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Groups with a few nonabelian centralizers

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Abstract. For a group G, let $\mathrm{cent}(G)$ denote the set of centralizers of single elements of G and $\mathrm{nacent}(G)$ denote the set of all nonabelian centralizers belonging to $\mathrm{cent}(G)$. We first characterize all finite groups G with $|\mathrm{nacent}(G)| = 2$. We denote by $\omega(G)$, the maximum possible size of a subset of pairwise noncommuting elements of a finite group G. In this article we find a necessary and sufficient condition for some finite groups G satisfying $|\mathrm{cent}(G)| = |\mathrm{nacent}\ (G)| + \omega(G)$. In particular we show that this equality is valid for some simple groups.

1. Introduction and main results

Throughout this paper G is a finite nonabelian group and $Z(G)$ is its center. We denote by $\mathrm{cent}(G) = \{C_G(g) : g \in G\}$ where $C_G(g)$ is the centralizer of the element g in G and $\mathrm{nacent}(G)$ denotes the set of all nonabelian centralizers belonging to $\mathrm{cent}(G)$. A subgroup H of G is called a proper centralizer of G if $H = C_G(x)$ for some $x \in G \setminus Z(G)$. In recent years many authors have studied the influence of $|\mathrm{cent}(G)|$ on the structure of the group G (see [1], [10], [11] and [14], [15], [16]).

Schmidt [13] characterized all groups G with $|\mathrm{nacent}(G)| = 1$ (such groups are called CA-groups). In this article we determine all groups G with $|\mathrm{nacent}(G)| = 2$.

Theorem 1.1. *Let G be a group such that $|\mathrm{nacent}(G)| = 2$. If $C_G(a)$ is a proper nonabelian centralizer for some $a \in G$, then one of the following holds:*

(1) $\frac{G}{Z(G)}$ *is a p-group for some prime p.*

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(2) *$C_G(a)$ is the Fitting subgroup of G of prime index p, p divides $|C_G(a)|$ and $|\mathrm{cent}(G)| = |\mathrm{cent}(C_G(a))|+j+1$, where j is the number of distinct centralizers $C_G(g)$ for $g \in G \setminus C_G(a)$.*

(3) *$\frac{G}{Z(G)}$ is a Frobenius group with cyclic Frobenius complement $\frac{C_G(x)}{Z(G)}$ for some $x \in G$.*

Let G be a finite group and let X be a subset of pairwise noncommuting elements of G such that $|X| \geq |Y|$ for any other set of pairwise noncommuting elements Y in G. Then the subset X is said to have the maximum size and this size is denoted by $\omega(G)$. Various attempts have been made to find $\omega(G)$ for some groups G (see for example [3–6] and [12]).

The following result states the relation among $|\mathrm{cent}(G)|$,$\omega(G)$ and $|\mathrm{nacent}(G)|$ for some groups.

Theorem 1.2. *Let G be a finite group and $k \leq 6$ be a positive integer. Then $|\mathrm{nacent}(G)| = k$ if and only if $|\mathrm{cent}(G)| = \omega(G) + k$.*

So the natural question is to find the largest integer k such that if $|\mathrm{nacent}(G)| \leq k$ for some group G, then $|\mathrm{cent}(G)| - \omega(G) = |\mathrm{nacent}(G)|$. It seems that $k = 7$.

If $n > 0$ is an integer and q is a power of a prime p, then we denote by $\mathrm{PSL}(n, q)$ and $\mathrm{Sz}(q)$, the projective special linear group of degree n over the finite field of size q and the Suzuki group over the field with q elements, respectively. In what follows we give some simple groups G where $|\mathrm{cent}(G)| = |\mathrm{nacent}(G)|+\omega(G)$.

We notice that there is a minor error in Theorem 2.1 of [15] when $|\mathrm{cent}(\mathrm{Sz}(q))|$ was computed and we correct it in Theorem 1.3 (2).

Theorem 1.3.

