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A variant of Wilson's functional equation

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Abstract. In the present paper we determine the complex-valued solutions (f, g) of the functional equation

$$f(xy) + f(\sigma(y)x) = 2f(x)g(y),$$

in the setting of groups and monoids that need not be abelian, where σ is an involutive automorphism.

1. Introduction

In [7] Wilson dealt with functional equations related to and generalizing the cosine functional equation $g(x + y) + g(x - y) = 2g(x)g(y)$ on the real line. He generalized the cosine equation to

$$f(x + y) + f(x - y) = 2f(x)g(y), \quad x, y \in \mathbb{R}, \tag{1.1}$$

that contains the two unknown functions f and g. In [8] he introduced his second generalization

$$f(x + y) + f(x - y) = 2g(x)h(y), \quad x, y \in \mathbb{R}, \tag{1.2}$$

that contains the three unknown functions f, g and h. These functional equations have been extended to abelian groups: You just replace the domain of definition $\mathbb{R}$ by an abelian group $(G, +)$. They have been solved in that setting. The equation (1.1) has even been extended to general groups (see, e.g., [5, Chapter 11]).

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The purpose of the present paper is to solve the following equation

$$f(xy)+f(\sigma(y)x)=2f(x)g(y), \quad x,y\in M, \tag{1.3}$$

where M is a possibly non-abelian group or monoid (that is, a semigroup with identity) and $\sigma : M \to M$ is an involutive automorphism, for unknown functions $f,g : M \to \mathbb{C}$. A special case of (1.3) is the variant $f(xy)+f(\sigma(y)x)=2f(x)$ of Jensen's functional equation.

As a consequence we obtain the solutions $f,g,h : M \to \mathbb{C}$ of the more general functional equation

$$f(xy)+f(\sigma(y)x)=2g(x)h(y), \quad x,y\in M. \tag{1.4}$$

Note that the variant (1.3) of Wilson's functional equation is a generalization of the equation

$$g(xy)+g(\sigma(y)x)=2g(x)g(y), \quad x,y\in M, \tag{1.5}$$

which was introduced and solved on semigroups by H. Stetkær in [6]. Taking M an abelian group, our equation becomes

$$f(x+y)+f(x+\sigma(y))=2f(x)g(y), \quad x,y\in M, \tag{1.6}$$

which was solved by H. Stetkær in [4].

By elementary methods we find all solutions of (1.3) on monoids that are generated by their squares and on groups, in terms of multiplicative and additive functions. This contrasts the solutions of the functional equation $f(xy)+f(y^{-1}x)=2f(x)g(y)$, where the non-abelian phenomena like 2-dimensional irreducible representations may occur (see [1]). Our formulas for the solutions of (1.3) on groups are the same as those on abelian groups, so that our results constitute a natural extension of earlier results of, e.g., [4], from the abelian to the non-abelian case. Finally, we note that the sine addition law on semigroups given in [2] is a key ingredient of the proof of our main results (Theorem 3.6 and Theorem 3.7).

2. Notation and terminology

To formulate our results we introduce the following notation and assumptions that will be used throughout the paper:
Let G be a group and M a monoid, that is a semigroup (a set with an associative composition rule) with an identity element that we denote e. The map

$\sigma : M \to M$ denotes an involutive automorphism. That it is involutive means that $\sigma(\sigma(x)) = x$ for all $x \in M$. It is easy to derive that $\sigma(e) = e$ for any involutory automorphism $\sigma : M \to M$ (see [5, Lemma A.31]). If $(G, +)$ is an abelian group, then the inversion $\sigma(x) := -x$ is an example of an involutive automorphism. Another example is the complex conjugation map on the multiplicative group of nonzero complex numbers. For more examples of involutive automorphisms we refer, e.g., to [2].

For any complex-valued function F on M we let F_e and F_o denote the even and odd parts of F with respect to σ, i.e.,

$$F_e = \frac{F + F \circ \sigma}{2} \quad \text{and} \quad F_o = \frac{F - F \circ \sigma}{2}.$$

We say that F is even if $F = F_e$, and odd if $F = F_o$.

