] Chpt. 3.1 C), which is given by the formula:

$$\bar{\psi}_1(x)=-\sum_{j=0}^{\infty}\frac{\eta(f^j(x))\phi_0(f^j(x))+\mathbf{i}\mathcal{T}\psi_2(f^j(x))}{\mu(f^j(x))G_{j+1}(x)\phi_0(f^j(x))},\quad x\in I.$$

In view of the form (12) of the function $\bar{\psi}_1(x)$, we get (11). This completes the proof. The function ψ_1 in the formula (11) is a sum of two terms, first of which is a solution of the inequality (1), and the second is a solution of the equation (4).

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(Received October 25, 1993)