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On C-automorphisms of finite groups admitting a strongly 2-embedded subgroup

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Abstract. Let G be a finite group with a strongly 2-embedded subgroup. It is proved that every C-automorphism of G is inner. In particular, the normalizer property holds for G. As a consequence, it is also obtained that every C-automorphism of finite (TI)-groups is inner.

1. Introduction

All groups considered in this paper are finite. Let G be a finite group, $\mathbb{Z}G$ its integral group ring and $\mathrm{U}(\mathbb{Z}G)$ the unit group of $\mathbb{Z}G$. The normalizer problem ([16, Problem 43], see also [6, Question 3.7]) of integral group rings asks whether $\mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(G) = G \cdot \mathrm{Z}(\mathrm{U}(\mathbb{Z}G))$ for any group G, where $\mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(G)$ and $\mathrm{Z}(\mathrm{U}(\mathbb{Z}G))$ denote the normalizer of G in $\mathrm{U}(\mathbb{Z}G)$ and the center of $\mathrm{U}(\mathbb{Z}G)$ respectively. If the equality is valid for G, then G is said to have the normalizer property.

C-automorphisms of groups have intimate connection with the normalizer problem. Recall that an automorphism σ of G is called a C-automorphism if σ satisfies the following three conditions: (I) the restriction of σ to each Sylow subgroup of G equals the restriction of some inner automorphism of G; (II) σ is a class-preserving automorphism; (III) $\sigma^2 \in \mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ is the inner

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automorphism group of G. This definition was firstly introduced by Marciniak and Roggenkamp in [14]. Denote by $\mathrm{Aut_C}(G)$ the group of all C-automorphisms of G. Set $\mathrm{Out_C}(G) := \mathrm{Aut_C}(G)/\mathrm{Inn}(G)$. Additionally, denote by $\mathrm{Aut_{Col}}(G)$ the group of all automorphisms of G which satisfy only condition (I) above. It is known that $\mathrm{Out_C}(G) = 1$ implies that G has the normalizer property. In this direction, a lot of results on C-automorphisms have appeared in the literature, see [2]–[7], [9]–[15] for instance.

The aim of this paper is to study C-automorphisms of finite groups with a strongly 2-embedded subgroup. Recall that a subgroup M of a group G is said to be a strongly 2-embedded subgroup in G if the following conditions are satisfied: (i) $M < G$ and 2 divides $|M|$; (ii) If $g \in G \setminus M$, then 2 does not divide $|M \cap M^g|$. A characterization of groups with a strongly 2-embedded subgroup can be found in [1] or the appendix in [8]. Our main result is as follows.

Theorem A. *Let G be a group with a strongly 2-embedded subgroup. Then $\mathrm{Out_C}(G) = 1$. In particular, the normalizer property holds for G.*

A group G of even order is called a (TI)-group if two different Sylow 2-subgroups contain only the identity element in common. This notion was introduced by Suzuki [17]. As a consequence of Theorem A, we have the following result.

Corollary B. *Let G be a (TI)-group. Then $\mathrm{Out_C}(G) = 1$. In particular, the normalizer property holds for G.*

2. Notation and preliminaries

Let σ be an automorphism of a group G, and $N \trianglelefteq G$ with $N^\sigma = N$. Write $\sigma|_N$ for the restriction of σ to N, and $\sigma|_{G/N}$ for the automorphism of G/N induced by σ in the natural way. For a fixed element $y \in G$, write $\mathrm{conj}(y)$ for the inner automorphism of G induced by y via conjugation. Our other notation is standard and follows that in [8].