(1) *If $G = \mathrm{PSL}(2, q)$ or $\mathrm{PSL}(3, 3)$, then $|\mathrm{cent}(G)| = |\mathrm{nacent}(G)| + \omega(G)$.*

(2) *If $G = \mathrm{Sz}(q)$ where $q = 2^{2m+1}$ and $m > 0$, then*

$$|\mathrm{cent}(G)| = (q^2 + 1)(q - 1) + \frac{q^2(q^2 + 1)}{2} + \frac{q^2(q^2 + 1)(q - 1)}{4(q + 2r + 1)} + \frac{q^2(q^2 + 1)(q - 1)}{4(q - 2r + 1)} + (q^2 + 2)$$

where $r = 2^m$ and also $|\mathrm{nacent}(G)| = |\mathrm{cent}(G)| - \omega(G) = q^2 + 2$.

Note that there are some simple groups G for which the equality $|\mathrm{cent}(G)| - \omega(G) = |\mathrm{nacent}(G)|$ does not hold. For example, it can be checked by GAP [8] that $|\mathrm{cent}(A_8)| = 5448$, $|\omega(A_8)| = 4201$ and $|\mathrm{nacent}(A_8)| = 1562$ where A_8 is the alternating group of degree 8.

2. Proof of the main results

Definition 2.1. Let G be a group. A set $\prod = \{H_1, \ldots, H_n\}$ of subgroups $H_i (i = 1, \ldots, n)$ is called a partition of G if every element $x \in G \setminus \{1\}$ belongs to one and only one subgroup $H_i \in \prod$.

For a finite group G, let $\mathrm{Fit}(G)$ denote the Fitting subgroup of G, i.e., the largest normal nilpotent subgroup of G. In what follows we determine all groups with $|\mathrm{nacent}(G)| = 2$.

Proof of Theorem 1.1. Suppose that $X = \{x_1, \ldots, x_{\omega(G)}\}$ is a set of pairwise non-commuting elements of G having maximum size. Then $C_G(x_i)$ is abelian for each i by Proposition 2.5 (a)(1) of [1]. Put $Y = \{x_i \in X : ax_i = x_i a\}$. Then it is easy to see that $C_G(y) \subset C_G(a)$ for every $y \in Y$ and $C_G(x) \cap C_G(a) = Z(G)$ for every $x \in X \setminus Y$. Also we have $C_G(g) \cap C_G(g') = Z(G)$ for every $g \neq g' \in X \setminus Y$. Therefore $\Pi = \left\{\frac{C_G(a)}{Z(G)}, \frac{C_G(x)}{Z(G)} : x \in X \setminus Y\right\}$ is a partition of $\frac{G}{Z(G)}$. Since $Z(C_G(a))$ is a normal subgroup of G, we have $\mathrm{Fit}\left(\frac{G}{Z(G)}\right) \neq 1$. It follows from Satz 3 of [7] that one of the following holds:

(i) $\frac{G}{Z(G)}$ is a p-group for some prime p.

(ii) $\mathrm{Fit}\left(\frac{G}{Z(G)}\right) \in \Pi$ and $\left|\frac{G}{Z(G)} : \mathrm{Fit}\left(\frac{G}{Z(G)}\right)\right| = p$, p is a prime divisor of $\left|\mathrm{Fit}\left(\frac{G}{Z(G)}\right)\right|$.

(iii) $\frac{G}{Z(G)}$ is a Frobenius group.

(iv) $\frac{G}{Z(G)} \cong S_4$.

We first prove that $\frac{G}{Z(G)} \ncong S_4$. Suppose, for a contradiction, that $\frac{G}{Z(G)} \cong S_4$. Since $C_G(a)$ is a proper normal subgroup of G (note that $C_G(a)$ is the unique proper nonabelian centralizers in G), we have $\frac{C_G(a)}{Z(G)} \cong A_4$ or $C_2 \times C_2$. It follows that $\frac{C_G(a)}{Z(C_G(a))}$ is cyclic which is a contradiction.

In case (ii), we conclude that $\mathrm{Fit}(G) = C_G(a)$ and $\left|\frac{G}{C_G(a)}\right| = p$. Therefore $\left|\frac{C_G(x)}{Z(G)}\right| = p$ for each $x \in X \setminus Y$. Also we have $|\mathrm{cent}(G)| = |\mathrm{cent}(C_G(a))| + j + 1$ where $j = |X \setminus Y|$, as wanted.

