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Fixed points on trivial surface bundles over a connected CW-complex

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Abstract. The main purpose of this work is to study fixed points of fibre-preserving maps over a connected CW-complex B on the trivial surface bundles $B \times S$ where S is a closed surface of negative Euler characteristic. The case where $B = S^1$ and S is equal to S_2, i.e., the closed orientable surface of genus 2, is already known. We classify all such maps that can be deformed fibrewise to a fixed point free map.

Introduction

Given a fibration $E \to B$ and $h : E \to E$ a fibre-preserving map over B, the question if h can be deformed over B (by a fibrewise homotopy) to a fixed point free map has been considered for several years by many authors. Among others, see for example [Dol74], [FH81], [Gon87], [Pen97], [GPV04], [GPV09I], [GPV09II] and [GLPV13]. More recently also the fibrewise coincidence case has been considered in [Kos11], [GK09], [GPV10], [SV12], [Vie] and [GKLN], which certainly has intersection with the fixed point case.

The present work is a continuation of the work [GLPV13] and the main stream is the study of fixed point theory of surface bundles. There, the problem has been solved in the case where the surface bundle is the trivial bundle $S^1 \times S_2$

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where S_2 is the closed orientable surface of genus 2. Here we extend the work [GLPV13] where we now consider the case of surface trivial bundle $B \times S$ over a connected CW-complex B (not necessarily S^1) and the fibre S is a closed surface of negative Euler characteristic. So S is either S_g or N_g, where S_g is the closed orientable surface of genus $g > 1$ and N_g is the closed non-orientable surface of genus $g > 2$ (i.e. the sum of $g + 1$ projective planes). More precisely we study fibrewise maps of the trivial bundles $B \times S_g$ and $B \times N_g$.

This paper is organized into 3 sections. In Section 1 we review an approach to study fixed points of fibrewise maps and we adapt it for the case to be analyzed. In Section 2 we show that if an element $z \in \pi_1(S \times S - \Delta)$, is in the image of the homomorphism θ it must satisfy certain algebraic conditions. This is given by Theorem 2.3, which proof is in the appendix. In Section 3 let $h : B \times S \to B \times S$ be a fibrewise map given by $h(x, y) = (x, f(x, y))$. Then we prove the main result of this paper, which is:

Theorem 3.3 A fibrewise map h can be fibrewise deformed over B to a fixed point free map if and only if h is fibrewise homotopic to $id \times g$ where $g : S \to S$ is a fixed point free map homotopic to f restricted to $x_0 \times S$.

1. Preliminaries

Let $h : E \to E$ be a fibre-preserving map over a connected CW-complex B, i.e., $p \circ h = p$ where $p : E \to B$ is a fibre bundle with fibre a surface denoted by S. When is h deformable over B to a fixed point free map h' by a fibrewise homotopy over B ? We remark that in order to have a positive answer a necessary condition is that the map f restricted to a fibre is deformable to a fixed point free map.

Now we review an approach which was used in [GPV04] and [GPV09II]. Assuming the necessary condition, h is deformable over B to a fixed point free map g by a fibrewise homotopy over B if and only if there exists a lifting k such that the following diagram is commutative, up to homotopy:

$$\begin{array}{ccc} & & \mathcal{F} \\ & & \downarrow \\ & & \mathcal{E}(E \times_B E - \Delta) \\ & \overset{k}{\nearrow} & \downarrow e_1 \\ E & \xrightarrow[(h,1)]{} & E \times_B E \end{array} \tag{1.1}$$

Here $E \times_B E$ is the pullback of p by p, Δ is the diagonal in $E \times_B E$ and the inclusion $E \times_B E - \Delta \hookrightarrow E \times_B E$ is changed into the fibration $e_1 : \mathcal{E}(E \times_B E - \Delta) \to E \times_B E$ with fibre $\mathcal{F}$, where $\pi_i(\mathcal{F}) \simeq \pi_{i+1}(E \times_B E, E \times_B E - \Delta)$. Also $\mathcal{E}(E \times_B E - \Delta)$ is the pullback of the fibration $e_0 : (E \times_B E)^{[0,1]} \to E \times_B E$ by the inclusion $E \times_B E - \Delta \to E \times_B E$. The fibration $e_0 : (E \times_B E)^{[0,1]} \to E \times_B E$ is the evaluation at 0 and $e_1 : \mathcal{E}(E \times_B E - \Delta) \to E \times_B E$ is the evaluation at 1.

Let us observe that if E, B and S are closed manifolds then $\pi_{i+1}(E \times_B E, E \times_B E - \Delta) \simeq \pi_{i+1}(S, S - y_0)$, see [FH81].

