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Integers with large practical component

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Abstract. A positive integer n is called practical if all integers between 1 and n can be written as a sum of distinct divisors of n. We give an asymptotic estimate for the number of integers $\leq x$ which have a practical divisor $\geq y$.

1. Introduction

A positive integer n is called *practical* if all integers between 1 and n can be written as a sum of distinct divisors of n. In 1948, Srinivasan [8] began the study of practical numbers, which have been the source of a fair amount of research activity ever since. Let $P(x)$ denote the number of practical numbers $\leq x$. Increasingly precise estimates for $P(x)$ have been obtained by Erdős and Loxton [2], Hausman and Shapiro [3], Margenstern [4], Tenenbaum [10] and Saias [6], who found that the order of magnitude of $P(x)$ is $x/\log x$. In [12] we showed that there is a positive constant c such that

$$P(x) = \frac{cx}{\log x}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right), \tag{1}$$

confirming a conjecture by Margenstern [4]. In this note we want to generalize (1) to integers which have a large practical divisor.

Let $g(n)$ denote the *practical component* of n, i.e. the largest divisor of n which is practical. We have $g(n) = n$ if and only if n is practical, hence we can think of $g(n)$ as a measure for how close n is to being practical. Let $M(x, y)$ be

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the number of integers $\le x$ whose practical component is at least y, i.e.

$$M(x,y) := \#\{n \le x : g(n) \ge y\}.$$

A closely related arithmetic function is $f(n)$, the largest integer with the property that all integers in the interval $[1, f(n)]$ can be written as a sum of distinct divisors of n. Clearly, n is practical if and only if $f(n) \ge n$. Thus $f(n)$ represents another measure for how close n is to being practical. Pollack and Thompson [5] call an integer n a *practical pretender* (or a *near-practical number*) if $f(n)$ is large. More precisely, they define

$$N(x,y) := \#\{n \le x : f(n) \ge y\}$$

and show that there are two positive constants c_1, c_2 such that

$$c_1 \frac{x}{\log y} \le N(x,y) \le c_2 \frac{x}{\log y} \quad (4 \le y \le x).$$

In [5, Lemma 2.1] they find that $f(n)$ satisfies $f(n) = \sigma(g(n))$, where $\sigma(m)$ denotes the sum of the positive divisors of m.

To describe the asymptotic behavior of $M(x,y)$ and $N(x,y)$ we need the following notation. Let c be the positive constant in (1), $\chi(n)$ be the characteristic function of the set of practical numbers,

$$u = \frac{\log x}{\log y},$$

and $\omega(u)$ be Buchstab's function, i.e. the unique continuous solution to the equation

$$(u\omega(u))' = \omega(u-1) \quad (u > 2)$$

with initial condition $\omega(u) = 1/u$ for $1 \le u \le 2$.

Theorem 1. *For $x \ge y \ge 2$ we have*

(i) $M(x,y) = \frac{c(x\omega(u)-y)}{\log y} + O\big(\frac{x \log\log 2y}{(\log y)^2}\big)$,

(ii) $N(x,y) = \frac{cx\omega(u)}{\log y} + O\big(\frac{y}{\log y} + \frac{x\log\log 2y}{(\log y)^2}\big)$,

(iii) $M(x,y) = x\mu_y + O(2^y)$,

(iv) $N(x,y) = x\nu_y + O(2^y)$,

where

$$\mu_y := 1 - \sum_{n<y} \frac{\chi(n)}{n} \prod_{p \le \sigma(n)+1} \left(1 - \frac{1}{p}\right)$$

and

$$\nu_y := 1 - \sum_{\sigma(n)<y} \frac{\chi(n)}{n} \prod_{p \le \sigma(n)+1} \left(1 - \frac{1}{p}\right).$$

