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Rizza-negativity of holomorphic vector bundles

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Abstract. In the present paper, we shall introduce the notion of *Rizza-negativity* of complex Finsler structures, and we show that Rizza-negativity implies the negativity of holomorphic vector bundles, i.e., the ampleness of its dual E^*. Further we shall show that any Rizza-negative Finsler structure induces a Griffiths-negative Hermitian structure.

1. Introduction

A holomorphic line bundle L over a compact complex manifold M is said to be *very ample* if there exists a basis for holomorphic sections of L such that it defines a holomorphic imbedding $f : M \hookrightarrow \mathbb{P}^N$ into a complex projective space $\mathbb{P}^N$. If L is very ample, then L is isomorphic to $f^*\mathbb{H}$ for the hyperplane bundle $\mathbb{H}$ over $\mathbb{P}^N$. A holomorphic line bundle L is said to be *ample* if there exists a positive integer k such that $L^{\otimes k}$ is very ample. Since the metric on $\mathbb{H}$ induced from the Fubini-Study metric on $\mathbb{P}^N$ has positive curvature, the induced k-th root metric on L is a Hermitian metric of positive curvature. Therefore any ample line bundle is *positive*. The converse is also true by the well-known Kodaira's embedding theorem.

A holomorphic line bundle L is said to be *negative* if its dual L^* is positive or ample. A holomorphic line bundle L is negative if and only if L admits a

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Hermitian metric h of negative curvature. It is natural to generalize the notion of ampleness or negativity to higher rank vector bundles.

Let $\pi : E \to M$ be a holomorphic vector bundle of rank r (≥ 2) over a compact complex manifold M of $\dim_{\mathbb{C}} M = m$. Denoted by E^0 the set of non-zero elements of E, the multiplicative group $\mathbb{C}^* = \mathbb{C} - \{0\}$ acts on E^0 by scalar multiplication. Then the projective bundle $\phi : \mathbb{P}(E) \to M$ associated with E is defined by $\mathbb{P}(E) = E^0/\mathbb{C}^*$. The fiber $\phi^{-1}(z) = \mathbb{P}_z$ is the complex projective space of dimension $r - 1$. The *tautological line bundle* $\mathbb{L}(E)$ is a holomorphic line bundle over $\mathbb{P}(E)$ defined by $\mathbb{L}(E) = \{(V, v) \in \mathbb{P}(E) \times E \mid v \in V\}$, and the dual line bundle $\mathbb{H}(E) := \mathbb{L}(E)^*$ is called the *hyperplane bundle*. A holomorphic vector bundle E is said to be *ample* if the bundle $\mathbb{H}(E^*) = \mathbb{L}(E^*)^{-1}$ over $\mathbb{P}(E^*)$ is ample, where E^* is the dual of E ([Ha]).

A holomorphic vector bundle E over a compact complex manifold M is said to be *Griffiths-positive* if E admits a Hermitian metric of positive curvature. It is well-known that any Griffiths-positive vector bundle is ample (see e.g., [Sh-So]). If E is ample, then the determinant bundle $\det E = \wedge^r E$ is an ample line bundle over M. Hence, if there exists an ample vector bundle over M, the base manifold M is projective, i.e., M is holomorphically embedded into $\mathbb{P}^N$ for some sufficiently large N (see e.g., [Zh]).

Definition 1.1 ([Ko1]). A holomorphic vector bundle E is said to be *negative* if its dual E^* is ample, i.e., E is negative if $\mathbb{L}(E)$ is negative.

A *complex Finsler structure* in E is a function $F : E \to \mathbb{R}$ satisfying the following conditions:

(F1) F is smooth on E^0, and F is continuous on E;

(F2) $F(z, \zeta) \geq 0$ and $= 0$ if and only if $\zeta = 0$;

(F3) F satisfies the homogeneity condition $F(z, \lambda \cdot \zeta) = |\lambda|^2 F(z, \zeta)$ for any $\lambda \in \mathbb{C}$.

The geometry of complex Finsler structures was started by [Ri] and [Ru1], and the connection theory in a complex manifold with a complex Finsler structure was developed by [Ru2].

In [Ko1], any complex Finsler structure F in E is identified with a Hermitian structure in $\mathbb{L}(E)$ by identifying $v \in E^0$ with $([v], v) \in \mathbb{P}(E) \times E^0$. Since the curvature of $(\mathbb{L}(E), F)$ is given by $\bar{\partial}\partial \log F$, a characterization of negative holomorphic vector bundles is given by

Theorem 1.1. *A holomorphic vector bundle E over a compact complex* *manifold is negative if and only if E admits a complex Finsler structure F such that* $\sqrt{-1}\bar{\partial}\partial\log F < 0$.

From the expression (2.13) of $\sqrt{-1}\bar{\partial}\partial\log F$ in the below, it follows that the negativity of E is easier to describe than the ampleness of E from the viewpoint of differential geometry. Hence, in this present paper, we will be concerned with negativity instead of ampleness of vector bundles, and we shall show some results obtained from Kobayashi's characterization.

2. Complex Finsler geometry

Denoted by T_M and T_E the holomorphic tangent bundles over M and E respectively, we obtain a short exact sequence of holomorphic vector bundles over E:

$$\mathbb{O} \longrightarrow V \xrightarrow{\iota} T_E \xrightarrow{\widetilde{\pi}} \widetilde{T_M} \longrightarrow \mathbb{O}, \tag{2.1}$$

where $\widetilde{T_M} = \pi^* T_M$ is the pull-back of T_M, and the vertical sub-bundle is defined by $V := \ker(\widetilde{\pi})$ and $\widetilde{\pi} = (\pi, \pi_*)$ for the push-forward $\pi_* : T_{(z,\zeta)}E \to T_z M$ at $(z,\zeta) \in E$. The fiber $V_{(z,\zeta)}$ of V over $(z,\zeta) \in E$ is the tangent space $T_\zeta E_z$ at $\zeta \in E_z := \pi^{-1}(z)$.