When $E = B \times S$ is the trivial bundle and $h : B \times S \to B \times S$ is a fibre-preserving map over B which can be written in the form $h(x, y) = (x, f(x, y))$ where $f : B \times S \to S$, the diagram 1.1 can be modified and becomes equivalent to the following diagram:

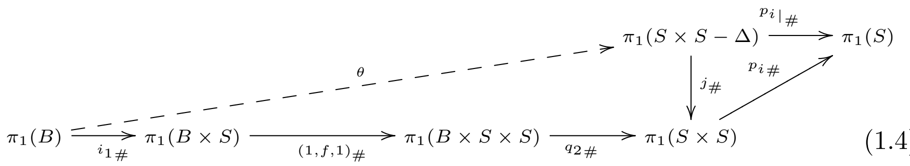
$$\begin{array}{ccc} & & \mathcal{F} \\ & & \downarrow \\ & & \mathcal{E}(B \times (S \times S - \Delta)) \\ & \overset{k}{\nearrow} & \downarrow e_1 \\ B \times S & \xrightarrow[(1,f,1)]{} & B \times S \times S \end{array} \tag{1.2}$$

Now we reduce the above lifting problem to an equivalent algebraic problem. We will follow the same steps as was done in [GLPV13] for the case where $B = S^1$ and $S = S_2$. Let x_0 and y_0 be base points of B and S, respectively, and $f : (B \times S, (x_0, y_0)) \longrightarrow (S, f(x_0, y_0))$, with $f(x_0, y_0) \neq y_0$. From the map f we obtain the maps $g = f|_{\{x_0\} \times S}$ and $l = f|_{B \times \{y_0\}}$. Recall that we are assuming the necessary condition: *the map g is deformable to a fixed point free map.* The existence of a lifting k in the above diagram is equivalent to the existence of an algebraic lifting ψ in the diagram

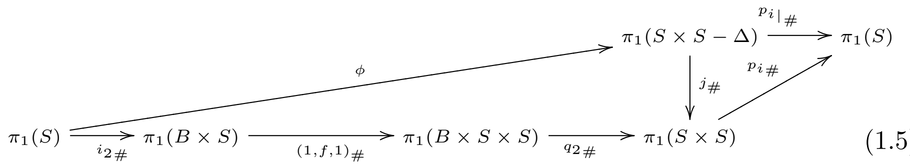
$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ & & \pi_1(\mathcal{F}) \simeq \pi_2(S, S - x_0) \\ & & \downarrow \\ & & \pi_1(\mathcal{E}(B \times (S \times S - \Delta))) \simeq \pi_1(B \times (S \times S - \Delta)) \\ & \overset{\psi}{\nearrow} & \downarrow q_{\#} = 1_{\pi_1(B)} \times j_{\#} \\ \pi_1(B \times S) & \xrightarrow[(1,f,1)_{\#}]{} & \pi_1(B \times S \times S) \\ & & \downarrow \\ & & 1 \end{array} \tag{1.3}$$

where $\pi_1(\mathcal{F}) \simeq \pi_1(S \times S - \triangle)$ is the pure braid group of S on 2-strings.

The existence of the lifting ψ above is equivalent to finding liftings θ and ϕ which are in diagrams 1.4 and 1.5 below. Since we are assuming the necessary condition then the lifting ϕ exists. So, we have the following two diagrams, where $i_{1\#}$, $i_{2\#}$ and $j_\#$ are induced homomorphisms on fundamental groups from the inclusion maps $i_1 : B \to B \times S$, $i_2 : S \to B \times S$ and $j : S \times S - \Delta \to S \times S$, respectively, and $q_{2\#}$ and $p_{i\#}$ are induced homomorphisms from the projection maps $q_2 : B \times S \times S \to S \times S$ and $p_i : S \times S \to S$, respectively.



(1.4)



(1.5)

We remark that in these diagrams we are omitting base points.

The following theorem provides necessary and sufficient conditions for the existence of the lifting θ and ϕ. So these conditions are equivalent to a positive solution of the fixed point problem for the trivial bundle.

Theorem 1.1. *There exists ψ on the diagram* (1.3) *if and only if there exist θ and ϕ in the diagrams* (1.4) *and* (1.5)*, respectively, such that* $\operatorname{Im}\theta$ *commutes with* $\operatorname{Im}\phi$.

Proof. It follows *mutatis mutandis* of the proof of Theorem 2.1 in [GLPV13]. □

2. The algebraic problem for $\chi(S) < 0$

In this section we will show that if an element $z \in \pi_1(S \times S - \Delta)$ is in the image of the homomorphism θ it must satisfy certain algebraic conditions. This is given by Theorem 2.3, whose detailed proof is in the appendix.

If ϕ is a lifting of the diagram 1.5, to discuss the existence of the lifting θ let us fix once and for all a presentation of the group $\pi_1(S \times S - \Delta)$.

For $S = S_g$ or $S = N_g$ observe that from the fibration $p_2 |: S \times S - \Delta \longrightarrow S$ we get the following exact sequence:

$$1 \longrightarrow \pi_1(S - y_0) \longrightarrow \pi_1(S \times S - \Delta) \xrightarrow{p_2|_\#} \pi_1(S) \longrightarrow 1 \ . \qquad (2.1)$$

Let $a_i = \rho_{1,i} \in \pi_1(S \times S - \Delta)$ and $b_i = \rho_{2,i} \in \pi_1(S \times S - \Delta)$ where $1 \leq i \leq 2g$ if S is orientable and $1 \leq i \leq g + 1$ if S is non-orientable.

