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Character of the solution of difference equations in the field of Mikusiński operators

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Abstract. In this paper a class of differential equations of the first order in the field of Mikusiński operators, $\mathcal{F}$, is considered. This class may correspond to a class of partial differential equations with constant coefficients.

We construct a discrete analogue for these differential equations in $\mathcal{F}$, and obtain difference equations of the first order. We find conditions for the existence of solutions of difference equations, construct them and show that they can be treated as approximate solutions of the corresponding differential equations in $\mathcal{F}$. For that case, we also estimate the error of approximation.

1. Notations and notions

For the convenience of the reader we shall rewrite some of the basic notions from the Mikusiński operational calculus; it is completely exposed in the monographs [2] and [3].

The set of continuous functions with supports in $[0, \infty)$, denoted by $\mathcal{C}_+$, with the usual addition and the multiplication given by the convolution

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \quad t \geq 0,$$

is a ring. By the Titchmarsh theorem, $\mathcal{C}_+$ has no divisors of zero, hence its quotient field can be defined (see [2]). The elements of this field, the Mikusiński operator field, $\mathcal{F}$, are called operators. They are quotients of the form

$$\frac{f}{g}, \quad f \in \mathcal{C}_+, \ 0 \not\equiv g \in \mathcal{C}_+,$$

where the last division is observed in the sense of convolution.

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(It is easy to show that if one starts, instead from $\mathcal{C}_+$, with the ring of locally integrable functions with supports in $[0, \infty)$, then one gets the same field $\mathcal{F}$.)

Among the most important operators are the integral operator ℓ and its inverse operator, the differential operator, s, while I is the identical operator. We have

$$\ell s = I, \quad \ell^\alpha = \left\{ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right\}, \quad \alpha > 0.$$

If the function $f \in \mathcal{C}_+$ has a continuous n-th derivative, then it holds

$$\{f^{(n)}(t)\} = s^n f - s^{n-1} f(0) - ... - f^{(n-1)}(0) I.$$

Let us denote by $\mathcal{F}_c$ the subset of $\mathcal{F}$ consisting of the operators representing continuous functions. If $a \in \mathcal{F}_c$ represents the continuous function $a(t), t \geq 0$, then we put $a = \{a(t)\}$. We denote by $\mathcal{F}_I$ the subset of $\mathcal{F}$ consisting of the elements αI, for some numerical constant α.

An operational function $u(x)$ will be said to be continuous ([2], p. 191) in a finite open interval J if it can be represented as

$$u(x) = q\{u(x,t)\},$$

where $u(x,t)$ is a continuous function (in the usual sense) in the domain $D = \{(x,t),\ x \in J,\ t \geq 0\}$, and q is some nonzero operator from $\mathcal{F}$. Moreover, an operational function $u(x)$ will be said to have a continuous first order derivative $u'(x)$ in an interval $J \subset \mathbb{R}$ if the function $u(x,t)$ has a continuous partial derivative $\frac{\partial}{\partial x} u(x,t)$ in the domain D, and by definition it holds

$$u'(x) = q \left\{ \frac{\partial}{\partial x} u(x,t) \right\}.$$

In this manner one also defines the derivatives of higher order of an operational function.

The absolute value of an operator a from $\mathcal{F}_c$, $a = \{a(t)\}$, denoted by $|a|$, is the operator $|a| = \{|a(t)|\}$. Also, we put $|a(x)| = \{|a(x,t)|\}$.

In [2], p. 237, J. Mikusiński proposed the following comparison of operators from $\mathcal{F}_c$. For two operators $a = \{a(t)\}$ and $b = \{b(t)\}$ from $\mathcal{F}_c$ he defined the relation "$\leq$" by

$$a \leq b \quad \text{iff} \quad a(t) \leq b(t) \ \text{ for each } \ t \geq 0.$$

Analogously, we shall say for two operator functions $a(x)$ and $b(x)$ that

$$a(x) \leq_T b(x), \quad x \in [c,d],$$

if $a(x)$ and $b(x)$ are representing continuous real valued functions of two variables, $a(x) = \{a(x,t)\}$, $b(x) = \{b(x,t)\}$ and

$$a(x,t) \leq b(x,t), \quad \text{for} \ \ t \in [0,T], \quad x \in [c,d].$$

In this paper we shall use only the so called type I convergence (shortly: convergence) in the field $\mathcal{F}$ (see [2], p. 155). By definition, a sequence of operators $(a_n)_{n\in\mathbb{N}}$ converges to an operator a iff there exists an operator $g \neq 0$, such that $(ga_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions on $[0,\infty)$ which converges uniformly on every finite interval to the continuous function ag.