Let $z = (z^1, \dots, z^m)$ be a local complex coordinate system in an open subset U of M, and $s = \{s_1, \dots, s_r\}$ a local holomorphic frame field of E over U. The induced coordinate system in the fiber E_z will be denoted by $\zeta = (\zeta^1, \dots, \zeta^r)$ so that $(z,\zeta) = (z^1, \dots, z^m, \zeta^1, \dots, \zeta^r)$ is a local coordinate system in $\pi^{-1}(U) \subset E$. Given a complex Fnsler structure F, we set

$$g_{i\bar{j}} := \partial_i \partial_{\bar{j}} F, \tag{2.2}$$

where $\partial_i = \partial/\partial\zeta^i$ and $\partial_{\bar{j}} = \partial/\partial\bar{\zeta}^j$. Usually, if $(g_{i\bar{j}})$ is positive-definite along the fibers, then F is said to be *strongly pseudoconvex*. If F is strongly pseudoconvex, then $(g_{i\bar{j}})$ defines a Hermitian structure g in V by

$$g(Z,W) = \sum g_{i\bar{j}} Z^i \overline{W^j} \tag{2.3}$$

for all $Z = \sum Z^i(\partial/\partial\zeta^i), W = \sum Z^j(\partial/\partial\zeta^j) \in \Gamma(V)$. We shall call F a *Rizza structure* if F is strongly pseudoconvex to shorten the terminology.

Let μ be the action of the multiplier group $\mathbb{C}^*$ on E defined by $\mu : \mathbb{C}^* \times E_z \ni (\lambda, (z,\zeta)) \longmapsto (z, \lambda\cdot\zeta) := \mu_\lambda(z,\zeta) \in E_z$ for all $\lambda \in \mathbb{C}^*$. The condition

(F3) shows $g_{i\bar{j}} \circ \mu = g_{i\bar{j}}$[1], and the action μ induces a section $\mathcal{E}$ of V such that $(\mu_\lambda)_*(\mathcal{E}(z,\zeta)) = \mathcal{E}(z, \lambda\cdot\zeta)$:

$$\mathcal{E} = \sum \zeta^i \frac{\partial}{\partial \zeta^i}. \tag{2.4}$$

From the assumption (F3), it follows that

$$g(\mathcal{E}, \mathcal{E}) = F(z, \zeta). \tag{2.5}$$

2.1. Partial connection in (V, g)**.** Any complement $H_{(z,\zeta)}$ of $V_{(z,\zeta)} = T_\zeta E_z$ defines a horizontal vector sub-bundle $H \subset T_E$. If H is smooth on E and is invariant by the action μ, then H is called an *Ehresmann connection* in E. If a horizontal sub-bundle H is smooth on E^0, and H is continuous on E, then H is called a *nonlinear connection* in E.

Any nonlinear connection H is defined by $H = \ker(P)$ for a smooth morphism $P : T_E \to V$ satisfying $P \circ \iota = id$. Since P may be considered as a $(1,0)$-form with values in V, P can be written as $P = \sum(\partial/\partial\zeta^i) \otimes (d\zeta^i + \sum N^i_\alpha dz^\alpha)$ for some local functions N^i_α such that they are homogeneous of degree one in the fiber coordinate $\zeta^1, \dots, \zeta^r$, i.e., $N^i_\alpha \circ \mu_\lambda = \lambda N^i_\alpha$. Then the horizontal lifts X_α with respect to H of local frame fields $\{\partial/\partial z^\alpha\}$ are given by

$$X_\alpha = \frac{\partial}{\partial z^\alpha} - \sum N^i_\alpha \frac{\partial}{\partial \zeta^i},$$

and H is locally spanned by $\{X_1, \dots, X_m\}$ satisfying

$$(\mu_\lambda)_*(X_\alpha(z,\zeta)) = X_\alpha(z, \lambda\cdot\zeta). \tag{2.6}$$

For any tangent vector X in M and its horizontal lift X^H with respect to H, we set

$$D_{X^H} Z := P \circ \mathcal{L}_{X^H} Z, \tag{2.7}$$

where $\mathcal{L}_{X^H}$ is the Lie derivative by X^H. Then $D_{X^H}Z$ is linear in X^H and satisfies the Leibniz rule $D_{X^H}(fZ) = X^H(f)Z + fD_{X^H}Z$, i.e., D is a *partial connection* of $(1,0)$-type in V in the horizontal direction H. We wish to fix a nonlinear connection H so that D satisfies

$$D_{X^H} g = 0 \tag{2.8}$$

for all $X \in \Gamma(T_M)$, i.e., we will be concerned with H such that the parallel displacement preserves the Hermitian structure g. The assumption (2.8) can be

[1]The matrix $(g_{i\bar{j}})$ defines a Hermitian structure g in the induced bundle ϕ^*E over $\mathbb{P}(E)$. In [Ko1] and [Ko2], the Hermitian connection of (ϕ^*E, g) was discussed to characterize the negativity of E by using its curvature.

written as

$$\frac{\partial}{\partial \zeta^i}\left(\frac{\partial^2 F}{\partial z^\alpha \partial \bar{\zeta}^j} - \sum g_{i\bar{j}} N^i_\alpha\right) = 0.$$

Hence, if we define H by

$$N^i_\alpha = \sum g^{\bar{j}i} \frac{\partial^2 F}{\partial z^\alpha \partial \bar{\zeta}^j} \tag{2.9}$$

for the the components $g^{\bar{j}i}$ of the inverse of $(g_{i\bar{j}})$, the partial connection D satisfies (2.8).