The group $\pi_1(S - y_0)$ is free and, from the sequence above, it is identified with the subgroup of $\pi_1(S \times S - \Delta)$ freely generated by $a_1, a_2, a_3, a_4, \ldots, a_{2g}$ in the orientable case and freely generated by $a_1, a_2, a_3, a_4, \ldots, a_{g+1}$ in the non-orientable case. Also the image of the set of elements $b_1, b_2, b_3, b_4, \ldots, b_{2g-1}, b_{2g}$ in the orientable case and $b_1, b_2, \ldots, b_{g+1}$ in the non-orientable projects to a set of generators of $\pi_1(S)$ and we have the standard presentation:

$$\pi_1(S) = \begin{cases} \langle \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{2g-1}, \bar{b}_{2g} \mid [\bar{b}_1, \bar{b}_2^{-1}] \cdots [\bar{b}_{2g-1}, \bar{b}_{2g}^{-1}] \rangle & \text{if } S = S_g \\ \langle \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{g+1} \mid \bar{b}_1^2 \cdots \bar{b}_{g+1}^2 \rangle & \text{if } S = N_g. \end{cases}$$

In the case $S = S_g$ (the orientable surface of genus g) let us consider the presentation of $\pi_1(S_g \times S_g - \Delta)$ given in [FH82]:
generators: a_i, b_i, where $i = 1, \ldots, 2g$.
relations:

(I) $[a_1, a_2^{-1}][a_3, a_4^{-1}] \cdots [a_{2g-1}, a_{2g}^{-1}] =: B_{1,2} = B_{2,1}^{-1} := [b_1, b_2^{-1}][b_3, b_4^{-1}] \cdots \cdots [b_{2g-1}, b_{2g}^{-1}]$ (which defines the elements $B_{1,2}$ and $B_{2,1}^{-1}$).

(II) $b_l a_j b_l^{-1} = a_j$ where $1 \leq j, l \leq 2g$, and $j < l$ (resp. $j < l - 1$) if l is odd (resp. l is even).

(III) $b_k a_k b_k^{-1} = a_k [a_k^{-1}, B_{1,2}]$ and $b_k^{-1} a_k b_k = a_k [B_{1,2}^{-1}, a_k]$ for all $1 \leq k \leq 2g$.

(IV) $b_k a_{k+1} b_k^{-1} = B_{1,2} a_{k+1} [a_k^{-1}, B_{1,2}]$ and
$b_k^{-1} a_{k+1} b_k = B_{1,2}^{-1} [B_{1,2}, a_k] a_{k+1} [B_{1,2}^{-1}, a_k]$, for all k odd, $1 \leq k \leq 2g$.

(V) $b_{k+1} a_k b_{k+1}^{-1} = a_k B_{1,2}^{-1}$, and $b_{k+1}^{-1} a_k b_{k+1} = a_k B_{1,2} [B_{1,2}^{-1}, a_{k+1}]$, for all k odd, $1 \leq k \leq 2g$.

(VI) $b_l a_j b_l^{-1} = [B_{1,2}, a_l^{-1}] a_j [a_l^{-1}, B_{1,2}]$ and $b_l^{-1} a_j b_l = [a_l, B_{1,2}^{-1}] a_j [B_{1,2}^{-1}, a_l]$ for all $1 \leq l < j \leq 2g$ and $(j, l) \neq (2t, 2t - 1)$ for all $t \in \{1, \ldots, g\}$.

In the case $S = N_g$ (the closed non-orientable surface of genus $g \geq 2$, i.e. the connected sum of $g + 1$ projective planes), let us consider the presentation of $\pi_1(N_g \times N_g - \Delta)$ obtained from [Sco69]:
generators: a_i, b_i, where $i = 1, \ldots, g + 1$.
relations:

(I) $a_1^2 \cdots a_{g+1}^2 = b_1^2 \cdots b_{g+1}^2 = B_{1,2}$

(II) $b_l a_j b_l^{-1} = a_j$ if $j < l$.

(III) $b_k a_k b_k^{-1} = a_k B_{1,2}^{-1}$ all k.

(IV) $b_l a_j b_l^{-1} = B_{1,2} a_l^{-1} B_{1,2}^{-1} a_l a_j a_l^{-1} B_{1,2} a_l B_{1,2}^{-1} = [B_{1,2}, a_l^{-1}] a_j [a_l^{-1}, B_{1,2}]$ if $j > l$.

(VI) $b_k B_{1,2} b_k^{-1} = B_{1,2} a_k^{-1} B_{1,2}^{-1} a_k B_{1,2}^{-1} = [B_{1,2}, a_k] B_{1,2}^{-1}$,

where the convention on how to multiply two braids is the one used in [FH82].

Given a group G the central series of G is defined recursively by

$$G_1 = G, G_{n+1} = [G, G_n], \quad n = 1, 2, \ldots.$$

For any group G we have that G_m is a normal subgroup of G_n for all $n \leq m$.

In case G is free group of finite rank r then it is well known that G_n/G_{n+1} is a free abelian group of rank

$$N_n = \frac{1}{n} \sum_{d|n} \mu(d) r^{\frac{n}{d}}$$

(see [[MKS76], Theorem 5.11, p. 330]). Here $\mu(d)$ denotes the *Moebius function* defined for all positive integers by $\mu(1) = 1$, $\mu(p) = -1$ if p is a prime number, $\mu(p^k) = 0$ for $k > 1$, and $\mu(b \cdot c) = \mu(b) \cdot \mu(c)$ if b and c are coprime integers.

For any group G denote the commutator $[[a, b], c]$ by (a, b, c). If a, b, c are elements of a group G and k, m, n are positive integers such that $a \in G_k$, $b \in G_m, c \in G_n$ then $(a, b, c) \cdot (b, c, a) \cdot (c, a, b) \equiv 1 \mod G_{k+m+n+1}$ (see [[MKS76], Theorem 5.3, p. 293]).