The convergence of an infinite sum in the field of Mikusiński operators is defined accordingly. An example that will be used several times in this paper is the infinite series

$$\sum_{i=1}^{\infty} \phi^i,$$

where $\phi \in \mathcal{F}_c$. It is important to note that this series converges and its sum is an operator from $\mathcal{F}_c$.

(This follows from the estimations in the field $\mathcal{F}$:

$$|\phi| \leq_T \Phi_T \ell \implies |\phi|^n \leq_T (\Phi_T)^n \ell^n \leq_T (\Phi_T)^n \frac{T^{n-1}}{(n-1)!}\ell,$$

with $\phi = \{\phi(t)\}$ and $\Phi_T = \max\limits_{0\leq t\leq T} |\phi(t)|$.)

2. Introduction

We analyze the following problem in the field of Mikusiński operators

$$\sum_{m=0}^{p} k_m s^m u'(x) + \sum_{j=0}^{r} q_j s^j u(x) = f(x), \quad x \in \mathbb{R}, \tag{1}$$

with the initial condition

$$u(0) = R. \tag{2}$$

In relation (1) (and throughout the paper) we suppose that p and r are natural numbers, the coefficients $k_m,\ m = 0, 1, .., p$, and $q_j,\ j = 0, 1, ..., r$, are numerical constants, $k_p \neq 0, q_r \neq 0$, and $f(x)$ is a given, while $u(x)$ is the unknown operational function. In relation (2) R is a given operator. In this paper we shall mostly consider two cases: the first when R represents a continuous function (i.e., R is from $\mathcal{F}_c$), and the other when R is from $\mathcal{F}_I$. (In distribution theory, the last case would mean that R is a multiple of the Dirac delta measure.)

The significance of equation (1) lies in the fact that if $f(x)$ and $u(x)$ are operators representing continuous functions $f(x,t)$ and $u(x,t)$ respectively, then equation (1) corresponds to the partial differential equation

$$\sum_{m=0}^{p} k_m \frac{\partial^{m+1} u(x,t)}{\partial x \partial t^m} + \sum_{j=0}^{r} q_j \frac{\partial^j u(x,t)}{\partial t^j} = f_1(x,t),$$

with certain given conditions. In the last equation the right-hand side function $f_1(x,t)$ is expressed via the function $f(x,t)$ from (1) and the imposed conditions.

(As an example for the imposed conditions, we can have the following. If in the problem (1), (2) it holds that $r > p$ and R is from $\mathcal{F}_c$, then the conditions can have the form

$$\frac{\partial^j u(x,0)}{\partial t^j} = 0, \quad j = 0, 1, .., r-1, \ x \in \mathbb{R},$$

(Cauchy conditions), and

$$u(0,t) = R(t), \quad t \geq 0,$$

where $R = \{R(t)\}$.)

In the field of Mikusiński operators, the solution of the homogeneous equation for the differential equation (1) has the form $e^{x\omega}$, where ω is the solution of the characteristic equation of equation (1)

$$\sum_{m=0}^{p} k_m s^m \omega + \sum_{j=0}^{r} q_j s^j = 0.$$

It is well known that in the field of Mikusiński operators one can apply the algebraic operations (like, e.g., addition, multiplication, division, etc.) in the same way as when one deals with real numbers. So, the solution of the characteristic equation has the form

$$\omega = \frac{-\sum\limits_{j=0}^{r} q_j s^j}{\sum\limits_{m=0}^{p} k_m s^m} = \frac{-\sum\limits_{j=0}^{r} q_j \ell^{p-j}}{\sum\limits_{m=0}^{p} k_m \ell^{p-m}} = \frac{-\sum\limits_{j=0}^{r} q_j \ell^{p-j}}{k_p I + \sum\limits_{m=0}^{p-1} k_m \ell^{p-m}}.$$