Lemma 2.1 ([15, Theorem 7]). *If G is a group with a Sylow 2-subgroup of order 2, then $\mathrm{Out_C}(G) = 1$.*

Lemma 2.2. *Let G be a group of odd order. Then $\mathrm{Out_C}(G) = 1$.*

Proof. This is a consequence of Proposition 1 in [5]; see also Theorem 3.4 in [6]. □

Lemma 2.3. *Let G be a simple group. Then $\mathrm{Out_C}(G) = 1$.*

Proof. This is a consequence of Theorem 14 in [5]. □

Lemma 2.4 ([5, Lemma 6]). *Let $\sigma \in \mathrm{Aut}(G)$ and $M \trianglelefteq G$ with $M^\sigma = M$. Suppose that $\sigma|_Q = \mathrm{conj}(h)|_Q$ for some $h \in G$, where Q is a Sylow subgroup of M. Then σ fixes $H = M\mathrm{C}_G(Q) \trianglelefteq G$ and $\sigma|_{G/H} = \mathrm{conj}(h)|_{G/H}$.*

Lemma 2.5 ([2, Lemma 2]). *Let $N \trianglelefteq G$ and let $\sigma \in \mathrm{Aut}(G)$ be of p-power order, where p is a prime. Suppose that $\sigma|_N = \mathrm{id}|_N$ and $\sigma|_{G/N} = \mathrm{id}|_{G/N}$. Then $\sigma|_{G/\mathrm{O}_p(\mathrm{Z}(N))} = \mathrm{id}|_{G/\mathrm{O}_p(\mathrm{Z}(N))}$. Furthermore, if σ fixes a Sylow p-subgroup of G element-wise, then $\sigma \in \mathrm{Inn}(G)$.*

Lemma 2.6. *Let N be a normal subgroup of a group G such that G/N is of odd order. If $\mathrm{Out_C}(N) = 1$, then $\mathrm{Out_C}(G) = 1$. In particular, this is the case when G has a normal Sylow 2-subgroup.*

Proof. This is a consequence of Corollary 3 in [5]; see also Theorem 3.6 in [6]. □

Lemma 2.7. *Suppose that G possesses a strongly 2-embedded subgroup. Then one of the following statements holds.*

(i) *The Sylow 2-subgroups of G are either cyclic or quaternion groups.*

(ii) *G possesses a normal series $1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G$ such that G_1 and G/G_2 are of odd order, and G_2/G_1 is isomorphic to $\mathrm{PSL}_2(2^n)$, $\mathrm{Sz}(2^{2n-1})$ or $\mathrm{PSU}_3(2^n)$ with $n \geq 2$.*

Lemma 2.8 ([3, Theorem] and [4, Proposition 4.7]). *Let G be a group whose Sylow 2-subgroups are either cyclic, dihedral or generalized quaternion. Then $\mathrm{Out_C}(G) = 1$.*

Lemma 2.9. *Let G be a (TI)-group. Then G has either a normal Sylow 2-subgroup or a strongly 2-embedded subgroup.*

Proof. Assume that G has no normal Sylow 2-subgroups. Let P be a Sylow 2-subgroup of G. We claim that $M := \mathrm{N}_G(P)$ is a strongly 2-embedded subgroup of G. It is clear that 2 divides $|M|$ and $M < G$. In addition, since by hypothesis G is a (TI)-group, it follows that 2 does not divide $|M \cap M^g|$ for any $g \in G \setminus M$. We are done. □

Lemma 2.10 ([16, Lemma 1]). *Let G be a (TI)-group. Then any subgroup of even order of G is a (TI)-group. If N is a normal subgroup of odd order of G, then G/N is a (TI)-group.*

3. Proof of Theorem A

In this section, we shall present a proof of Theorem A. To do this, we first prove the following result.

Theorem 3.1. *Let G be an extension of an odd order group by a simple group. Then $\mathrm{Out}_{\mathrm{C}}(G) = 1$. In particular, the normalizer property holds for G.*

Proof. Let M be a normal subgroup of odd order of G such that G/M is a simple group. According to whether G/M is abelian or not, we divide the proof of Theorem 3.1 into two cases.

Case 1. G/M is abelian.

If the quotient G/M is of order 2, then G must have a Sylow 2-subgroup of order 2, and thus the assertion follows from Lemma 2.1. If the quotient G/M is of odd prime order, then G is of odd order, and thus the assertion follows from Lemma 2.2.

Case 2. G/M is non-abelian.

Let $\sigma \in \mathrm{Aut}_{\mathrm{C}}(G)$. Next we shall show that $\sigma \in \mathrm{Inn}(G)$. Since σ^2 is inner, it follows that σ is inner provided that an odd power of σ is inner. Noticing this, by replacing σ with an odd power of it (if necessary), we may assume without loss that σ is of 2-power order.