Lemma 2.1. *In the case $S = S_g$ and $G_1 = \pi_1(S - y_0)$ we have in G_1/G_3 the relations:*

a) $b_k a_{k+1}^n b_k^{-1} = B_1^n a_{k+1}^n$ *for all k odd, $1 \leq k \leq 2g$.*

b) $b_{k+1} a_k^n b_{k+1}^{-1} = a_k^n B_1^{-n}$ *for all k odd, $1 \leq k \leq 2g$.*

c) $b_k a_j^n b_k^{-1} = a_j^n$, *$j \neq k+1$ for all k odd, $1 \leq k \leq 2g$ and $j \neq k-1$ for all k even, $1 \leq k \leq 2g$.*

and in the case $S = N_g$ we have in G_1/G_2 the relations:

a) $b_k a_k^n b_k^{-1} = a_k^n B_1^{-n}$.

b) $b_k a_j^n b_k^{-1} = a_j^n$, $j \neq k$.

Proof. These relations follow from the presentation of $\pi_1(S_g \times S_g - \Delta)$ and $\pi_1(N_g \times N_g - \Delta)$, respectively. □

Lemma 2.2. *In G_1/G_3 we have:*

a) $[a_ia_j, x] = [a_ja_i, x]$, *where a_i, a_j are generators of G and $x \in G$.*

b) *The element $[a_i^{x_i}, a_j^{x_j}]$ is given in the following form:*

$$[a_i^{x_i}, a_j^{x_j}] = \begin{cases} 0 & \text{if } i = j \\ x_ix_j[a_i, a_j] & \text{if } i < j \\ -x_ix_j[a_j, a_i] & \text{if } i > j \end{cases}$$

c) *If $B_1 = [a_1, a_2^{-1}][a_3, a_4^{-1}] \cdots [a_{2g-1}, a_{2g}^{-1}] \in G_1$, then $B_1 = -[a_1, a_2] - [a_3, a_4] - \cdots - [a_{2g-1}, a_{2g}]$ on G_1/G_3.*

Proof. Since $[a_ia_j, x] = [a_i, [a_j, x]][a_j, x][a_i, x]$ and $[a_ja_i, x] = [a_j, [a_i, x]][a_i, x][a_j, x]$, item *a*) follows by observing that in G_1/G_3 we have that $[a_i, [a_j, x]] = 0 = [a_j, [a_i, x]]$. Items *b*) and *c*) are easy. □

Theorem 2.3. *If $z \in \mathrm{Im}(\theta)$ and $(p_2|_\#)(C(z)) = \pi_1(S)$ then $z \in G_2 = [G_1, G_1]$, where $C(z)$ denotes the centralizer of z in $\pi_1(S \times S - \Delta)$.*

Proof. See the appendix. □

3. Main result

Let $h : B \times S \to B \times S$ be given by $h(x, y) = (x, f(x, y))$. Let us consider $l : (B, x_0) \to (S, f(x_0, y_0))$ and $g : (S, y_0) \to (S, f(x_0, y_0))$ defined by $l(x) = f(x, y_0)$ and $g(y) = f(x_0, y)$, respectively. Without loss of generality we are assuming that g is a fixed point free map.

To prove our main result we will first prove two lemmas.

Lemma 3.1. *Let $t : S \to S$ be a continuous map and $t_\# : \pi_1(S) \to \pi_1(S)$ its induced homomorphism. Suppose that $t_\#(\bar{b}_i) = \alpha^{n_i}$ for some $\alpha \in \pi_1(S)$ and for all i where the $\bar{b}_i'$s form a set of canonical generators of $\pi_1(S)$. If the map t can be deformed to a fixed point free map then $\sum_i n_i|\alpha|_i = 1$ where $|\alpha|_i$ denotes the sum of the exponents of $\bar{b}_i$ in the word α.*

Proof. Let $\iota : S^1 \to S$ be a map which represents the element $\alpha \in \pi_1(S)$. We can define $t' : S \to S^1$ such that $\iota \circ t' = t$. By the commutativity property we know that the Nielsen number of t is the same as the Nielsen number of $t' \circ \iota$, which is a self map of the circle. So if t is deformable to a fixed point free map then we have that the Nielsen number of $t' \circ \iota$ is trivial which is equivalent to saying that the Lefschetz number of $t' \circ \iota$ is 0, which is the same as $\sum_i n_i|\alpha|_i = 1$. So the result follows. □

Lemma 3.2. *The fibrewise map h is deformable to a fixed point free map over B if and only if* $\mathrm{Im}(l_\#) = \{e\}$, *where* $l_\# : \pi_1(B; x_0) \to \pi_1(S; f(x_0, y_0))$ *and* $h|_S$ *can be deformed to a fixed point free map.*

Proof. Let h be a fibrewise map where $h(x, y) = (x, f(x, y))$. Suppose that $\mathrm{Im}(l_\#) = \{e\}$ and $h|_S$ can be deformed to fixed point free. Then from Theorem 1.1 it follows that h can be deformed fibrewise to a fixed point free map.

Conversely, let h be a map deformable to a fixed point free map over S^1. Then by Theorem 1.1 there exist ϕ and θ such that the image of θ commutes with the image of ϕ. From diagrams 1.4 and 1.5, $p_{1|_\#} \circ \theta = l_\#$ and $p_{1|_\#} \circ \phi = g_\#$. It is known that $g_\#(\pi_1(S))$ is a subgroup of $\pi_1(S)$ isomorphic to one of the following groups:

(1) $\{e\}$.