A function $f : M \to \mathbb{C}$ is abelian, if

$$f(x_{\pi(1)} x_{\pi(2)} \dots x_{\pi(n)}) = f(x_1 x_2 \dots x_n)$$

for all $x_1, x_2, \dots, x_n \in M$, all permutations π of n elements and all $n = 2, 3, \dots$. On abelian monoids all functions are abelian. Any abelian function f is central, meaning $f(xy) = f(yx)$ for all $x, y \in M$.

A function $a : M \to \mathbb{C}$ is called additive, if it satisfies $a(xy) = a(x) + a(y)$ for all $x, y \in M$.

A multiplicative function on M is a map $\chi : M \to \mathbb{C}$ such that $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in M$. A character on a group G is a homomorphism from G into the multiplicative group of non-zero complex numbers. While a non-zero multiplicative function on a group can never take the value 0, it is possible for a multiplicative function on a monoid to take the value 0 on a proper, non-empty subset of M. If $\chi : M \to \mathbb{C}$ is multiplicative and $\chi \neq 0$, then

$$I_\chi = \{x \in M \mid \chi(x) = 0\}$$

is either empty or a proper subset of M. The fact that χ is multiplicative establishes that I_χ is a two-sided ideal in M if not empty (for us an ideal is never the empty set). It follows also that $M \setminus I_\chi$ is a subsemigroup of M. These ideals play an essential role in our discussion of equation (1.3) on monoids.

If M is a topological space, then we let $C(M)$ denote the algebra of continuous functions from M into $\mathbb{C}$.

3. Main results

We first give a result for the sine addition law on monoids. All of this comes directly from Lemma 3.4 in [2]. For the notation I_χ see the section Notation and terminology.

Lemma 3.1. *Let M be a monoid, and suppose $f, g : M \to \mathbb{C}$ satisfy the sine addition law*

$$f(xy) = f(x)g(y) + f(y)g(x), \quad x, y \in M,$$

with $f \neq 0$. Then there exist multiplicative functions $\chi_1, \chi_2 : M \to \mathbb{C}$ such that

$$g = \frac{\chi_1 + \chi_2}{2}.$$

Additionally we have the following.

(i) *If $\chi_1 \neq \chi_2$, then $f = c(\chi_1 - \chi_2)$ for some constant $c \in \mathbb{C} \setminus \{0\}$.*

(ii) *If $\chi_1 = \chi_2$, then letting $\chi := \chi_1$ we have $g = \chi \neq 0$.*

If M is a group, then there is an additive function $a : M \to \mathbb{C}$, $a \neq 0$, such that $f = a\chi$.

If M is a monoid which is generated by its squares, then there exists an additive function $a : M \setminus I_\chi \to \mathbb{C}$ for which

$$f(x) = \begin{cases} a(x)\chi(x) & \text{for } x \in M \setminus I_\chi \\ 0 & \text{for } x \in I_\chi. \end{cases}$$

Furthermore, if M is a topological group, or if M is a topological monoid generated by its squares, and $f, g \in C(M)$, then $\chi_1, \chi_2, \chi \in C(M)$. In the group case $a \in C(M)$ and in the second case $a \in C(M \setminus I_\chi)$.

In the following Lemma we derive some properties of solutions of (1.3). We prove these results on a monoid M.

Lemma 3.2. *Let M be a monoid, let σ be an involutive automorphism on M, and let the pair $f, g : M \to \mathbb{C}$ be a solution of the functional equation (1.3) such that $f \neq 0$.*

(a) *The even part $(f + f \circ \sigma)/2$ of f is $f(e)g$, so it is proportional to g.*

(b) *f is odd if and only if $f(e) = 0$.*

(c) *(f_o, g) is a solution of the sine addition law, i.e.,*

$$f_o(xy) = f_o(x)g(y) + f_o(y)g(x) \quad \text{for all } x, y \in M.$$

(d) *Both f and g are abelian functions.*

(3) *g is even.*

PROOF. (a) and (b) Taking $x = e$ in (1.3) we get $f(y)+f(\sigma(y)) = 2f(e)g(y)$ which is (a). Using the same identity we get (b).