If we denote

$$P = \sum_{m=0}^{p-1} k_m \ell^{p-m} \quad \text{and} \quad Q = \sum_{j=0}^{r} q_j \ell^{p-j}, \tag{3}$$

then we have

$$\omega = \frac{-Q}{k_p I + P} = \frac{Q}{k_p} \sum_{i=0}^{\infty} (-1)^{i+1} (\frac{P}{k_p})^i.$$

The operator P is from $\mathcal{F}_c$; for $p > r$ the operator Q represents a continuous function too. In view of what was said on the convergence of the geometric series at the end of Section 1, the last series converges in $\mathcal{F}$ and hence ω is from $\mathcal{F}_c$.

If $p = r$, then $\omega = \alpha_1 I + \beta_c$, where α_1 is a numerical constant and β_c is an operator representing a continuous function.

Hence, in the case $p \geq r$ the solution ω of the characteristic equation is logarithmic, i.e. the exponential function $e^{x\omega}$ exists (see [3], p. 18). Then the solution of the homogeneous equation corresponding to the problem (1), (2), $R \cdot e^{x\omega}$, is from $\mathcal{F}_c$, provided that $R \in \mathcal{F}_c$ and $x \neq 0$.

If, however, $p < r$, i.e. $r = p + \nu$, $\nu \in \mathbb{N}$, then we have

$$\omega = \frac{-q_r s^\nu - \cdots - q_p I - \sum\limits_{n=0}^{p-1} q_n \ell^{p-n}}{k_p I + \sum\limits_{m=0}^{p-1} k_m \ell^{p-m}} =$$

$$= \frac{I}{k_p}(-q_r s^\nu - \cdots - q_p I - \sum_{n=0}^{p-1} q_n \ell^{p-n}) \sum_{i=0}^{\infty} (\frac{-P}{k_p})^i.$$

So we obtain the equality

$$\omega = \frac{-I}{k_p}(q_r s^\nu + a_{r-1} s^{\nu-1} + \ldots + a_0 I) + \beta_{c,1}, \tag{4}$$

where $a_j,\ j = 0, 1, .., r-1$, are numerical constants and $\beta_{c,1}$ is an operator representing a continuous function. Therefore if $\nu = r - p > 1$, the solution of the homogeneous equation $e^{x\omega}$ does not exist as an operator from the Mikusiński operator field. In this case, we cannot apply some general method (which uses the solution of the homogeneous equation) for obtaining the particular solution of equation (1).

In the special case $\nu = 1$, the expression $\exp(-\frac{xq_r}{k_p})s$ for $\frac{xq_r}{k_p} > 0$ gives the shift operator from $\mathcal{F}$.

We now look for the approximate solution of the problem (1), (2). In fact, in Theorems 1, 2 and 3 we shall find the approximate solution for the case when $r > p$. Unfortunately, this method cannot always be applied in the case $p > r$. (See Theorems 5 and 6 below.) Let us mention that the case $p = r$ was considered in the papers [4]–[7].

From now on h will be a positive number. As is usual in numerical analysis, we replace the derivative $u'(x)$ with the quotient

$$\frac{u(x+h) - u(x)}{h}.$$

Putting

$$A=\sum_{m=0}^{p}k_m s^m \quad \text{and} \quad B=\sum_{j=0}^{r}q_j s^j,$$

we obtain the following difference equation in the field $\mathcal{F}$ (see [1], p. 14):

$$(5)\qquad A\frac{u(x+h)}{h}+(B-\frac{A}{h})u(x)=f(x).$$

Let $x_0=0$, put $x_n=x_{n-1}+h,\ n\in\mathbb{Z}$, and define the operator f_n by

$$f_n=f(x_n), \qquad n\in\mathbb{Z}.$$

Then instead of equation (5), we shall have the following difference equation in $\mathcal{F}$:

$$(6)\qquad au_n+bu_{n+1}=f_n, \quad n\in\mathbb{Z},$$

where

$$(7)\qquad a=\sum_{j=0}^{r}q_j s^j-\frac{I}{h}\sum_{m=0}^{p}k_m s^m,$$

and

$$(8)\qquad b=\frac{I}{h}\sum_{m=0}^{p}k_m s^m.$$

We shall impose the condition $u_0=R$, where the operator R is the one given in (2).