(2) a free group of rank ≥ 2.

(3) $\pi_1(S)$

(4) $\mathbb{Z} = \langle \beta \rangle$

Case (1) does not occur, because otherwise g is homotopic to the constant map so it can not be deformed to a fixed point free map.

In cases (2) and (3), from above an element $z \in \mathrm{Im}(l_\#)$ commutes with all elements of $g_\#(\pi_1(S))$. But in these cases the centralizer of $g_\#(\pi_1(S))$ is trivial, therefore $\mathrm{Im}(l_\#) = \{e\}$.

For the last case we have that $g_\#(\pi_1(S)) = \mathbb{Z} = \langle \beta \rangle = \langle \alpha^k \rangle$ where $\alpha \neq 0$, α has no roots and $\alpha^k = \beta$. Since $\mathrm{Im}(l_\#)$ commutes with the elements of $g_\#(\pi_1(S))$ then $\mathrm{Im}(l_\#) = \langle \alpha^r \rangle$. If $r = 0$ the proof follows. So suppose that $r \neq 0$.

Writing $g_\#(\bar{b}_i) = \alpha^{n_i}$ by Lemma 3.1 if g is homotopic to a fixed point free map then $\sum_i n_i |\alpha|_i \neq 0$.

We have that $p_{1|_\#} \circ \theta(\pi_1(B)) = \mathrm{Im}(l_\#)$. We also have that $\mathrm{Im}\,\phi \subset C(\mathrm{Im}(\theta))$, and $p_{2|_\#}(\mathrm{Im}(\phi))) = \pi_1(S)$. Therefore $p_{2|_\#}(C(\mathrm{Im}(\theta))) = \pi_1(S)$ and by Theorem 2.3 it follows that $\mathrm{Im}(\theta) \subset G_2 = [G_1, G_1]$ and then $|\alpha|_i = 0$ for all i. Since $\sum_i n_i |\alpha|_i \neq 0$ we have a contradiction and the result follows. □

Now we can state the main result.

Theorem 3.3. *A fibrewise map h can be deformed fibrewise over a connected CW-complex B to a fixed point free map if and only if h is fibrewise homotopic to $id \times g$ where $g : S \to S$ is a fixed point free map homotopic to f restricted to $x_0 \times S$.*

Proof. The "if part" is clear since $id \times g$ is a fibrewise fixed point free map. Conversely, suppose that h can be deformed over B to a fixed point free map h' and denote by g the restriction of h' to the fibre $x_0 \times S$. It suffices to show that h' is homotopic to $id \times g$. If $h'(x,y) = (x, f(x,y))$ where $f(x_0,y) = g(y)$, this is equivalent to show that $f, f' : B \times S \to S$ are homotopic where $f'(x,y) = g(y)$.

Because S is a $K(\pi,1)$ the two maps are homotopic if the induced homomorphisms on the fundamental group are conjugate (see [[Whi78], V,4, Theorem 4.3] and [[Whi78], V,4, Corollary 4.4]). Because $\pi_1(B \times S) = \pi_1(B) \times \pi_1(S)$ let us consider first the restriction of the two homomorhisms to $\pi_1(B)$. From Lemma 3.2 they coincide and it is the trivial homomorphism. By the definition of f and f' the two maps coincide when restricted to $x_0 \times S$, so the induced homomorphisms coincide. Therefore the two maps are homotopic and the result follows. □

Final comments: It is natural to study fixed points of fibre maps of surface bundles which are not trivial, so let E be a $S-$bundle over a connected CW-complex B, not necessarily trivial. As a first step in studying the problem above let us consider the case where $B = S^1$ and make some comments which are relevant for this new situation. The space E is the mapping torus of a homeomorphism of the surface S and the level of difficulty seems to us to be much higher than the case when the bundle is trivial. First of all the formulation of the problem is already more elaborate. More precisely, let us consider the map $\phi : [E,E]_{S^1} \to [S,S]$ which associates to a homotopy class of a fibre preserve map $[f]$ the homotopy class of the restriction $f|_S : S \to S$. Then one would like first to know which homotopy classes $[g] \in [S,S]$ which contain a fixed point free map are in the image of ϕ. This problem is related to some questions on the mapping class group of a surface. The answer is clear when the bundle is trivial but not in general. Second, for a class $[g]$ in the image of ϕ, as above, we would like to compute $\phi^{-1}[g]$. It is very difficult to determine which ones are fibrewise fixed point free classes, i.e. classes which admit a representative which is a fixed point free fibrewise map. In the case that we solved in this work, we were able to take advantage of the fact that the class $[g]$ is always in the image of ϕ for all maps g which are fixed point free and the pre-image consists of exactly one class where this class has the property that it contains a representative which is a fibrewise fixed point free map.

The study of the case above is in progress and it should appear somewhere.

4. Appendix

Proof of Theorem 2.3. Given a word w in a free group $F(x_1, \dots, x_n)$ we denote by $|w|_{x_i}$ the sum of the exponents of the letter x_i which appear in the word w.