Finally if $\frac{G}{Z(G)}$ is a Frobenius group, then $\frac{C_G(a)}{Z(G)}$ is contained in Frobenius kernel and $\frac{C_G(x)}{Z(G)}$ is a Frobenius complement for some $x \in X \setminus Y$. Since $\frac{C_G(x)}{Z(G)}$ is abelian, we have the result by Corollary 6.17 of [9]. This completes the proof. $\square$

In the following result we give a lower bound for $|\mathrm{nacent}(G)|$.

Proposition 2.2. *Let G be a group. Then $|\mathrm{cent}(G)| - \omega(G) \leq |\mathrm{nacent}(G)|$. The equality occurs if and only if $\omega(G)$ is the number of abelian centralizers belonging to $\mathrm{cent}(G)$.*

Proof. Put $s = |\mathrm{cent}(G)| - |\mathrm{nacent}(G)|$. If $s = 0$, then we have the result. So suppose that $s > 0$ and $\{C_G(x_1), \ldots, C_G(x_s)\}$ is the set of all abelian centralizers which belong to $\mathrm{cent}(G)$. Then $\{x_1, \ldots, x_s\}$ is a set of pairwise noncommuting elements of G and so $\omega(G) \geq s$. The proof is now complete. □

Proposition 2.2 motivates us to ask when the equality $|\mathrm{cent}(G)| = \omega(G) + |\mathrm{nacent}(G)|$ is valid. We will prove Theorem 1.2 by Propositions 2.3, 2.4, 2.5, 2.7, 2.9 and 2.10.

Proposition 2.3. *Let G be a finite group. Then $|\mathrm{nacent}(G)| = 1$ if and only if $|\mathrm{cent}(G)| = \omega(G) + 1$.*

Proof. The result follows from Lemma 2.6 of [1]. □

Proposition 2.4. *Let G be a finite group. Then $|\mathrm{nacent}(G)| = 2$ if and only if $|\mathrm{cent}(G)| = \omega(G) + 2$.*

Proof. Let $X = \{x_1, \ldots, x_{\omega(G)}\}$ be a set of pairwise noncommuting elements of G having maximum size and $\Gamma = \{C_G(x_i) : i = 1, \ldots, w(G)\}$. Suppose that $|\mathrm{nacent}(G)| = 2$. Then G has only one proper nonabelian centralizer $C_G(y)$ for some $y \in G$ and so Proposition 2.2 yields $|\mathrm{cent}(G)| - \omega(G) \leq 2$. Now we have the result by Proposition 2.3.

Conversely suppose that $|\mathrm{cent}(G)| = \omega(G) + 2$. Then every element of Γ is abelian and every abelian centralizer belonging to $\mathrm{cent}(G)$ is in Γ by Proposition 2.5 (a-1) of [1]. It follows that $|\mathrm{nacent}(G)| = 2$, as wanted. □

Proposition 2.5. *Let G be a finite group. Then $|\mathrm{nacent}(G)| = 3$ if and only if $|\mathrm{cent}(G)| = \omega(G) + 3$.*

Proof. Let $X = \{x_1, \ldots, x_{\omega(G)}\}$ be a set of pairwise noncommuting elements of G having maximum size and $\Gamma = \{C_G(x_i) : i = 1, \ldots, w(g)\}$. Suppose that $|\mathrm{nacent}(G)| = 3$. Then G has only two proper nonabelian centralizers $C_G(y)$ and $C_G(z)$ for some $y, z \in G$ and so by Proposition 2.2 we have $|\mathrm{cent}(G)| - \omega(G) \leq 3$. Now Propositions 2.3 and 2.4 imply that $|\mathrm{cent}(G)| = \omega(G) + 3$, as wanted.