(c) The method used here is closely related to and inspired by the one in [6, Proof of Theorem 2.1]. Let $a, b, c \in M$ be arbitrary. With $x = ab$ and $y = c$ the equation (1.3) becomes

$$f(abc) + f(\sigma(c)ab) = 2f(ab)g(c). \tag{3.1}$$

To get rid of σ in the second term on the left hand side of (3.1) we take $x = \sigma(c)a$ and $y = b$ which gives us

$$f(\sigma(c)ab) + f(\sigma(b)\sigma(c)a) = 2f(\sigma(c)a)g(b) = 2g(b)[2f(a)g(c) - f(ac)]. \tag{3.2}$$

We reformulate the second term on the left hand side of (3.2) as follows

$$f(\sigma(b)\sigma(c)a) = f(\sigma(bc)a) = 2f(a)g(bc) - f(abc),$$

which turns the identity (3.2) into

$$f(\sigma(c)ab) + 2f(a)g(bc) - f(abc) = 4f(a)g(b)g(c) - 2f(ac)g(b).$$

Subtracting this from (3.1) we get after some simplifications that

$$f(abc) - f(a)g(bc) = [f(ab) - f(a)g(b)]g(c) + [f(ac) - f(a)g(c)]g(b). \tag{3.3}$$

With the notation $h_x(y) := f(xy) - f(x)g(y)$ we can reformulate (3.3) to

$$h_a(bc) = h_a(b)g(c) + h_a(c)g(b).$$

This shows that the pair (h_a, g) satisfies the sine addition law for any $a \in M$. In particular for $a = e$. Since $h_e = f - f(e)g = f - f_e = f_o$ then (f_o, g) is a solution of the sine addition law.

(d) If $f_o \neq 0$, we know from [5, Theorem 4.1] that both f_o and g are abelian functions. Since $f = f_e + f_o = f(e)g + f_o$, then f is also an abelian function.

If $f_o = 0$, then we see that $f = f_e = f(e)g$ and $f(e) \neq 0$. Indeed, $f(e) = 0$ would entail $f = 0$, contradicting our assumption. So g is a solution of the functional equation

$$g(xy) + g(\sigma(y)x) = 2g(x)g(y) \quad \text{for all } x, y \in M.$$

According to [6, Theorem 2.1], there exists a multiplicative function $\chi : M \to \mathbb{C}$ such that $g = (\chi + \chi \circ \sigma)/2$. Then g is an abelian function and hence so is $f = f(e)g$.

(e) Since f is abelian, then f is central. So the functional equation (1.3) becomes

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in M, \tag{3.4}$$

Replacing y by $\sigma(y)$ does not change the left hand side of (1.3). Thus we arrive at $f(x)g(y) = f(x)g(\sigma(y))$ for all $x, y \in M$. Since $f \neq 0$, then $g = g \circ \sigma$ i.e. g is even. $\square$

In the following corollary, we solve the variant of Jensen's functional equation, namely

$$f(xy) + f(\sigma(y)x) = 2f(x), \quad x, y \in M, \tag{3.5}$$

on monoids.

Corollary 3.3. *Let M be a monoid and let σ be an involutive automorphism on M. The solutions $f : M \to \mathbb{C}$ of* (3.5) *are the functions of the form $f = a + \alpha$, where $a : M \to \mathbb{C}$ is an additive map such that $a \circ \sigma = -a$, and where α is a complex constant.*

Proof. It is clear that $f \equiv 0$ is a solution of (3.5), so we suppose that $f \neq 0$. On putting $g = 1$ in Lemma 3.2 we get that $f_e = f(e)$ and f_o is additive, so $f = f_e + f_o$ has the desired form. The other direction of the proof is trivial to verify. $\square$

Proposition 3.4. *Let M be a monoid, let σ be an involutive automorphism on M, and let the pair $f, g : M \to \mathbb{C}$ be a solution of the functional equation* (1.3) *such that $f \neq 0$.*

(a) *g is a solution of the variant* (1.5) *of d'Alembert's functional equation.*