Proposition 2.1. *Let F be a Rizza structure in E. There exists a non-linear connection H of E so that the partial connection D associated with H satisfies* (2.8).

2.2. Curvature K of D and Rizza-negativity. Since D is compatible with g in the horizontal direction H, D is given by

$$D_\alpha \frac{\partial}{\partial \zeta^i} := D_{X_\alpha} \frac{\partial}{\partial \zeta^i} = \sum \Gamma^l_{i\alpha} \frac{\partial}{\partial \zeta^l}, \quad \Gamma^l_{i\alpha} = \sum g^{l\bar{m}} X_\alpha(g_{i\bar{m}}). \tag{2.10}$$

The following proposition will be easily proved from (2.5) and (2.8).

Proposition 2.2. *The partial connection D in (V, g) satisfies*

$$D_\alpha \mathcal{E} = 0, \tag{2.11}$$

and the Rizza structure F is constant along H, i.e.,

$$X_\alpha F = 0. \tag{2.12}$$

The curvature form Ω^i_j of D is defined by

$$D^2 \frac{\partial}{\partial \zeta^j} = \sum \frac{\partial}{\partial \zeta^i} \otimes \Omega^i_j,$$

and the curvature tensor $K^i_{j\alpha\bar{\beta}}$ is defined by $\Omega^i_j = \sum K^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$. Then (2.8) implies

$$K_{i\bar{j}\alpha\bar{\beta}} := \sum g_{l\bar{j}} K^l_{i\alpha\bar{\beta}} = -X_{\bar{\beta}} X_\alpha g_{i\bar{j}} + \sum g_{k\bar{l}} \Gamma^k_{i\alpha} \overline{\Gamma^l_{j\beta}}.$$

Definition 2.1. We say that (E, F) is *Rizza-negative* if

$$K(Z \otimes X^H) := \sum K_{i\bar{j}\alpha\bar{\beta}} Z^i X^\alpha \overline{Z^j X^\beta} < 0$$

at any point $(z, \zeta) \in E$ for all non-zero $Z \in V_{(z,\zeta)}$ and $X^H \in H_{(z,\zeta)}$.

We will investigate the negativity of E using an expression of $\bar{\partial}\partial \log F$ in terms of curvature tensor of the partial connection D in (V, g). Such an expression is given in [Ai]:

Proposition 2.3. *Let F be a Rizza structure in E. Then*

$$\sqrt{-1}\bar{\partial}\partial \log F = \frac{\sqrt{-1}}{F}\sum \Psi_{\alpha\bar{\beta}} dz^{\alpha} \wedge d\bar{z}^{\beta} - \sqrt{-1}\sum (\log F)_{i\bar{j}} P^i \wedge \overline{P^j}, \qquad (2.13)$$

where we put $\Psi_{\alpha\bar{\beta}} = \sum K_{i\bar{j}\alpha\bar{\beta}}\zeta^i\overline{\zeta^j}$, $(\log F)_{i\bar{j}} = \partial_i\partial_{\bar{j}}(\log F)$ *and* $P^i = d\zeta^i + \sum N^i_{\alpha} dz^{\alpha}$.

We denote by $\rho : E^0 \to \mathbb{P}(E)$ the natural projection, and by $(z, [\zeta]) \in \mathbb{P}(E)$ corresponding to (z, ζ), $\zeta \neq 0$. Since ρ satisfies

$$\rho_*\left(\sum \zeta^i \frac{\partial}{\partial \zeta^i}\right) = \rho_*\mathcal{E} = 0,$$

$\ker(\rho_*)$ is spanned by the tautological section $\mathcal{E}$. Further,

$$\sum (\log F)_{i\bar{j}} Z^i \overline{W^j} = \frac{1}{F}\Big[(g(Z, W) - \frac{1}{F} g(Z, \mathcal{E}) g(\mathcal{E}, W)\Big] := g^{\perp}(Z, W) \qquad (2.14)$$

for all $Z, W \in \Gamma(V)$. From (2.13) and Theorem 1.1, we get

Theorem 2.1. *Let E be a holomorphic vector bundle admitting a Rizza structure F. If (E, F) is Rizza-negative, then E is negative.*

Proof. Since F is a Rizza structure, the identity (2.5) and Schwarz inequality assure the negativity of the second term of (2.13) in each $T_{[\zeta]}\mathbb{P}_z$. Further, if (E, F) is Rizza-negative, then $\sum \Psi_{\alpha\bar{\beta}} X^{\alpha}\overline{X^{\beta}} = K(\mathcal{E} \otimes X^H)$ implies the negativity of the first term of (2.13). Hence, if (E, F) is Rizza-negative, then E is negative. □

The Rizza negativity of (E, F) concludes the negativity of $K(\mathcal{E} \otimes X^H)$, but the converse is not true in general. Thus it is an open problem whether a Rizza-negative structure exists in a negative vector bundle.