Conversely suppose that $|\mathrm{cent}(G)| = \omega(G) + 3$. Then $|\mathrm{nacent}(G)| = 3$ by Proposition 2.5 (a-1) of [1] and Proposition 2.2, as desired. □

Lemma 2.6. *Let $X = \{x_1, \ldots, x_{\omega(G)}\}$ be a set of pairwise noncommuting elements of a group G having maximum size. If $|\mathrm{cent}(G)| = \omega(G) + 4$, then at most one of the $C_G(x_i)$'s is nonabelian.*

Proof. Set $X_i = C_G(x_i)$ for each i and $\Gamma = \{X_1, X_2, \dots, X_{\omega(G)}\}$. Without loss of generality suppose, for a contradiction, that X_1 and X_2 are nonabelian. Then there are $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ of pairwise noncommuting elements of X_1 and X_2, respectively. Therefore $C_G(a_i) \neq X_j \neq C_G(b_i)$ for each i, j. Since $|\mathrm{cent}(G)| = \omega(G) + 4$, we may assume that $C_G(b_i) = C_G(a_i)$ for $i = 1, 2, 3$. Now we claim that $C_G(a_1x_1) \notin \Gamma \cup \{G, C_G(a_i) : 1 \leq i \leq 3\}$.

If $C_G(a_1x_1) = X_1$, then $a_1 \in Z(X_1)$ which is impossible since $a_1 \in X_1$. If $C_G(a_1x_1) = X_j$ for some $j > 1$, then $x_1 \in X_j$, a contradiction. Also $C_G(a_1x_1) \neq C_G(a_i)$ for each i since $x_2 \in C_G(b_i) = C_G(a_i)$. This proves the claim and so we have $|\mathrm{cent}(G)| > \omega(G) + 4$ which is a contradiction. □

Proposition 2.7. *Let G be a finite group. Then $|\mathrm{nacent}(G)| = 4$ if and only if $|\mathrm{cent}(G)| = \omega(G) + 4$.*

Proof. Let $X = \{x_1, \dots, x_{\omega(G)}\}$ be a set of pairwise noncommuting elements of G having maximum size and set $X_i = C_G(x_i)$ for $1 \leq i \leq \omega(G)$. Let $\Gamma = \{X_1, \dots, X_{\omega(G)}\}$ and $|\mathrm{cent}(G)| = \omega(G)+4$. Suppose that, for a contradiction, that $|\mathrm{nacent}(G)| \geq 5$. Then X_k is nonabelian for some k by Proposition 2.2. Without loss of generality, assume that X_1 is nonabelian. It follows from Lemma 2.6 that X_j is abelian for each $j > 1$. So there exist $a_1, a_2, a_3 \in X_1$ such that $a_ia_j \neq a_ja_i$ for all $i \neq j$. Since $\{a_1, a_2, a_3, x_2, \dots, x_{\omega(G)}\}$ is not a set of pairwise noncommuting elements of G, we have $a_ix_j = x_ja_i$ for some i, j. Without loss of generality assume that $a_1x_2 = x_2a_1$. Since X_2 is abelian, $X_2 \subset C_G(a_1)$. It is not hard to see that $C_G(a_1x_1) \notin \Gamma \cup \{G, C_G(a_1), C_G(a_2), C_G(a_3)\}$. Therefore $|\mathrm{cent}(G)| > \omega(G) + 4$ which is a contradiction.

Conversely let $|\mathrm{nacent}(G)| = 4$. Then by Proposition 2.2, we conclude that that $|\mathrm{cent}(G)| - \omega(G) \leq 4$. On the other hand $|\mathrm{cent}(G)| > \omega(G) + 3$ by Propositions 2.3, 2.4 and 2.5 which implies that $|\mathrm{cent}(G)| = \omega(G) + 4$. The claim is established. □

Lemma 2.8. *Let $X = \{x_1, \dots, x_{\omega(G)}\}$ be a set of pairwise noncommuting elements of G. If $|\mathrm{cent}(G)| = \omega(G) + 5$, then at most one of the $C_G(x_i)$'s is nonabelian.*