It may seem a little surprising to see Buchstab's function appear in the asymptotic formulas for $M(x,y)$ and $N(x,y)$. The reason for this is that $M(x,y)$ and $N(x,y)$ satisfy functional equations (see Lemma 1 below) which closely resemble the functional equation

$$\Phi(x,y) = 1 + \sum_{y<p\le x} \Phi(x/p, p-0) \tag{2}$$

satisfied by

$$\Phi(x,y) := \#\{n \le x : P^-(n) > y\}.$$

Here $P^-(n)$ denotes the smallest prime factor of n and $P^-(1) = \infty$. The main difference is that the primes in (2) are replaced by the practical numbers in Lemma 1, which explains the constant factor c in Theorem 1. With Lemma 2 (ii) we find that $M(x,y) \sim c\,\Phi(x,y)$ for $y \le (1-\varepsilon)x$ and $y \to \infty$.

Moreover, combining (1), Theorem 1, Lemma 2 and the prime number theorem, we have

$$\frac{P(x)}{M(x,y)} \sim \frac{\pi(x)}{\Phi(x,y)} \sim \frac{1}{u\omega(u)} \quad (y \to \infty,\ x/y \to \infty).$$

Hence the probability that a random integer $n \le x$ is practical, given that $g(n) \ge y$, is asymptotically equivalent to the probability that a random integer $n \le x$ is prime, given that $P^-(n) > y$, as $y \to \infty$, $x/y \to \infty$.

The rapid convergence of $\omega(u)$ to $e^{-\gamma}$ (see Lemma 3 (ii)) and Theorem 1 imply that, for $x \ge y \ge 2$,

$$M(x,y), N(x,y) = \frac{ce^{-\gamma}x}{\log y}\left(1 + O\left(\frac{1}{\Gamma(u+1)} + \frac{\log\log 2y}{\log y}\right)\right), \tag{3}$$

where Γ denotes the usual gamma function. Combining (3) with (iii) and (iv) gives the estimate

$$\mu_y, \nu_y = \frac{ce^{-\gamma}}{\log y}\left(1 + O\left(\frac{\log\log y}{\log y}\right)\right).$$

The following table shows $\mu_y = \lim_{x\to\infty} M(x,y)/x$ and $\nu_y = \lim_{x\to\infty} N(x,y)/x$ for small values of y:

$y \in$	μ_y
$[0,1]$	1
$(1,2]$	$1/2$
$(2,4]$	$1/3$
$(4,6]$	$29/105$

$y \in$	ν_y
$[0,1]$	1
$(1,3]$	$1/2$
$(3,7]$	$1/3$
$(7,12]$	$29/105$

From part (iii) of Theorem 1 we obtain the natural density of integers whose practical component is equal to m.

Corollary 1. *Let $m \geq 1$ and*

$$\alpha_m := \mu_m - \mu_{m^+} = \frac{\chi(m)}{m} \prod_{p \leq \sigma(m)+1} \left(1 - \frac{1}{p}\right).$$

For $x \geq 1$ we have $\#\{n \leq x : g(n) = m\} = x\alpha_m + O(2^m)$.

Pollack and Thompson [5, Corollary 1.2] found that the set of integers n with $f(n) = m$ has a natural density ρ_m. Part (iv) of Theorem 1 implies

Corollary 2. *Let $m \geq 1$ and*

$$\rho_m := \nu_m - \nu_{m^+} = \sum_{\sigma(n)=m} \frac{\chi(n)}{n} \prod_{p \leq \sigma(n)+1} \left(1 - \frac{1}{p}\right) = \sum_{\sigma(n)=m} \alpha_n.$$

For $x \geq 1$ we have $\#\{n \leq x : f(n) = m\} = x\rho_m + O(2^m)$.