3. Smooth family of Kähler metrics

Let $\phi : \mathbb{P}(E) \to M$ be the projective bundle associated with E. Then we obtain a short exact sequence of holomorphic vector bundles over $\mathbb{P}(E)$:

$$\mathbb{O} \longrightarrow \mathscr{V} \xrightarrow{\iota} T_{\mathbb{P}(E)} \xrightarrow{\widetilde{\phi}} \widetilde{T_M} \longrightarrow \mathbb{O}, \qquad (3.1)$$

where $\widetilde{T_M} = \phi^* T_M$ is the pull-back of T_M, and $\mathscr{V} := \ker(\widetilde{\phi})$ with $\widetilde{\phi} := (\phi, \phi_*)$ for the push-forward $\phi_* : T_{(z,[\zeta])}\mathbb{P}(E) \to T_z M$. Each fiber $\mathscr{V}_{(z,[\zeta])}$ of $\mathscr{V}$ is the tangent space $T_{[\zeta]}\mathbb{P}_z = \rho_*(T_\zeta E_z^0)$ of $\mathbb{P}_z = \phi^{-1}(z)$ at $[\zeta] \in \mathbb{P}_z$, and $\mathscr{V}_{(z,[\zeta])}$ is spanned by $\{\widetilde{s}_1 := \rho_*\left(\partial/\partial\zeta^1\right), \ldots, \widetilde{s}_r := \rho_*(\partial/\partial\zeta^r)\}$ satisfying the relation

$$\sum \zeta^i \widetilde{s}_i = \rho_* \mathcal{E} = 0. \tag{3.2}$$

Since the metric $g^\perp$ defined in (2.14) is invariant by the action μ, i.e., $\mu_\lambda^* g^\perp = g^\perp$ for any $\lambda \in \mathbb{C}^*$, there exists a Hermitian structure $\mathfrak{g}$ in $\mathscr{V}$ such that

$$(\rho^*\mathfrak{g})(Z, W) = g^\perp(Z, W) = \frac{1}{F}\Big[(g(Z,W) - \frac{1}{F} g(Z,\mathcal{E}) g(\mathcal{E}, W)\Big], \tag{3.3}$$

and $\mathfrak{g}$ defines a Kähler metric $\mathfrak{g}_z := \mathfrak{g} \restriction_{\mathbb{P}_z}$ in each fiber $\mathbb{P}_z$. Hence any Rizza structure F in E makes $\phi : \mathbb{P}(E) \to M$ a smooth family of Kähler manifolds $\{\mathbb{P}_z, \mathfrak{g}_z\}$.

Conversely, we suppose that $\phi : \mathbb{P}(E) \to M$ is a smooth family of Kähler manifolds $\{\mathbb{P}_z, \mathfrak{g}_z\}$. Let ω_z be the Kähler form in each $\mathbb{P}_z$ defined by $\mathfrak{g}_z$. Then E admits a Rizza structure F such that $\rho^*\omega_z = \sqrt{-1}\sum(\log F)_{i\bar{j}} d\zeta^i \wedge d\bar{\zeta}^j$ (see the last section).

Proposition 3.1. *A holomorphic vector bundle E admits a Rizza structure if and only if the projective bundle $\phi : \mathbb{P}(E) \to M$ is a smooth family of Kähler manifolds.*

Let H be the nonlinear connection in E determined from the given Rizza structure F. Since H is invariant by the action μ, we can define a horizontal sub-bundle $\mathscr{H} \subset T_{\mathbb{P}(E)}$ by $\mathscr{H} = \rho_* H$. Then (2.6) shows that $\rho_*(X_\alpha(z, \lambda\cdot\zeta)) = \rho_*(X_\alpha(z,\zeta))$ for any $\lambda \in \mathbb{C}^*$ and $(z,\zeta) \in E^0$, and this implies that

$$\mathcal{X}_\alpha(z, [\zeta]) := (\rho_* X_\alpha)(z, [\zeta]) \tag{3.4}$$

makes sense for the basis $\{X_1, \ldots, X_m\}$ of H. Hence $\mathscr{H}$ is spanned by $\{\mathcal{X}_1, \ldots, \mathcal{X}_m\}$. Denoted by $\mathscr{P} : T_{\mathbb{P}(E)} \to \mathscr{V}$ the morphism with $\ker(\mathscr{P}) = \mathscr{H}$, we define a partial connection $\mathscr{D}$ of $(1,0)$-type in $\mathscr{V}$ by the Lie derivative in the horizontal direction $\mathscr{H}$ as in (2.7):

$$\mathscr{D}_{X^{\mathscr{H}}} := \mathscr{P} \circ \mathcal{L}_{X^{\mathscr{H}}}. \tag{3.5}$$

Further, since the projection P and $\mathscr{P}$ are commutative with ρ_*, i.e., $\mathscr{P} \circ \rho_* = \rho_* \circ P$ implies

$$\mathscr{D}_{X^{\mathscr{H}}} = \rho_* D_{X^H}. \tag{3.6}$$

From (2.11) and (2.12), it follows that the partial connection D in (V, g) satisfies $Dg^{\perp} = 0$. Using this fact and (3.6), we get

$$\begin{aligned}(\mathscr{D}_{X^{\mathscr{H}}}\mathfrak{g})(\widetilde{s_i}, \widetilde{s_j}) &= \mathcal{X}_\alpha\left(\mathfrak{g}\left(\rho_*\frac{\partial}{\partial\zeta^i}, \rho_*\frac{\partial}{\partial\zeta^j}\right)\right) \\ &\quad - \mathfrak{g}\left(\rho_* D_{X^H}\frac{\partial}{\partial\zeta^i}, \rho_*\frac{\partial}{\partial\zeta^j}\right) - \mathfrak{g}\left(\rho_*\frac{\partial}{\partial\zeta^i}, \rho_* D_{X^H}\frac{\partial}{\partial\zeta^j}\right) \\ &= X_\alpha\left(g^{\perp}\left(\frac{\partial}{\partial\zeta^i}, \frac{\partial}{\partial\zeta^j}\right)\right) - g^{\perp}\left(D_{X^H}\frac{\partial}{\partial\zeta^i}, \frac{\partial}{\partial\zeta^j}\right) - g^{\perp}\left(\frac{\partial}{\partial\zeta^i}, D_{X^H}\frac{\partial}{\partial\zeta^j}\right) \\ &= (D_\alpha g^{\perp})\left(\frac{\partial}{\partial\zeta^i}, \frac{\partial}{\partial\zeta^j}\right) = 0.\end{aligned}$$