(b) *Both f_e and f_o satisfy* (1.3) *with the same g as for f.*

Proof. (a) Choose $x_0 \in M$ such that $f(x_0) \neq 0$. Using (1.3) and the fact that f is abelian we get

$$\begin{aligned} 2f(x_0)[g(xy) + g(\sigma(y)x)] &= f(x_0xy) + f(\sigma(x)\sigma(y)x_0) + f(x_0\sigma(y)x) + f(y\sigma(x)x_0) \\ &= f(x_0xy) + f(x_0\sigma(y)x) + f(y\sigma(x)x_0) + f(\sigma(x)\sigma(y)x_0) \\ &= f(x_0xy) + f(\sigma(y)x_0x) + f(\sigma(x)x_0y) + f(\sigma(y)\sigma(x)x_0) \\ &= 2f(x_0x)g(y) + 2f(\sigma(x)x_0)g(y) \\ &= 2[f(x_0x) + f(\sigma(x)x_0)]g(y) = 4f(x_0)g(x)g(y), \end{aligned}$$

for all $x, y \in M$, which implies that g is a solution of (1.5).

(b) Since $f_e = f(e)g$ and g satisfies (1.5), then f_e satisfies (1.3) with the same g as for f and hence also $f_o = f - f_e$. □

As a consequence of Proposition 3.4, in the special case of $\sigma = \mathrm{id}$, we have the following result.

Corollary 3.5. *Let M be a monoid and let the pair $f, g : M \to \mathbb{C}$ be a solution of the functional equation*

$$f(xy) + f(yx) = 2f(x)g(y), \quad x, y \in M, \tag{3.6}$$

such that $f \neq 0$. Then there exists a multiplicative function χ on M such that $f = \alpha\chi$ and $g = \chi$ for some $\alpha \in \mathbb{C}$.

Proof. On putting $\sigma = \mathrm{id}$ in Proposition 3.4 (a) we see that g satisfies the symmetrized multiplicative Cauchy equation, that is

$$\frac{g(xy) + g(yx)}{2} = g(x)g(y), \quad x, y \in M.$$

Then g is a multiplicative function (see [5, Theorem 3.21]) . Interchanging x and y in (3.6) does not change the left hand side. Thus we arrive at $f(x)g(y) = f(y)g(x)$ for all $x, y \in M$. Since $f \neq 0$, then f is proportional to g. This finishes the proof. □

The following theorem solves the variant (1.3) of Wilson's functional equation on an arbitrary group. For abelian groups it generalizes many results (see, e.g., [3, Lemma 4.2] and [4, Theorem III.4]).

Theorem 3.6. *Let G be a group, let σ be an involutive automorphism on G, and let the pair $f, g : G \to \mathbb{C}$ be a solution of the functional equation* (1.3) *such that $f \neq 0$. Then there exists a character χ of G such that $g = (\chi + \chi \circ \sigma)/2$. Furthermore, we have the following possibilities:*

(i) *If $\chi \neq \chi \circ \sigma$, then*

$$f = \alpha\frac{\chi + \chi \circ \sigma}{2} + \beta\frac{\chi - \chi \circ \sigma}{2},$$

for some $\alpha, \beta \in \mathbb{C}$.

(ii) *If $\chi = \chi \circ \sigma$, then there exists an additive function $a : G \to \mathbb{C}$ with $a \circ \sigma = -a$ such that*

$$f = \alpha\chi + a\chi,$$

for some $\alpha \in \mathbb{C}$.

Conversely, the formulas above for g and f define solutions of (1.3).

Moreover, if G is a topological group, and $f, g \in C(G)$, then $\chi, \chi\circ\sigma, a \in C(G)$.

The monoid version (Theorem 3.7) differs from Theorem 3.6 only when $\chi = \chi \circ \sigma$ (case (ii)), where the formulations are more complicated. The conclusions of the two versions agree if χ vanishes nowhere, which is the case on groups.