Since $\sigma \in \mathrm{Aut}_{\mathrm{C}}(G)$, it follows that $\sigma|_{G/M} \in \mathrm{Aut}_{\mathrm{C}}(G/M)$. By hypothesis G/M is non-abelian simple, so by Lemma 2.3 $\sigma|_{G/M} \in \mathrm{Inn}(G/M)$, and thus $\sigma|_{G/M} = \mathrm{conj}(x)|_{G/M}$ for some $x \in G$. Modifying σ with a suitable inner automorphism, we may assume that

$$\sigma|_{G/M} = \mathrm{id}\,|_{G/M}. \tag{3.1}$$

Next we shall show that $\sigma|_M \in \mathrm{Aut}_{\mathrm{Col}}(M)$. Let P be an arbitrary Sylow subgroup of M. Since $\sigma \in \mathrm{Aut}_{\mathrm{C}}(G)$, it follows that there exists some $y \in G$ such that

$$\sigma|_P = \mathrm{conj}(y)|_P. \tag{3.2}$$

Write $H = M\mathrm{C}_G(P)$. Then by Lemma 2.4 H is normal in G and $H^\sigma = H$. Moreover,

$$\sigma|_{G/H} = \mathrm{conj}(y)|_{G/H}. \tag{3.3}$$

Note that $H/M \trianglelefteq G/M$ and G/M is simple, so either $H/M = G/M$ or H/M=1. It follows that $G = H$ or $H = M$.

If $G = H$, then $G = M\mathrm{C}_G(P)$, and thus we may write y as $y = cm$ with $c \in \mathrm{C}_G(P)$ and $m \in M$. It follows from (3.2) that

$$\sigma|_P = \mathrm{conj}(y)|_P = \mathrm{conj}(m)|_P. \tag{3.4}$$

If $H = M$, then (3.3) may be rewritten as

$$\sigma|_{G/M} = \mathrm{conj}(y)|_{G/M}. \tag{3.5}$$

Combining (3.1) with (3.5), we obtain

$$\mathrm{conj}(y)|_{G/M} = \sigma|_{G/M} = \mathrm{id}\,|_{G/M}, \tag{3.6}$$

which implies that $yM \in \mathrm{Z}(G/M)$. Since by hypothesis G/M is non-abelian simple, it follows that $\mathrm{Z}(G/M) = \bar{1}$, and thus $yM = M$, i.e., $y \in M$.

Consequently, no matter which case appears, by what we have just proved, the element y in the equality (3.2) can always be assumed to lie in M. As P is an arbitrary Sylow subgroup of M, we obtain $\sigma|_M \in \mathrm{Aut}_{\mathrm{Col}}(M)$. Note that M is of odd order, but $\sigma|_M$ is of even order. So by Proposition 1 in [5], we obtain that

$$\sigma|_M = \mathrm{id}\,|_M. \tag{3.7}$$

Then Lemma 2.6, (3.1) and (3.7) yield that

$$\sigma|_{G/\mathrm{O}_2(\mathrm{Z}(M))} = \mathrm{id}\,|_{G/\mathrm{O}_2(\mathrm{Z}(M))}. \tag{3.8}$$

But note that M is of odd order, so (3.8) implies that $\sigma = \mathrm{id}$.

This completes the proof of Theorem 3.1. □

Proof of Theorem A. In view of Lemma 2.7, we divide the proof of Theorem A into two cases.

Case 1. G has either a cyclic or a quaternion Sylow 2-subgroup.

The assertion follows from Lemma 2.8.

Case 2. G possesses a normal series $1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G$ such that G_1 and G/G_2 are of odd order, and G_2/G_1 is isomorphic to $\mathrm{PSL}_2(2^n)$, $\mathrm{Sz}(2^{2n-1})$ or $\mathrm{PSU}_3(2^n)$ with $n \geq 2$.

By Lemma 2.6, it suffices to show that $\mathrm{Out}_{\mathrm{C}}(G_2) = 1$. But note that G_2 is an extension of an odd order group G_1 by a simple group, so by Theorem 3.1 $\mathrm{Out}_{\mathrm{C}}(G_2) = 1$.