Proposition 3.2. *The partial connection $\mathscr{D}$ is compatible with the Hermitian structure $\mathfrak{g}$:*

$$\mathscr{D}_{X^{\mathscr{H}}}\mathfrak{g} = 0. \tag{3.7}$$

For any smooth relative (r, r)-form η on $\mathbb{P}(E)$,

$$\frac{\partial}{\partial z^\alpha}\int_{\mathbb{P}_z}\eta = \int_{\mathbb{P}_z}\mathcal{L}_{\mathcal{X}_\alpha}\eta \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^\alpha}\int_{\mathbb{P}_z}\eta = \int_{\mathbb{P}_z}\mathcal{L}_{\mathcal{X}_{\overline{\alpha}}}\eta, \tag{3.8}$$

where $\mathcal{X}_{\overline{\alpha}} := \overline{\mathcal{X}_\alpha}$ denotes the complex conjugate of $\mathcal{X}_\alpha$.

From (3.7), it follows that the Hermitian structure $\mathfrak{g}$ is preserved by the parallel translation with respect to $\mathscr{H}$, i.e.,

$$(\mathcal{L}_{\mathcal{X}_\alpha}\mathfrak{g}) \restriction_{\mathbb{P}_z} = 0. \tag{3.9}$$

Let ω_z be the Kähler form in each $\mathbb{P}_z$ defined by $\mathfrak{g}_z$, and let $dv = \omega_z^{r-1}/(r-1)!$ be the volume form in $\{\mathbb{P}_z, \mathfrak{g}_z\}$. Then (3.9) implies $(\mathcal{L}_{\mathcal{X}_\alpha}\omega_z) \restriction_{\mathbb{P}_z} = 0$, and so the relative volume form dv is also parallel with respect to $\mathscr{H}$, i.e., $(\mathcal{L}_{\mathcal{X}_\alpha}dv) \restriction_{\mathbb{P}_z} = 0$. Thus

$$\frac{\partial}{\partial z^\alpha}\int_{\mathbb{P}_z} dv = \int_{\mathbb{P}_z}\mathcal{L}_{\mathcal{X}_\alpha} dv = 0. \tag{3.10}$$

Proposition 3.3. *The volume of each fiber $\{\mathbb{P}_z, \mathfrak{g}_z\}$ is constant.*

Remark 3.1. In [Ya], this proposition is proved by a direct method in case of $E = T_M$. A similar proposition for a smooth family of Einstein-Kähler manifolds has been proved in [Sc].

3.1. Averaged Hermitian structure and connections. In this section, we shall introduce the *averaged Hermitian structure* h of $\mathfrak{g}$ and *averaged connection* ∇ of $\mathscr{D}$ analogously to the real Finsler geometry (see [Ma-Ra-Tr-Ze] and [To-Et]).

For any $u = \sum u^i s_i \in \Gamma(E)$, we set $\widetilde{u} = \sum u^i \widetilde{s}_i \in \Gamma(\mathscr{V})$. We define a Hermitian structure h in E by the L^2-inner product

$$h(u,v) := \int_{\mathbb{P}_z} \mathfrak{g}(\widetilde{u},\widetilde{v})dv = \int_{\mathbb{P}_z} \sum (\log F)_{i\bar{j}}\, u^i \overline{v^j} dv. \tag{3.11}$$

Further we define $\nabla : \Gamma(T_M) \times \Gamma(E) \ni (X,u) \longmapsto \nabla_X u \in \Gamma(E)$ by

$$h(\nabla_X u, v) := \int_{\mathbb{P}_z} \mathfrak{g}(\mathscr{D}_{X^{\mathscr{H}}}\widetilde{u},\widetilde{v})dv. \tag{3.12}$$

It is obvious that $\nabla_X u$ is linear in X and the Leibniz rule for ∇ will be checked easily.

Proposition 3.4. *The connection ∇ is the Hermitian connection in (E,h).*

Proof. Since $\mathscr{D}$ is of $(1,0)$-type, the connection ∇ is also of $(1,0)$-type. Further, from (2.3) and Proposition 3.3, we have

$$\begin{aligned}(\nabla_X h)(u,v) &= X(h(u,v)) - h(\nabla_X u, v) - h(u,\nabla_X v)\\ &= \int_{\mathbb{P}_z} \left\{X^{\mathscr{H}}(\mathfrak{g}(\widetilde{u},\widetilde{v})) - \mathfrak{g}(\mathscr{D}_{X^{\mathscr{H}}}\widetilde{u},\widetilde{v}) - \mathfrak{g}(\widetilde{u},\mathscr{D}_{X^{\mathscr{H}}}\widetilde{v})\right\} dv\\ &= \int_{\mathbb{P}_z} (\mathscr{D}_{X^{\mathscr{H}}}\mathfrak{g})(\widetilde{u},\widetilde{v})dv = 0. \qquad \square\end{aligned}$$