Theorem 3.7. *Let M be a monoid which is generated by its squares, let σ be an involutive automorphism on M, and let the pair $f, g : M \to \mathbb{C}$ be a solution of the functional equation* (1.3) *such that $f \neq 0$. Then there exists a multiplicative function $\chi : M \to \mathbb{C}$, $\chi \neq 0$, such that $g = (\chi + \chi \circ \sigma)/2$. Furthermore, we have the following possibilities:*

(i) *If $\chi \neq \chi \circ \sigma$, then*

$$f = \alpha \frac{\chi + \chi \circ \sigma}{2} + \beta \frac{\chi - \chi \circ \sigma}{2},$$

for some $\alpha, \beta \in \mathbb{C}$.

(ii) *If $\chi = \chi \circ \sigma$, then there exists an additive function $a : M \setminus I_\chi \to \mathbb{C}$ with $a \circ \sigma = -a$ such that*

$$f(x) = \begin{cases} \alpha\chi(x) + a(x)\chi(x) & \text{for } x \in M \setminus I_\chi \\ 0 & \text{for } x \in I_\chi \end{cases}$$

for some $\alpha \in \mathbb{C}$.

Conversely, the formulas above for g and f define solutions of (1.3).

Moreover, if M is a topological monoid generated by its squares, and $f, g \in C(M)$, then $\chi, \chi \circ \sigma \in C(M)$, while $a \in C(M \setminus I_\chi)$.

Proof of Theorems 3.6 and 3.7. From Proposition 3.4 (a) we see that g is a solution of (1.5). According to [6, Theorem 2.1], there exists a multiplicative function $\chi : M \to \mathbb{C}$ such that $g = (\chi + \chi \circ \sigma)/2$. If $\chi = 0$, we get $g = 0$ which implies that $f = 0$ contradicting our assumption. Hence $\chi \neq 0$. Additionally, by [5, Corollary 3.19] we see that χ is unique except that it can be interchanged by $\chi \circ \sigma$.

Let f_e and f_o denote the even and the odd parts of f. We see from Lemma 3.2 that $f_e = f(e)g$, and that (f_o, g) is a solution of the sine addition law.

Assume first f even. Hence $f_o = 0$, so that

$$f = f_e = f(e)g = f(e)\frac{\chi + \chi \circ \sigma}{2},$$

so we are in case (i) or (ii).

Assume next f not even. Hence $f_o \neq 0$. Since $g = (\chi + \chi \circ \sigma)/2$ and χ is unique, except that it can be interchanged by $\chi \circ \sigma$, we may apply Lemma 3.1 with $\chi_1 = \chi$ and $\chi_2 = \chi \circ \sigma$ to find the form of f_o.

(i) Assume that $\chi \neq \chi \circ \sigma$, then we get that

$$f_o = \beta \frac{\chi - \chi \circ \sigma}{2},$$

for some $\beta \in \mathbb{C} \setminus \{0\}$, so that

$$f = f_e + f_o = f(e)g + \beta \frac{\chi - \chi \circ \sigma}{2} = \alpha \frac{\chi + \chi \circ \sigma}{2} + \beta \frac{\chi - \chi \circ \sigma}{2},$$

where $\alpha = f(e)$.

(ii) Assume that $\chi = \chi \circ \sigma$. If M is a group we get from Lemma 3.1 that $f_o = a\chi$ for some additive function a. From f_o being odd with respect to σ we see that $a \circ \sigma = -a$. Thus

$$f = f_e + f_o = f(e)g + f_o = \alpha\chi + a\chi,$$

where $\alpha = f(e)$.

If M is a monoid which is generated by its squares, we get from Lemma 3.1 that there exists an additive function $a : M \setminus I_\chi \to \mathbb{C}$ for which

$$f_o(x) = \begin{cases} a(x)\chi(x) & \text{for } \ x \in M \setminus I_\chi \\ 0 & \text{for } \ x \in I_\chi. \end{cases}$$

From f_o being odd with respect to σ we see that $a \circ \sigma = -a$. Since $f = f_e + f_o = f(e)g + f_o$ and $\chi(x) = 0$ for $x \in I_\chi$ we have

$$f(x) = \begin{cases} \alpha\chi(x) + a(x)\chi(x) & \text{for } \ x \in M \setminus I_\chi \\ 0 & \text{for } \ x \in I_\chi \end{cases}$$

where $\alpha = f(e)$.