Proof. Let without losing generality $C_G(x_1)$ and $C_G(x_2)$ be nonabelian. Suppose that $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are subsets of pairwise noncommuting elements of $C_(x_1)$ and $C_G(x_2)$, respectively. Then we have $C_G(a_i) \neq C_G(x_j) \neq C_G(b_i)$ for each $i \in \{1, 2, 3\}$ and $j \in \{1, \dots, \omega(G)\}$. Since $|\mathrm{cent}(G)| = \omega(G) + 5$, we conclude that $C_G(a_{i_1}) = C_G(b_{j_1})$ and $C_G(a_{i_2}) = C_G(b_{j_2})$ for some i_1, i_2, j_1, j_2. There is no loss of generality in assuming that $i_1 = j_1 = 1$ and $i_2 = j_2 = 2$. But it is easy to see that

$$C_G(x_1), C_G(x_2), \ldots, C_G(x_{\omega(G)}), C_G(a_1), C_G(a_2), C_G(a_3), C_G(a_1x_1), C_G(a_2x_1)$$

are distinct, in contradiction to $|\mathrm{cent}(G)|$. This completes the proof. $\square$

Proposition 2.9. *Let G be a finite group. Then $|\mathrm{nacent}(G)| = 5$ if and only if $|\mathrm{cent}(G)| = \omega(G) + 5$.*

Proof. Suppose that $|\mathrm{cent}(G)| = \omega(G) + 5$ and $X = \{x_1, \ldots, x_{\omega(G)}\}$ is a set of pairwise noncommuting elements of G. By Proposition 2.2 it is enough to show that $C_G(x_i)$ is abelian for each $1 \leq i \leq \omega(G)$.

Suppose, for a contradiction, that $C_G(x_i)$ is nonabelian for some i, say $i = 1$. Then $C_G(x_j)$ is abelian for each $j > 1$ by Lemma 2.8. Assume that $\{b_1, b_2, b_3\}$ is a subset of pairwise noncommuting elements of $C_G(x_1)$. Then $C_G(b_i) \neq C_G(x_j)$ for every $i \in \{1, 2, 3\}$ and $j \in \{1, \ldots, \omega(G)\}$. Since $\{b_1, b_2, x_2, \ldots, x_{\omega(G)}\}$ is not a set of pairwise noncommuting elements of G, we have $x_i \in C_G(b_1) \cup C_G(b_2)$ for some i. Without loss of generality, assume that $x_2 \in C_G(b_1)$. Similarly by considering $\{b_2, b_3, x_2, \ldots, x_{\omega(G)}\}$, we can assume that $x_3 \in C_G(b_2)$ since $C_G(x_2)$ and $C_G(x_3)$ are abelian and also $b_1b_2 \neq b_2b_1$. Therefore $C_G(x_2) \subseteq C_G(b_1)$ and $C_G(x_3) \subseteq C_G(b_2)$. Now it is easy to see that

$$C_G(x_1), \ldots, C_G(x_{\omega(G)}), C_G(b_1), C_G(b_2), C_G(b_3), C_G(b_1x_1), C_G(b_2x_1)$$

are distinct proper centralizers belonging to $\mathrm{cent}(G)$ which is impossible since $|\mathrm{cent}(G)| = \omega(G) + 4$.

Conversely suppose that $|\mathrm{nacent}(G)| = 5$. Then $|\mathrm{cent}(G)| - \omega(G) \leq 5$ by Proposition 2.2. It follows from Propositions 2.3-2.7 that $|\mathrm{cent}(G)| = \omega(G) + 5$. This completes the proof. $\square$

Proposition 2.10. *Let G be a finite group. Then $|\mathrm{nacent}(G)| = 6$ if and only if $|\mathrm{cent}(G)| = \omega(G) + 6$.*

Proof. The proof is similar to Proposition 2.9. $\square$

Remark 2.11. Note that if G is a group such that $|\mathrm{nacent}(G)| = 7$, then $|\mathrm{cent}(G)| = \omega(G) + 7$ by Proposition 2.2 and Theorem 1.2. But we can not prove the converse of it and also we can not find any counterexample. So we conjecture that the converse is not true. Although there is a group $G = \mathrm{SmallGroup}(32, 50) = (C_2 \times D_8) \rtimes C_2$ such that $|\mathrm{cent}(G)| - \omega(G) = 16 - 5 = 11 \neq |\mathrm{nacent}(G)|$.