This completes the proof of Theorem A. □

Proof of Corollary B. Let G be a TI-group. Then by Lemma 2.9 G has either a normal Sylow 2-subgroup or a strongly 2-embedded subgroup. In the former case, the result follows from Lemma 2.6. In the latter case, the result follows from Theorem A. This completes the proof of Corollary B. □

As a direct consequence of Corollary B, we have the following more general result.

Corollary 3.2. *Let G be a (TI)-group. Then the following statements hold.*

(i) *Let H be an arbitrary subgroup of G. Then $\mathrm{Out}_{\mathrm{C}}(H) = 1$.*

(ii) *Let K be an arbitrary homomorphic image of G. Then $\mathrm{Out}_{\mathrm{C}}(K) = 1$.*

Proof. (i) Let H be an arbitrary subgroup of G. We have to show that $\mathrm{Out}_{\mathrm{C}}(H) = 1$. If H is of odd order, then the assertion follows immediately from Lemma 2.2. If H is of even order, then the assertion follows from Lemma 2.10 and Corollary B.

(ii) Let N be a normal subgroup of G. We have to show that $\mathrm{Out}_{\mathrm{C}}(G/N) = 1$. If N is of odd order, then by Lemma 2.10 G/N is a (TI)-group, and hence the assertion follows from Corollary B. Hereafter, we assume that N is of even order. According to Lemmas 2.7 and 2.9, we may divide the proof into three cases.

Case 1. G has a normal Sylow 2-subgroup.

It is easy to see that G/N also has a normal Sylow 2-subgroup and hence by Lemma 2.6 $\mathrm{Out}_{\mathrm{C}}(G/N) = 1$.

Case 2. G has a cyclic or generalized quaternion Sylow 2-subgroup.

Note that a factor group of a cyclic 2-group or a generalized quaternion 2-group is either cyclic, or a generalized 2-group, or $C_2 \times C_2$ (the dihedral group of order 4). Thus by Lemma 2.8 $\mathrm{Out}_{\mathrm{C}}(G/N) = 1$.

Case 3. G contains a normal series: $1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G$, where G/G_2 and G_1 are of odd order and G_2/G_1 is isomorphic to one of the groups $\mathrm{L}_2(q)$, $\mathrm{U}_3(q)$ or $\mathrm{G}(q)$. Here the notation of simple groups follows that of [15].

We claim that G/N is of odd order. Considering all possible relationships between N and G_2, we divide the proof of the preceding claim into three subcases.

Subcase 3.1. $N \geq G_2$

It is clear that G/N is of odd order since by hypothesis G/G_2 is.

Subcase 3.2. $N \leq G_2$.

Since G_2/G_1 is a simple group, it follows that either $NG_1/G_1 = 1$ or $NG_1/G_1 = G_2/G_1$. In the former case, we have $NG_1 = G_1$, and thus $N \leq G_1$, which is impossible since N is of even order and G_1 is of odd order. So we must have $NG_1/G_1 = G_2/G_1$, i.e., $G_2 = NG_1$. It follows that $G_2/N = NG_1/N \cong G_1/N \cap G_1$. By hypothesis G_1 is of odd order, so is G_2/N. Note that $G/N/G_2/N \cong G/G_2$ and G/G_2 is of odd order, so G/N is of odd order.

Subcase 3.3. $N \nleq G_2$ and $N \ngeq G_2$.

Write $M = N \cap G_2$. Then $1 \neq M \trianglelefteq G$. Note that $N/M = N/N \cap G_2 \cong NG_2/G_2 \leq G/G_2$ and G/G_2 is of odd order, so M must be of even order. Note further that $M \leq G_2$. Then by Subcase 3.2 G/M is of odd order. Note that $G/N \cong G/M/N/M$, so G/N is of odd order.

Thus in any subcase G/N is of odd order, as claimed. If follows from Lemma 2.2 that $\mathrm{Out}_{\mathrm{C}}(G/N) = 1$.

This completes the proof of Corollary 3.2. □

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