By hypothesis for each i there is a word of the form $w_i b_i$ which commutes with z, where w_i belongs to the subgroup generated by the $a_i's$. So $b_i z b_i^{-1} = w_i^{-1} z w_i$.

a) The case where S is the non-orientable surface N_g: In this case let us consider the equation $w_i b_i z = z w_i b_i$ on G_1/G_2. It follows easily that this is equivalent to $b_i z b_i^{-1} = z$. But from Lemma2.1 it follows that $b_i z b_i^{-1} = z B_1^{-|z|_{a_i}}$ on G_1/G_2. However the element B_1 is non trivial on G_1/G_2. Therefore $|z|_{a_i} = 0$ for all $i = 1, \dots, g+1$ and the result follows.

b) The orientable case: For this case we need to look at our equations on the group G_1/G_3. But from Lemma2.1 follows that in G_1/G_3

$$[z^{-1}, b_i] = \begin{cases} B_1^{|z|_{a_{i+1}}} & \text{if } i \text{ odd}, \ 1 \le i \le 2g \\ B_1^{-|z|_{a_{i-1}}} & \text{if } i \text{ even}, \ 1 \le i \le 2g. \end{cases}$$

So since $[z^{-1}, b_i] = [z^{-1}, w_i^{-1}]$ we obtain a system with $g(2g-1)$ equations and $2g$ variables $z : (i_1, i_2, \dots, i_{2g-1}, i_{2g})$ corresponding to $z = a_1^{i_1} a_2^{i_2} \cdots a_{2g-1}^{i_{2g-1}} a_{2g}^{i_{2g}}$ and for each i, there are $2g$ variables $(j_{1i}, j_{2i}, \dots, j_{(2g-1)i}, j_{(2g)i})$ corresponding to $w_i^{-1} = a_1^{j_{1i}} a_2^{j_{2i}} \cdots a_{2g-1}^{j_{(2g-1)i}} a_{2g}^{j_{(2g)i}}$.

We calculate $[w^{-1}, z^{-1}]$, observing that the exponents of $w^{-1} = a_1^{j_1} a_2^{j_2} \cdots \cdots a_{2g-1}^{j_{2g-1}} a_{2g}^{j_{2g}}$ should appear with sub-index i. For ease of notation we are omitting the sub-index.

Then $[w^{-1}, z^{-1}] = (i_1 j_2 - i_2 j_1)[a_1, a_2] + (i_1 j_3 - i_3 j_1)[a_1, a_3] + \cdots + (i_1 j_{2g} - i_{2g} j_1)[a_1, a_{2g}] + (i_2 j_3 - j_2 i_3)[a_2, a_3] + \cdots + (i_2 j_{2g} - i_{2g} j_2)[a_2, a_{2g}] + \cdots + (i_{2g-2} j_{2g-1} - i_{2g-1} j_{2g-2})[a_{2g-2}, a_{2g-1}] + (i_{2g-2} j_{2g} - i_{2g} j_{2g-2})[a_{2g-2}, a_{2g}] + (i_{2g-1} j_{2g} - i_{2g} j_{2g-1})$ $[a_{2g-1}, a_{2g}]$. So, by using Lemma 2.2, we obtain the following system

$$
\begin{cases}
-i_2 j_1 + i_1 j_2 = i_2, -i_1, \ldots, i_{2g}, -i_{2g-1} \\
\qquad\qquad\qquad \text{respectively for } b_1, b_2, \ldots, b_{2g-1}, b_{2g} \\
-i_3 j_1 + i_1 j_3 = 0 \\
\qquad\qquad \vdots \\
-i_{2g-2} j_1 + i_1 j_{2g-2} = 0 \\
-i_{2g-1} j_1 + i_1 j_{2g-1} = 0 \\
-i_{2g} j_1 + i_1 j_{2g} = 0 \\
-i_3 j_2 + i_2 j_3 = 0 \\
-i_4 j_2 + i_2 j_4 = 0 \\
\qquad\qquad \vdots \\
-i_{2g-2} j_2 + i_2 j_{2g-2} = 0 \\
-i_{2g-1} j_2 + i_2 j_{2g-1} = 0 \\
-i_{2g} j_2 + i_2 j_{2g} = 0 \\
-i_4 j_3 + i_3 j_4 = i_2, -i_1, \ldots, i_{2g}, -i_{2g-1} \\
\qquad\qquad\qquad \text{respectively for } b_1, b_2, \ldots, b_{2g-1}, b_{2g} \\
-i_5 j_3 + i_3 j_5 = 0 \\
\qquad\qquad \vdots \\
-i_{2g-2} j_{2g-3} + i_{2g-3} j_{2g-2} = i_2, -i_1, \ldots, i_{2g}, -i_{2g-1} \\
\qquad\qquad\qquad \text{respectively for } b_1, b_2, \ldots, b_{2g-1}, b_{2g} \\
-i_{2g-1} j_{2g-3} + i_{2g-3} j_{2g-1} = 0 \\
-i_{2g} j_{2g-3} + i_{2g-3} j_{2g} = 0 \\
-i_{2g-1} j_{2g-2} + i_{2g-2} j_{2g-1} = 0 \\
-i_{2g} j_{2g-2} + i_{2g-2} j_{2g} = 0 \\
-i_{2g} j_{2g-1} + i_{2g-1} j_{2g} = i_2, -i_1, \ldots, i_{2g}, -i_{2g-1} \\
\qquad\qquad\qquad \text{respectively for } b_1, b_2, \ldots, b_{2g-1}, b_{2g}
\end{cases}
\tag{4.1}
$$

understanding that in the letters $(j_1, j_2, \ldots, j_{2g-1}, j_{2g})$ should appear the sub-index i, but not in the letters $(i_1, i_2, \ldots i_{2g-1}, i_{2g})$.