Conversely, simple computations prove that the formulas above for g and f define solutions of (1.3).

The continuity statements follow from Lemma 3.1. $\square$

Example 3.8. For an application of our results on a non-abelian monoid, let $M = M(2, \mathbb{C})$ be the set of complex 2×2 matrices under matrix multiplication. Note that M is generated by its squares (see [2]). Let σ be the complex conjugation operator

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \quad \text{for } a, b, c, d \in \mathbb{C}.$$

We indicate here the corresponding continuous solutions of (1.3). We write $R(\lambda)$ (resp. $\mathrm{Im}(\lambda)$) for the real part of the complex number λ (resp. the imaginary part of λ).

The continuous non-zero multiplicative functions on M are (see [2, Example 5.6]): $\chi = 1$, or else

$$\chi(X) = \begin{cases} |\det(X)|^{\lambda-n}(\det(X))^n & \text{when } \det(X) \neq 0 \\ 0 & \text{when } \det(X) = 0 \end{cases}$$

where $\lambda \in \mathbb{C}$ with $R(\lambda) > 0$ and $n \in \mathbb{Z}$.

Let us first consider the case of $\chi \neq 1$. Since $|\det(X)| = |\det(\sigma(X))|$, we have $\chi \circ \sigma \neq \chi$ if and only if $n \neq 0$.

In the case $\chi \circ \sigma = \chi$ we have $n = 0$ so

$$\chi(X) = \begin{cases} |\det(X)|^{\lambda} & \text{when } \det(X) \neq 0 \\ 0 & \text{when } \det(X) = 0. \end{cases}$$

In this case we have $I_\chi = \{X \in M(2, \mathbb{C}| \det(X) = 0\}$.

In view of [2, Example 5.6], the continuous additive functions on $M \setminus I_\chi$ satisfy the condition $a \circ \sigma = -a$ only if $a = 0$. The same is true for $\chi = 1$, where $a : M \to \mathbb{C}$.

In conclusion, the continuous solutions $f, g : M(2, \mathbb{C}) \to \mathbb{C}$, where $f \neq 0$, of (1.3) are:

(a) $g = 1$ and $f = \alpha$, where α is a non-zero complex number;

(b)

$$g(X) = \begin{cases} |\det(X)|^{\lambda} & \text{when } \det(X) \neq 0 \\ 0 & \text{when } \det(X) = 0 \end{cases}$$

$$f(X) = \alpha g(X),$$

where α, λ are complex numbers such that $\alpha \neq 0$ and $R(\lambda) > 0$; and

(c)

$$g(X) = \begin{cases} |\det(X)|^{\lambda-n} R((\det(X))^n) & \text{when } \det(X) \neq 0 \\ 0 & \text{when } \det(X) = 0 \end{cases}$$

$$f(X) = \begin{cases} |\det(X)|^{\lambda-n}[\alpha R((\det(X))^n) & \text{when } \det(X) \neq 0 \\ \quad +\beta \mathrm{Im}((\det(X))^n)] & \\ 0 & \text{when } \det(X) = 0 \end{cases}$$

where α, β, λ are complex numbers such that $R(\lambda) > 0$ and $n \in \mathbb{Z} \setminus \{0\}$.

In the following lemma, we give a characterization of solutions of (1.4).

Lemma 3.9. *Let M be a monoid, let σ be an involutive automorphism on M, and let the triple $f, g, h : M \to \mathbb{C}$ be a solution of the functional equation* (1.4). *Then we have the following possibilities:*

(a) $f = 0$, $g = 0$, *h is arbitrary.*

(b) $f = 0$, $h = 0$, *g is arbitrary.*

(c) $f = h(e)g$, *where $h(e) \neq 0$, and g is a solution of* (1.3) *with companion function $h/h(e)$, i.e.,*

$$g(xy) + g(\sigma(y)x) = 2g(x)\frac{h(y)}{h(e)}, \quad x, y \in M.$$

Proof. The first two cases are obvious, so we suppose that $f \neq 0$. Taking $y = e$ in (1.4) we get $f = h(e)g$. If $h(e) = 0$ we get $f = 0$ contradicting our assumption. Hence $h(e) \neq 0$. Replacing f by $h(e)g$ in (1.4), we obtain the identity in (c). □

In view of Theorem 3.6 and Lemma 3.8 we find the complete solution of (1.4) on an arbitrary group.