The following table shows non-zero values of α_m and ρ_m for small m. Note that $\alpha_m > 0$ if and only if m is practical, while $\rho_m > 0$ if and only if $m = \sigma(n)$ for some practical number n.

m	α_m
1	1/2
2	1/6
4	2/35
6	32/1001

m	ρ_m
1	1/2
3	1/6
7	2/35
12	32/1001

The equality of α_m and $\rho_{\sigma(m)}$ does not always hold. For example, since $\sigma(54) = \sigma(56) = 120$ and both 54 and 56 are practical, we have $\rho_{120} = \alpha_{54} + \alpha_{56}$. Moreover, Pollack and Thompson [5, Theorem 1.3] show that the number of integers $m \leq x$ for which $\rho_m > 0$ is $\ll \frac{x}{(\log x)^A}$ for every fixed $A > 0$. Thus the support of ρ_m is a much thinner set than the support of α_m, the set of practical numbers.

The reader may have noticed that practical integers $n < y$ are not counted in $M(x, y)$. This suggests that we may want to consider replacing the parameter y by an increasing function of n, so that smaller values of n are not ignored. To this end, we define

$$M_\lambda(x) := \#\{n \leq x : g(n) \geq n^\lambda\}, \quad N_\lambda(x) := \#\{n \leq x : f(n) \geq n^\lambda\}.$$

Nevertheless, the following result shows that, for $x^\lambda \to \infty$, $x^{1-\lambda} \to \infty$,

$$M_\lambda(x) \sim M(x, x^\lambda) \sim N_\lambda(x) \sim N(x, x^\lambda) \sim \frac{cx\omega(1/\lambda)}{\log(x^\lambda)}.$$

Corollary 3. *For $x \geq y \geq 2$ we have*

(i) $M_{1/u}(x) = \frac{cx\omega(u)}{\log y} + O\big(\frac{x \log\log 2y}{(\log y)^2}\big)$,

(ii) $N_{1/u}(x) = \frac{cx\omega(u)}{\log y} + O\big(\frac{y}{\log y} + \frac{x \log\log 2y}{(\log y)^2}\big)$.

2. Proofs

Stewart [9] and Sierpinski [7] independently discovered the following characterization of practical numbers. An integer $n \geq 2$ with prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $p_1 < p_2 < \ldots < p_k$, is practical if and only if

$$p_j \leq 1 + \sigma\Bigg(\prod_{1 \leq i \leq j-1} p_i^{\alpha_i}\Bigg) \quad (1 \leq j \leq k).$$

It follows that the practical component of n is the largest practical divisor of n of the form $\prod_{1\leq i \leq j} p_i^{\alpha_i}$. If $j < k$, i.e. n is not practical, then we have $p_{j+1} > 1 + \sigma\big(\prod_{1 \leq i \leq j} p_i^{\alpha_i}\big)$.

Lemma 1. *For $x \geq 1$, $y \geq 1$ we have*

(i) $[x] = \sum_{n \leq x} \chi(n) \Phi(x/n, \sigma(n) + 1)$

(ii) $M(x,y) = \sum_{y \leq n \leq x} \chi(n)\Phi(x/n, \sigma(n)+1)$

(iii) $N(x,y) = \sum_{\substack{n \leq x \\ \sigma(n) \geq y}} \chi(n)\Phi(x/n, \sigma(n)+1)$

(iv) $M_\lambda(x) = \sum_{n \leq x} \chi(n)\Phi(\min(x/n, n^{1/\lambda - 1}), \sigma(n)+1)$

(v) $N_\lambda(x) = \sum_{n \leq x} \chi(n)\Phi(\min(x/n, \sigma(n)^{1/\lambda}/n), \sigma(n)+1)$

Proof. Each of these equations is based on the same principle, which is to count the integers m contributing to the left-hand side according to their practical component n. Part (i) is Lemma 2.3 of [12]. We only take a closer look at (ii). Every integer m counted in $M(x,y)$ factors uniquely as $m = nr$, where n is the practical component of m, $n \geq y$ and $P^-(r) > \sigma(n) + 1$. Given a practical component n, the number of admissible values of r is given by $\Phi\big(x/n, \sigma(n)+1\big)$. $\square$