Let $h_{i\bar{j}} = h(s_i,s_j)$ be the components of h with respect to $s = \{s_1,\dots,s_r\}$. We write the curvature form Ω^i_j of ∇ as $\Omega^i_j = \sum R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ and

$$R_{i\bar{j}\alpha\bar{\beta}} := \sum h_{l\bar{j}} R^l_{i\alpha\bar{\beta}} = -\partial_{\bar{\beta}}\partial_\alpha h_{i\bar{j}} + \sum h^{l\overline{m}} \partial_\alpha h_{i\overline{m}} \partial_{\bar{\beta}} h_{l\bar{j}}, \tag{3.13}$$

where $\partial_\alpha := \partial/\partial z^\alpha$ and $\partial_{\bar{\beta}} := \partial/\partial \bar{z}^\beta$. On the other hand, (3.7) and (3.10) imply

$$\partial_\alpha h_{i\bar{j}} = \int_{\mathbb{P}_z} \mathcal{X}_\alpha(\mathfrak{g}(\widetilde{s}_i,\widetilde{s}_j))dv = \int_{\mathbb{P}_z} \mathfrak{g}(\mathscr{D}_\alpha \widetilde{s}_i,\widetilde{s}_j)dv$$

and

$$\partial_{\bar{\beta}}\partial_\alpha h_{i\bar{j}} = \int_{\mathbb{P}_z} \mathcal{X}_{\bar{\beta}}(\mathfrak{g}(\mathscr{D}_\alpha\widetilde{s}_i,\widetilde{s}_j))dv = \int_{\mathbb{P}_z} \left\{\mathfrak{g}(\mathscr{D}_{\bar{\beta}}\mathscr{D}_\alpha\widetilde{s}_i,\widetilde{s}_j) + \mathfrak{g}(\mathscr{D}_\alpha\widetilde{s}_i,\mathscr{D}_\beta\widetilde{s}_j)\right\} dv,$$

where $\mathscr{D}_\alpha := \mathscr{D}_{\mathcal{X}_\alpha}$ and $\mathscr{D}_{\overline{\beta}} := \mathscr{D}_{\mathcal{X}_{\overline{\beta}}}$. Hence, from (3.13), we get

$$R_{i\bar{j}\alpha\bar{\beta}} - \sum h^{l\overline{m}}\partial_\alpha h_{i\overline{m}}\partial_{\bar{\beta}}h_{l\bar{j}} = \int_{\mathbb{P}_z} \left\{\mathscr{R}_{i\bar{j}\alpha\bar{\beta}} - \mathfrak{g}(\mathscr{D}_\alpha\widetilde{s}_i, \mathscr{D}_\beta\widetilde{s}_j)\right\} dv, \tag{3.14}$$

where we put $\mathfrak{g}(\mathscr{D}_{\overline{\beta}}\mathscr{D}_\alpha\widetilde{s}_i, \widetilde{s}_j) := -\mathscr{R}_{i\bar{j}\alpha\bar{\beta}}$.

We use the notations $R(u\otimes X)$ and $\mathscr{R}(\widetilde{u}\otimes X^{\mathscr{H}})$ for the Hermitian forms

$$R(u\otimes X) := \sum R_{i\bar{j}\alpha\bar{\beta}}u^iX^\alpha\overline{u^jX^\beta} \quad \text{and} \quad \mathscr{R}(\widetilde{u}\otimes X^{\mathscr{H}}) := \sum \mathscr{R}_{i\bar{j}\alpha\bar{\beta}}u^iX^\alpha\overline{u^jX^\beta}$$

respectively.

Definition 3.1. A Hermitian bundle (E,h) is said to be *Griffiths-negative* if $R(u\otimes X) < 0$ for all non-zero $u \in E_z$ and $X \in T_zM$ at any point $z \in M$[2]. We say that the partial connection $\mathscr{D}$ has *negative curvature* if $\mathscr{R}(\widetilde{u}\otimes X^{\mathscr{H}}) < 0$ is satisfied.

Remark 3.2. If a Rizza structure F comes from a Hermitian metric h, i.e., $F = \sum h_{i\bar{j}}(z)\zeta^i\overline{\zeta^j}$, then $\Psi_{\alpha\bar{\beta}}$ is given by $\Psi_{\alpha\bar{\beta}} = \sum R_{i\bar{j}\alpha\bar{\beta}}(z)\zeta^i\overline{\zeta^j}$ for the curvature tensor $R_{i\bar{j}\alpha\bar{\beta}}$ of (E,h). Hence Theorem 2.1 concludes that, if (E,h) is Griffiths-negative, then E is negative.

Theorem 3.1. *Let $\mathscr{D}$ be the partial connection in $(\mathscr{V},\mathfrak{g})$ determined by the horizontal sub-bundle $\mathscr{H}$, and let ∇ be the averaged connection of $\mathscr{D}$. Then the curvatures R of ∇ and $\mathscr{R}$ of $\mathscr{D}$ satisfy*

$$R(u\otimes X) \leq \int_{\mathbb{P}_z} \mathscr{R}(\widetilde{u}\otimes X^{\mathscr{H}})dv \tag{3.15}$$

for all $u \in E_z$ and $X \in T_zM$ at any point $z \in M$. Thus, if the partial connection $\mathscr{D}$ in $(\mathscr{V},\mathfrak{g})$ has negative curvature, then (E,h) is Griffiths-negative.