Initially we observe that from the last six lines of the system 4.1 we have $\begin{vmatrix} i_{2g-3} & i_{2g-1} \\ j_{2g-3} & j_{2g-1} \end{vmatrix} = 0$, $\begin{vmatrix} i_{2g-3} & i_{2g} \\ j_{2g-3} & j_{2g} \end{vmatrix} = 0$, $\begin{vmatrix} i_{2g-2} & i_{2g-1} \\ j_{2g-2} & j_{2g-1} \end{vmatrix} = 0$, $\begin{vmatrix} i_{2g-2} & i_{2g} \\ j_{2g-2} & j_{2g} \end{vmatrix} = 0$ and
$\begin{vmatrix} i_{2g-3} & i_{2g-2} \\ j_{2g-3} & j_{2g-2} \end{vmatrix} = \begin{vmatrix} i_{2g-1} & i_{2g} \\ j_{2g-1} & j_{2g} \end{vmatrix} = i_{2j}$ or $-i_{2j-1}$, $j = 1, 2, \ldots, g$.

So we have that $(j_{2g-3}, j_{2g-1}) = \alpha(i_{2g-3}, i_{2g-1})$, $(j_{2g-3}, j_{2g}) = \beta(i_{2g-3}, i_{2g})$, $(j_{2g-2}, j_{2g-1}) = \gamma(i_{2g-2}, i_{2g-1})$ and $(j_{2g-2}, j_{2g}) = \delta(i_{2g-2}, i_{2g})$ for some α, β, γ and δ.

To prove that $i_k = 0$ for $k = 1, 2 \ldots, (2g-5), (2g-4)$ we proceed as follows:

a-) If $i_{2g-1} = 0$ we have that $-i_{2g}j_{2g-1} = i_k$ for k even (or $-i_k$ for k odd) and $i_k j_{2g-1} = 0$. So $i_k(-i_{2g}j_{2g-1}) = 0$ for k even (or $i_k(i_{2g}j_{2g-1} = 0$ for k odd), which implies $i_k^2 = 0$ and therefore $i_k = 0$.

b-) If $i_{2g-1} \neq 0$, since $\alpha i_{2g-1} = j_{2g-1} = \gamma i_{2g-1}$ then $\alpha = \gamma$. So $(j_{2g-3}, j_{2g-2}) = \alpha(i_{2g-3}, i_{2g-2})$ and therefore $\begin{vmatrix} i_{2g-3} & i_{2g-2} \\ j_{2g-3} & j_{2g-2} \end{vmatrix} = 0$.

But $\begin{vmatrix} i_{2g-3} & i_{2g-2} \\ j_{2g-3} & j_{2g-2} \end{vmatrix} = i_k$ for k even (or $-i_k$ for k odd). Then $i_k = 0$, $k = 1, 2 \ldots, (2g-5), (2g-4)$.

To prove that $i_{2g-3} = 0$ we proceed as follows:

a-) If $i_{2g-1} = 0$ we have that $i_{2g-3}j_{2g-1} = 0$ and $-i_{2g}j_{2g-1} = -i_{2g-3}$. So $-i_{2g}(i_{2g-3}j_{2g-1}) = -i_{2g-3}^2$, which implies $-i_{2g-3}^2 = 0$ and therefore $i_{2g-3} = 0$.

b-) If $i_{2g-1} \neq 0$, since $\alpha i_{2g-1} = j_{2g-1} = \gamma i_{2g-1}$ then $\alpha = \gamma$. So $(j_{2g-3}, j_{2g-2}) = \alpha(i_{2g-3}, i_{2g-2})$ and therefore $\begin{vmatrix} i_{2g-3} & i_{2g-2} \\ j_{2g-3} & j_{2g-2} \end{vmatrix} = 0$.

But $\begin{vmatrix} i_{2g-3} & i_{2g-2} \\ j_{2g-3} & j_{2g-2} \end{vmatrix} = -i_{2g-3}$. Then $i_{2g-3} = 0$.

To prove that $i_{2g-2} = 0$ we proceed as follows:

a-) If $i_{2g} = 0$ we have that $i_{2g-1}j_{2g} = i_{2g-2}$ and $i_{2g-2}j_{2g} = 0$. So $i_{2g-2}(i_{2g-1}j_{2g}) = 0$, which implies $i_{2g-2} = 0$.

b-) If $i_{2g} \neq 0$, since $\beta i_{2g} = j_{2g} = \delta i_{2g}$ then $\beta = \delta$. So $(j_{2g-3}, j_{2g-2}) = \beta(i_{2g-3}, i_{2g-2})$ and therefore $\begin{vmatrix} i_{2g-3} & i_{2g-2} \\ j_{2g-3} & j_{2g-2} \end{vmatrix} = 0$.

But $\begin{vmatrix} i_{2g-3} & i_{2g-2} \\ j_{2g-3} & j_{2g-2} \end{vmatrix} = i_{2g-2}$. Then $i_{2g-2} = 0$.

To prove that $i_{2g-1} = 0$ we proceed as follows:

a-) If $i_{2g-3} = 0$ we have that $-i_{2g-2}j_{2g-3} = -i_{2g-1}$ and $-i_{2g-1}j_{2g-3} = 0$. So $i_{2g-1}(-i_{2g-2}j_{2g-3}) = 0$, which implies $-i_{2g-1}^2 = 0$ and therefore $i_{2g-1} = 0$.

b-) If $i_{2g-3} \neq 0$, since $\gamma i_{2g-3} = j_{2g-3} = \delta i_{2g-3}$ then $\gamma = \delta$. So $(j_{2g-1}, j_{2g}) = \gamma(i_{2g-1}, i_{2g})$ and therefore $\begin{vmatrix} i_{2g-1} & i_{2g} \\ j_{2g-1} & j_{2g} \end{vmatrix} = 0$.