Lemma 2. *We have*

(i) $\Phi(x,y) = x\prod_{p\le y}\left(1-\frac{1}{p}\right) + O\big(2^{\pi(y)}\big) \quad (x\ge 1, y\ge 2)$

(ii) $\Phi(x,y) = \frac{x\omega(u)-y}{\log y} + O\big(\frac{x}{(\log y)^2}\big) \quad (x\ge y\ge 2)$

(iii) $\Phi(x,y) = \frac{x\omega(u)}{\log y} + O\big(\frac{y}{\log y} + \frac{x}{(\log y)^2}\big) \quad (x\ge 1, y\ge 2)$

(iv) $\Phi(x,y) - 1 \ll \frac{x}{\log y} \quad (x\ge 1, y\ge 2)$

Proof. Part (i) is elementary (see e.g. de Bruijn [1]). For (ii) see Tenenbaum [11, Theorem III.6.3]. Parts (iii) and (iv) follow easily from (ii). □

Lemma 3. *We have*

(i) $|\omega'(u)| \le 1/\Gamma(u+1) \quad (u\ge 1)$

(ii) $|\omega(u) - e^{-\gamma}| \ll 1/\Gamma(u+1) \quad (u\ge 1)$

Proof. See Tenenbaum [11, Theorems III.5.5, III.6.4]. □

In the proof of Theorem 1 we will use the well-known fact (see for example [11, Theorem I.5.5]) $\limsup_{n\to\infty}\sigma(n)/(n\log\log n) = e^\gamma$.

Proof of Theorem 1. (i) We use Lemma 1(ii). If $\sqrt{x} < y \le x$, then $M(x,y) = P(x) - P(y-0)$ because $\Phi(x,y) = 1$ for $y\ge x\ge 1$. Thus the result follows from (1) in this case. If $y\le\sqrt{x}$ we have

$$M(x,y) = P(x) - P(\sqrt{x}) + \sum_{y\le n\le\sqrt{x}} \chi(n)\Phi\big(x/n, \sigma(n)+1\big).$$

We approximate Φ by Lemma 2(iii). The contribution from the error term $O(x/(\log y)^2)$ is

$$\sum_{y\le n\le\sqrt{x}} \chi(n)\frac{x/n}{(\log n)^2} \ll \frac{x}{(\log y)^2},$$

and from the error term $O(y/\log y)$ it is

$$\sum_{y\le n\le\sqrt{x}} \chi(n)\frac{\sigma(n)+1}{\log(\sigma(n)+1)} \ll \frac{\sqrt{x}\log\log x}{\log x}\sum_{y\le n\le\sqrt{x}}\chi(n) \ll \frac{x\log\log 2x}{(\log x)^2},$$

which is acceptable. The contribution from the main term is

$$x\sum_{y\le n\le\sqrt{x}} \frac{\chi(n)}{n\log(\sigma(n)+1)}\,\omega\left(\frac{\log x/n}{\log(\sigma(n)+1)}\right).$$

In the last sum, we replace the two occurrences of $\log(\sigma(n)+1)$ by $\log n + O(\log\log\log(8n))$. Lemma 3 and (1) show that the resulting error is $\ll x(\log\log 2y)/(\log y)^2$. We thus have

$$M(x,y) = P(x) + x \sum_{y\le n\le\sqrt{x}} \frac{\chi(n)}{n\log n}\,\omega\left(\frac{\log x}{\log n}-1\right) + O\left(\frac{x\log\log 2y}{(\log y)^2}\right).$$

Partial summation together with the estimates in Lemma 3 and (1) yields

$$M(x,y) = P(x) + x\int_y^{\sqrt{x}} \frac{c}{t(\log t)^2}\,\omega\left(\frac{\log x}{\log t}-1\right)\mathrm{d}t + O\left(\frac{x\log\log 2y}{(\log y)^2}\right).$$

The term with the integral simplifies to

$$\frac{cx}{\log x}\int_2^u \omega(s-1)\,\mathrm{d}s = \frac{cx}{\log x}\left(u\omega(u)-1\right).$$

The result now follows from (1).