Proof. Let $z_0 \in M$ be an arbitrary point, and let $s = \{s_1,\dots,s_r\}$ be a local frame field in (E,h) such that s is normal at $z_0 \in M$, i.e., $h_{i\bar{j}}(z_0) = \delta_{ij}$ and $\partial_\alpha h_{i\bar{j}}(z_0) = 0$. Then (3.14) implies

$$R_{i\bar{j}\alpha\bar{\beta}}(z_0) = \left[\int_{\mathbb{P}_z} \left\{\mathscr{R}_{i\bar{j}\alpha\bar{\beta}} - \mathfrak{g}(\mathscr{D}_\alpha\widetilde{s}_i, \mathscr{D}_\beta\widetilde{s}_j)\right\} dv\right]_{z=z_0},$$

and thus

$$\sum R_{i\bar{j}\alpha\bar{\beta}}(z_0)u^iX^\alpha\overline{u^jX^\beta}$$

[2]The Griffiths-negativity of (E,h) is equivalent to the existence of Griffiths-positive Hermitian structure in E^* (see e.g., [Sh-So])

$$= \left[\int_{\mathbb{P}_z} \left\{ \sum \mathscr{R}_{i\bar{j}\alpha\bar{\beta}} u^i X^\alpha \overline{u^j X^\beta} - \left\| \sum (\mathscr{D}_\alpha \widetilde{s}_i) u^i X^\alpha \right\|^2 \right\} dv \right]_{z=z_0}$$
$$\leq \left[\int_{\mathbb{P}_z} \left\{ \sum \mathscr{R}_{i\bar{j}\alpha\bar{\beta}} u^i X^\alpha \overline{u^j X^\beta} \right\} dv \right]_{z=z_0}$$

for all $(u^i) \in \mathbb{C}^r$ and $(X^\alpha) \in \mathbb{C}^m$, where we put

$$\left\| \sum (\mathscr{D}_\alpha \widetilde{s}_i) u^i X^\alpha \right\|^2 = \mathfrak{g} \left(\sum (\mathscr{D}_\alpha \widetilde{s}_i) u^i X^\alpha, \sum (\mathscr{D}_\beta \widetilde{s}_j) u^j X^\beta \right).$$

This inequality completes the proof. □

3.2. Main theorem. Let $\mathcal{E}^\perp$ be the orthogonal complement of $\mathcal{E}$ with respect to g, i.e., $\mathcal{E}^\perp = \{Z \in \Gamma(V) \mid g(Z, \mathcal{E}) = 0\}$. We denote by $Z \longmapsto Z^\perp$ the orthogonal projection. Then (3.3) implies

$$(\rho^* \mathfrak{g})(Z, W) = g^\perp(Z, W) = \frac{1}{F} g(Z^\perp, W^\perp).$$

Hence $(\mathscr{V}, \mathfrak{g})$ is interpreted as $(\mathcal{E}^\perp, g^\perp)$ through the projection ρ. From the definition of $\mathscr{D}$ and (3.6), it follows that

$$\mathscr{D}_{\bar{\beta}} \mathscr{D}_\alpha \widetilde{s}_i = \rho_* D_{\bar{\beta}} D_\alpha \left(\frac{\partial}{\partial \zeta^i} \right)^\perp$$

since $\rho_* \left(\partial / \partial \zeta^i \right)^\perp = \widetilde{s}_i$, and

$$D_{\bar{\beta}} D_\alpha \left(\frac{\partial}{\partial \zeta^i} \right)^\perp = D_{\bar{\beta}} D_\alpha \frac{\partial}{\partial \zeta^i} - \frac{1}{F} g \left(D_{\bar{\beta}} D_\alpha \frac{\partial}{\partial \zeta^i}, \mathcal{E} \right) \mathcal{E} = - \sum K^l_{i\alpha\bar{\beta}} \left(\frac{\partial}{\partial \zeta^l} \right)^\perp.$$

Therefore

$$\mathfrak{g} \left(\mathscr{D}_{\bar{\beta}} \mathscr{D}_\alpha \widetilde{s}_i, \widetilde{s}_j \right) = \mathfrak{g} \left(\rho_* D_{\bar{\beta}} D_\alpha \left(\frac{\partial}{\partial \zeta^i} \right)^\perp, \rho_* \left(\frac{\partial}{\partial \zeta^j} \right)^\perp \right)$$
$$= - \frac{1}{F} g \left(\sum K^l_{i\alpha\bar{\beta}} \left(\frac{\partial}{\partial \zeta^l} \right)^\perp, \left(\frac{\partial}{\partial \zeta^j} \right)^\perp \right).$$

For any $u = \sum u^i s_i(z) \in E_z$, we set

$$Z_u := \sum u^i \left(\frac{\partial}{\partial \zeta^i} \right)^\perp \in \mathcal{E}^\perp_{(z,\zeta)}.$$

Proposition 3.5. *Let D be the partial connection in (V, g), and let $\mathscr{D}$ be the partial connection in $(\mathscr{V}, \mathfrak{g})$ as above. Then the curvatures K of D and $\mathscr{R}$ of $\mathscr{D}$ satisfy the relation*

$$\mathscr{R}(\widetilde{u} \otimes X^{\mathscr{H}}) = \frac{1}{F} K(Z_u \otimes X^H) \tag{3.16}$$

at every point $(z, \zeta) \in E^0$ for any $u \in E_z$ and $X \in T_z M$.