But $\begin{vmatrix} i_{2g-1} & i_{2g} \\ j_{2g-1} & j_{2g} \end{vmatrix} = -i_{2g-1}$. Then $i_{2g-1} = 0$.

To prove that $i_{2g} = 0$ we proceed as follows:

a-) If $i_{2g-2}=0$ we have that $i_{2g-3}j_{2g-2}=i_{2g}$ and $-i_{2g}j_{2g-2}=0$. So $-i_{2g}(i_{2g-3}$ $j_{2g-2})=0$, which implies $-i_{2g}^2=0$ and therefore $i_{2g}=0$.

b-) If $i_{2g-2}\neq 0$, since $\alpha i_{2g-2}=j_{2g-2}=\beta i_{2g-2}$ then $\alpha=\beta$. So $(j_{2g-1},j_{2g})=\alpha(i_{2g-1},i_{2g})$ and therefore $\begin{vmatrix} i_{2g-1} & i_{2g} \\ j_{2g-1} & j_{2g}\end{vmatrix}=0$.

But $\begin{vmatrix} i_{2g-1} & i_{2g} \\ j_{2g-1} & j_{2g}\end{vmatrix}=i_{2g}$. Then $i_{2g}=0$.

From the considerations above we conclude that $|z|_{a_i}=0$ for all $i=1,\dots,2g$ and the result follows. □

References

[Dol74] A. Dold, The fixed point index of fibre-preserving maps, *Inventiones Math.* **25** (1974), 281–297.

[FH81] E. Fadell and S. Husseini, A Fixed Point Theory for Fiber-Preserving Maps, Vol. 886, Lecture Notes in Mathematics, *Springer Verlag*, 1981, 49–72.

[FH82] E. Fadell and S. Husseini, The Nielsen number on surfaces, in: Topological Methods in Non Linear Functional Analysis, Vol. 21, Toronto, Ont., 1982, in: Contemporary Mathematics,, *Amer. Math. Soc.*, *Providence, RI,*, 1983, 59–98.

[Gon87] D. L. Gonçalves, Fixed point of S^1- fibrations, *Pacific J. Math.* **129** (1987), 297–306.

[GK09] D. L. Gonçalves and U. Koschorke, Nielsen coincidence theory of fibre-preserving maps and Dold's fixed point index, Topological Methods in Nonlinear Analysis, Vol. 33, *Journ. of Juliusz Schauder Center*, 2009, 85–103.

[GKLN] D. L. Gonçalves, U. Koschorke, A. K. M. Libardi and O. M. Neto, Coincidences of fiberwise maps between sphere bundles over the circle, *Proc. Edinburgh Math. Soc.* **57** (2014), 713–735.

[GLPV13] D. L. Gonçalves, A. K. M. Libardi, D. Penteado and J. P. Vieira, Fixed points on trivial surface bundles, Vol. 6, Proceedings of the Institute of Mathematics of the National Academy of Sciences of Ukraine, Brazilian-Polish Topology Workshop, 2013, 67–85.

[GPV04] D. L. Gonçalves, D. Penteado and J. P. Vieira, Fixed points on torus fiber bundles over the circle, *Fund. Math.* **183** (2004), 1–38.

[GPV09I] D. L. Gonçalves, D. Penteado and J. P. Vieira, Abelianized obstruction for fixed points of fiber-preserving maps of surface bundles, *Topol. Meth. Nonl. Anal.* **33** (2009), 293–305.

[GPV09II] D. L. Gonçalves, D. Penteado and J. P. Vieira, Fixed points on Klein bottle fiber bundles over the circle, *Fund. Math.* **203** (2009), 263–292.

[GPV10] D. L. Gonçalves *et al.*, Coincidence points of fiber maps on S^n-bundles, *Topology and its Applications* (2010), doi:10.1016/j.topol.2010.02.025.

[Kos11] U. Koschorke, Fixed points and coincidences in torus bundles, *J. Topol. Anal.* **3** (2011), 177–212.

[MKS76] W. Magnus, A. Karrass and D. Solitar, Combinatorial Group Theory - Presentations of Groups in Terms of Generators and Relations, *Dover Publications, Inc.*, *New York*, 1976.

[Pen97] D. Penteado, Fixed points on surface fiber bundles, Vol. 13, Matemática Contemporânea, Proceedings of the 10th Brazilian Topology Meeting, 1997, 251–267.

[Sco69] G. P. Scott, Braid groups and the group of homeomorphisms of a surface, *Proc. Camb. Phil. Soc.* **68** (1969), 605–617.

[SV12] W. L. Silva and J. P. Vieira, Coincidences of self-maps on Klein bottle fiber bundles over the circle, *JP J. Geom. Topol.* **12** (2012), 55–97.

[Vie] J. P. Vieira, Coincidence of maps on torus fiber bundles over the circle (*to appear in* Topol. Meth. Nonl. Anal.).

[Whi78] G. W. Whitehead, Elements of Homotopy Theory, Vol. 61, Graduate texts in Mathematics, *Springer Verlag*, 1978.

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