(ii) Lemma 1 shows that

$$\begin{aligned} 0 \le N(x,y) - M(x,y) &= \sum_{\substack{n<y\\ \sigma(n)\ge y}} \chi(n)\Phi\big(x/n,\sigma(n)+1\big) \\ &\le \sum_{\frac{y}{A\log\log 2y}<n<y} \chi(n)\Phi\big(x/n,\sigma(n)+1\big), \end{aligned}$$

for some suitable constant A. Splitting the range by powers of 2 and using the estimate (1) and Lemma 2 (iv), the last sum is

$$\ll P(y) + \sum_{\frac{y}{A\log\log 2y}<n<y} \frac{x}{n(\log n)^2} \ll \frac{y}{\log y} + \frac{x\log\log 2y}{(\log y)^2}.$$

Hence (ii) follows from (i).

(iii) From Lemmas 1 and 2 we have

$$\begin{aligned} [x] - M(x,y) &= \sum_{n<y}\chi(n)\Phi\big(x/n,\sigma(n)+1\big) = \sum_{n<y}\chi(n)\Bigg(\frac{x}{n}\prod_{p\le\sigma(n)+1}\left(1-\frac{1}{p}\right) \\ &\quad + O\Big(2^{\pi(\sigma(n)+1)}\Big)\Bigg) = x(1-\mu_y) + O\Bigg(\sum_{n<y}2^{\pi(\sigma(n)+1)}\Bigg) \\ &= x(1-\mu_y) + O\big(2^{(1+o(1))e^{\gamma}y\log\log y/\log y}\big), \end{aligned}$$

since $\sigma(n)\le(1+o(1))e^{\gamma}n\log\log n$ and $\pi(y)\le(1+o(1))y/\log y$.

We omit the proof of (iv), since it is almost the same as that of (iii). $\square$

Proof of Corollary 3. (i) From Lemma 1 and Lemma 2 (iv) we have, with $\lambda = 1/u$,

$$\begin{aligned}M_\lambda(x) - M(x, x^\lambda) &= \sum_{n<x^\lambda} \chi(n)\Phi\Big(n^{1/\lambda-1}, \sigma(n)+1\Big) \\ &= P(y) + O\bigg(\sum_{n\le y} \chi(n)\frac{n^{u-1}}{\log 2n}\bigg) = P(y) + O\bigg(\frac{x}{(\log y)^2}\bigg),\end{aligned}$$

by partial summation. The result now follows from Theorem 1 and (1). The proof of (ii) follows the same idea. In the end we need an estimate for

$$\sum_{\sigma(n)<y} \chi(n)\frac{\sigma(n)^u}{n\log 2n}.$$

We split this sum into two parts. The contribution from large n is

$$\le \sum_{\frac{y}{A(\log y)^3}<n<y} \chi(n)\frac{y^u}{n\log 2n} \ll \sum_{\frac{y}{A(\log y)^3}<n<y} \frac{x}{n(\log 2n)^2} \ll \frac{x\log\log y}{(\log y)^2},$$

where A is a positive constant such that $\sigma(n) \le \frac{y}{(\log y)^2}$ whenever $n \le \frac{y}{A(\log y)^3}$ and $y \ge 2$. The contribution from small n is

$$\le \sum_{n\le\frac{y}{A(\log y)^3}} \chi(n)\frac{(y(\log y)^{-2})^u}{n\log 2n} \ll \frac{x}{(\log y)^2}\sum_{n\ge 1}\frac{1}{n(\log 2n)^2} \ll \frac{x}{(\log y)^2}. \qquad \square$$

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