The curvature R of the averaged connection ∇ satisfies the inequality (3.15). Thus, if (E, F) is Rizza-negative, then (3.15) and (3.16) conclude

$$R(u \otimes X) \leq \int_{\mathbb{P}_z} \frac{1}{F} K(Z_u \otimes X^H) dv < 0.$$

Hence we have

Theorem 3.2. *Let E be a holomorphic vector bundle admitting a complex Finsler structure F. If (E, F) is Rizza-negative, then E admits a Hermitian metric h such that (E, h) is Griffiths-negative.*

4. Appendix: Proof of Proposition 3.1

Let $\{\omega_z\}$ be a smooth family of Kähler forms in $\mathbb{P}(E)$. Let $\{U_{(j)}\}$ of $\mathbb{P}(E)$ be an open covering defined by $U_{(j)} = \{(z, [\zeta]) \in \phi^{-1}(U) \mid \zeta^j \neq 0\}$. Then $\{U_{z,(j)} := \mathbb{P}_z \cap U_{(j)}\}$ defines an open covering of $\mathbb{P}_z$. Denoted by $\omega_z = \sqrt{-1}\partial\bar{\partial}\mathfrak{G}_{(j)}$ for a smooth function $\mathfrak{G}_{(j)}$ in $U_{z,(j)}$,

$$\mathfrak{G}_{(j)} - \mathfrak{G}_{(i)} = k_{(ij)} + \overline{k_{(ij)}} \tag{4.1}$$

for some $k_{(ij)} \in Z^1(U_{z,(i)} \cap U_{z,(j)}, \mathcal{O}_{\mathbb{P}_z})$ since $\mathfrak{G}_{(j)} - \mathfrak{G}_{(i)}$ is pluri-harmonic on $U_{z,(i)} \cap U_{z,(j)} \neq \emptyset$[3]. Then $H^1(\mathbb{P}_z, \mathcal{O}_{\mathbb{P}_z}) = 0$ assures that we can take $k_{(j)} \in C^0(U_{z,(j)}, \mathcal{O}_{\mathbb{P}_z})$ satisfying

$$k_{(ij)} = (k_{(j)} - \log \zeta^j) - (k_{(i)} - \log \zeta^i),$$

where $\{k_{(i)}\}$ are smooth in $z \in U$. Then (4.1) implies

$$\mathfrak{G}_{(j)} - (k_{(j)} + \overline{k_{(j)}}) + \log |\zeta^j|^2 = \mathfrak{G}_{(i)} - (k_{(i)} + \overline{k_{(i)}}) + \log |\zeta^i|^2.$$

Putting $F_{(j)}(z, [\zeta]) = \exp[\mathfrak{G}_{(j)} - (k_{(j)} + \overline{k_{(j)}})]$, we have $|\zeta^j|^2 F_{(j)}(z, [\zeta]) = |\zeta^i|^2 F_{(i)}(z, [\zeta])$ on $U_{z,(i)} \cap U_{z,(j)}$. We define $F : E^0 \to \mathbb{R}$ by

$$F(z, \zeta) = |\zeta^j|^2 F_{(j)}(z, [\zeta]). \tag{4.2}$$

[3] $\mathcal{O}_{\mathbb{P}_z}$ denotes the sheaf of germs of holomorphic functions in $\mathbb{P}_z$

Since F satisfies (F3) for any $(z,\zeta) \in E^0$ and $\lambda \in \mathbb{C}^*$, we can extend F continuously on the whole of E by setting $F(z,0) = 0$. Thus any smooth family of Kähler forms $\{\omega_z\}$ in $\mathbb{P}(E)$ determines a complex Finsler structure F in E by (4.2).

Denoted by $F_z := F \restriction_{E_z^0}$ the restriction of F to the fiber $E_z^0 := E_z \backslash \{0\}$, we have

$$\sqrt{-1}\partial\bar{\partial}\log F_z = \sqrt{-1}\partial\bar{\partial}\left(\log F_{(j)} \restriction_{E_z^0}\right) = \sqrt{-1}\partial\bar{\partial}\mathfrak{G}_{(j)} \tag{4.3}$$

from the construction of F. Further

$$\partial\bar{\partial}F_z = F_z \cdot \partial\bar{\partial}\log F_z + \frac{1}{F_z}\partial F_z \wedge \bar{\partial}F_z \tag{4.4}$$

show that F is strongly pseudo-convex along E_z, i.e., F is a Rizza structure in E. This completes the proof of Proposition 3.1.

The Kähler potentials $\{\mathfrak{G}_{(j)}\}$ of ω_z are not uniquely determined. Let $\widetilde{F}$ be a Finsler structure obtained from another Kähler potentials $\{\widetilde{\mathfrak{G}}_{(j)}\}$. It follows from $\sqrt{-1}\partial\bar{\partial}\mathfrak{G}_{(j)} = \sqrt{-1}\partial\bar{\partial}\widetilde{\mathfrak{G}}_{(j)}$ that $\log \widetilde{F}_z - \log F_z$ is pluri-harmonic on $\mathbb{P}_z$, and thus it is a constant σ_z in $\mathbb{P}_z$. Consequently we have $\widetilde{F}_z = e^{\sigma_z}F_z$. The corresponding Finsler structures F and $\widetilde{F}$ satisfy the relation $\widetilde{F} = e^{\sigma(z)}F$ for a smooth function $\sigma(z)$ on M. Hence the Rizza structure obtained from $\{\omega_z\}$ is unique up to the conformal factor $e^{\sigma(z)}$ in M.

Corollary 4.1. *Any smooth family $\{\omega_z\}$ of Kähler forms in $\mathbb{P}_z$ determines the conformal class of a Rizza structure F in E.*

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