In what follows we will show that $|\mathrm{cent}(G)| - \omega(G) = |\mathrm{nacent}(G)|$ for some finite simple groups G. Also we correct a minor error in Theorem 1.2 of [15] since the author has not considered all conjugates of F, a Sylow 2-subgroup of the Suzuki group $\mathrm{Sz}(q)$, as centralizers of some elements in the group.

Proof of Theorem 1.3.

(1) Suppose that $G = \mathrm{PSL}(2, q)$. If $q \in \{2, 3, 5\}$ or $q \equiv 0 \mod 4$, then G is a CA-group and so we have the result by Proposition 2.3. Suppose that $q > 5$ is odd. By the proof of Theorem 1.1 (2–3) (or Lemma 3.21 of [3]), the number of abelian centralizers belonging to $\mathrm{cent}(G)$ is

$$q + 1 + \frac{q(q-1)}{2} + \frac{q(q+1)}{2} = q^2 + q + 1.$$

On the other hand it follows from Lemma 4.4 of [2] that $\omega(G) = q^2 + q + 1$ and so we have the result by Proposition 2.2.

If $G = \mathrm{PSL}(3, 3)$, then $\omega(G) = 1067$ by Theorem 1.3 of [2]. By Gap [8] we have $|\mathrm{cent}(G)| - \omega(G) = 1237 - 1067 = 170 = |\mathrm{nacent}(G)|$, as wanted.

(2) It is well-known that G has a partition $\mathcal{P} = \{A^x, B^x, C^x, F^x \mid x \in G\}$ where A, B, C are cyclic and F is a Sylow 2-subgroup of G of order q^2. Also we have $C_G(g) \leq H$ for every $g \in H$ and $H \in \mathcal{P}$. By the proof of Theorem 1.2 of [15], we see that F is a CA-group and $|\mathrm{cent}(F)| = q$. Since F has $q^2 + 1$ conjugates in G and they can be centralizers of some elements in G, the first part is established.

By the above argument all conjugates of F are the only proper nonabelian centralizers belonging to $\mathrm{cent}(G)$. On the other hand it follows from Theorem 1.2 of [2] that

$$\omega(G) = (q^2+1)(q-1) + \frac{q^2(q^2+1)}{2} + \frac{q^2(q^2+1)(q-1)}{4(q+2r+1)} + \frac{q^2(q^2+1)(q-1)}{4(q-2r+1)},$$

where $r = 2^m$. Thus $|\mathrm{cent}(G)| - \omega(G) = q^2 + 2$ which is the number of nonabelian centralizers belonging to $\mathrm{cent}(G)$. This completes the proof. □

Now we give a family of groups G for which $|\mathrm{cent}(G)| - \omega(G)$ is not the number of nonabelian centralizers belonging to $\mathrm{cent}(G)$. A group G is called an F-group if $C_G(x) \leq C_G(y)$ for $x, y \in G \setminus Z(G)$, then $C_G(x) = C_G(y)$.

Proposition 2.12. *Let G be an F-group such that $|\mathrm{nacent}(G)| > 1$. Then $|\mathrm{cent}(G)| - \omega(G) < |\mathrm{nacent}(G)|$.*

Proof. Let G be an F-group. If all centralizers of G are nonabelian, then the proof is clear. Assume that $\{C_G(x_i) : 1 \leq i \leq n\}$ are all distinct abelian centralizers of G. It is clear that $\{x_i : 1 \leq i \leq n\}$ is a set of pairwise noncommuting elements of G. Now since G is not a CA-group, G has a proper nonabelian centralizer such as $C_G(y)$ and so $\{y, x_i : 1 \leq i \leq n\}$ is a set of pairwise noncommuting elements of G. Therefore we have the result by Proposition 2.2. □

Finally our computation together with the computational group theory system GAP in investigating finite groups of small order suggests that probably every finite simple group is uniquely determined by the number of nonabelian centralizers. More precisely we pose the following question

Question 2.13. Let H and G be finite simple groups. Is it true that if $|\text{nacent}(G)| = |\text{nacent}(H)|$, then $G \cong H$?

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