Theorem 3.10. *Let G be a group, let σ be an involutive automorphism on G, and let the triple $f, g, h : G \to \mathbb{C}$ be a solution of the functional equation* (1.4). *Then we have the following possibilities:*

(a) $f = 0$, $g = 0$, *h is arbitrary.*

(b) $f = 0$, $h = 0$, *g is arbitrary.*

(c) *There exists a character χ of G such that $h = \frac{\gamma}{2}(\chi + \chi \circ \sigma)$ for some constant $\gamma \in \mathbb{C} \setminus \{0\}$. Furthermore:*

  (i) *If $\chi \neq \chi \circ \sigma$, then*

$$g = \alpha\frac{\chi + \chi \circ \sigma}{2} + \beta\frac{\chi - \chi \circ \sigma}{2},$$
$$f = \alpha\gamma\frac{\chi + \chi \circ \sigma}{2} + \beta\gamma\frac{\chi - \chi \circ \sigma}{2},$$

  for some $\alpha, \beta \in \mathbb{C}$.

  (ii) *If $\chi = \chi \circ \sigma$, then there exists an additive function $a : G \to \mathbb{C}$ with $a \circ \sigma = -a$ such that*

$$g = \alpha\chi + a\chi, \qquad f = \gamma(\alpha\chi + a\chi),$$

  for some $\alpha \in \mathbb{C}$.

Moreover, if G is a topological group, and $f, g, h \in C(G)$, then $\chi, \chi \circ \sigma, a \in C(G)$.

Conversely, the formulas above for f, g and h define solutions of (1.4).

The monoid version (Theorem 3.10) of Theorem 3.9 is a consequence of Lemma 3.8 and Theorem 3.7.

Theorem 3.11. *Let M be a monoid which is generated by its squares, let σ be an involutive automorphism on M, and let the triple $f, g, h : M \to \mathbb{C}$ be a solution of the functional equation* (1.4). *Then we have the following possibilities:*

(a) *$f = 0$, $g = 0$, h is arbitrary.*

(b) *$f = 0$, $h = 0$, g is arbitrary.*

(c) *There exists a multiplicative function $\chi : M \to \mathbb{C}$, $\chi \neq 0$, such that $h = \frac{\gamma}{2}(\chi + \chi \circ \sigma)$ for some constant $\gamma \in \mathbb{C} \setminus \{0\}$. Furthermore:*

(i) *If $\chi \neq \chi \circ \sigma$, then*

$$g = \alpha \frac{\chi + \chi \circ \sigma}{2} + \beta \frac{\chi - \chi \circ \sigma}{2},$$
$$f = \alpha\gamma \frac{\chi + \chi \circ \sigma}{2} + \beta\gamma \frac{\chi - \chi \circ \sigma}{2},$$

for some $\alpha, \beta \in \mathbb{C}$.

(ii) *If $\chi = \chi \circ \sigma$, then there exists an additive function $a : M \setminus I_\chi \to \mathbb{C}$ with $a \circ \sigma = -a$ such that*

$$g(x) = \begin{cases} \alpha\chi(x) + a(x)\chi(x) & \text{for } x \in M \setminus I_\chi \\ 0 & \text{for } x \in I_\chi \end{cases}$$

and

$$f(x) = \begin{cases} \gamma(\alpha\chi(x) + a(x)\chi(x)) & \text{for } x \in M \setminus I_\chi \\ 0 & \text{for } x \in I_\chi \end{cases}$$

for some $\alpha \in \mathbb{C}$.

Moreover, if M is a topological monoid generated by its squares, and $f, g, h \in C(M)$, then $\chi, \chi \circ \sigma \in C(M)$, while $a \in C(M \setminus I_\chi)$.

Conversely, the formulas above for f, g and h define solutions of (1.4).

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