The solution $u_n,\ n\in\mathbb{Z}$, of equation (6) will be treated in Section 4 as the approximate solution of the problem (1), (2).

The formal solution of equation (6) has the form

$$(9)\qquad u_n=\sum_{k=-\infty}^{\infty}G_{n-k}f_k\,,$$

where

$$(10)\qquad G_{n-k}=\begin{cases} R(-\frac{a}{b})^{n-k}, & n-k\le 0,\\ (R-\frac{I}{a})(-\frac{a}{b})^{n-k}, & n-k\ge 1,\end{cases}$$

and R is the operator given by the initial condition (2). In the next section we shall give conditions for the (type I) convergence of the series (9).

3. The character of the solution of the difference equation

Our goal is to give an approximate solution for equation (1) using the solution of the difference equation (6).

Lemma 1. *If in equation* (1) *it holds that* $p < r$ *and* a *and* b *are given by relations* (7) *and* (8), *then the operator*

$$\gamma_c := \frac{b}{a} \tag{11}$$

belongs to $\mathcal{F}_c$.

PROOF. The operator $\frac{b}{a}$ can be written as

$$\frac{b}{a} = \frac{\frac{I}{h}\sum\limits_{m=0}^{p} k_m s^m}{\sum\limits_{j=0}^{r} q_j s^j - \frac{I}{h}\sum\limits_{m=0}^{p} k_m s^m} = \frac{\sum\limits_{m=0}^{p} k_m \ell^{r-m}}{h\sum\limits_{j=0}^{r} q_j \ell^{r-j} - \sum\limits_{m=0}^{p} k_m \ell^{r-m}} =$$

$$= \frac{\sum\limits_{m=0}^{p} k_m \ell^{r-m}}{hq_r I + h\sum\limits_{j=0}^{r-1} q_j \ell^{r-j} - \sum\limits_{m=0}^{p} k_m \ell^{r-m}}.$$

If we denote by P_1 and Q_1 the following operators from $\mathcal{F}$:

$$P_1 = \sum_{m=0}^{p} k_m \ell^{r-m}, \qquad Q_1 = \sum_{j=0}^{r-1} q_j \ell^{r-j}, \tag{12}$$

then we have

$$\frac{b}{a} = \frac{P_1}{hq_r} \sum_{i=0}^{\infty} (-1)^i \left(\frac{hQ_1 - P_1}{hq_r}\right)^i.$$

Since the operators P_1 and Q_1 from (12) are from $\mathcal{F}_c$, it holds that the last infinite series converges and represents a continuous function. This implies that $\frac{b}{a}$ is in $\mathcal{F}_c$. □

Lemma 2. *If in equation* (1) $p < r$, *i.e.* $r = p + \nu$, $\nu \in \mathbb{N}$, *and* a *and* b *are given by relations* (7) *and* (8), *then the operator* $\frac{a}{b} \in \mathcal{F}$ *belongs neither to* $\mathcal{F}_c$ *nor to* $\mathcal{F}_I$.

PROOF. We have

$$\frac{a}{b} = \frac{h\sum\limits_{j=0}^{r} q_j \ell^{r-j} - \sum\limits_{m=0}^{p} k_m \ell^{r-m}}{\sum\limits_{m=0}^{p} k_m \ell^{r-m}} = \frac{hq_r I + h\sum\limits_{j=0}^{r-1} q_j \ell^{r-j} - \sum\limits_{m=0}^{p} k_m \ell^{r-m}}{\ell^{r-p}\sum\limits_{m=0}^{p} k_m \ell^{p-m}}.$$

Then, using notations (12) and (3), we obtain

$$\frac{a}{b} = s^{r-p}\frac{hq_r I + hQ_1 - P_1}{k_p I + P} =$$

$$= s^{r-p}\left(\frac{hq_r}{k_p}I + \frac{hQ_1 - P_1}{k_p} + \frac{hq_r + hQ_1 - P_1}{k_p}\sum_{i=1}^{\infty}(-1)^i(\frac{P}{k_p})^i\right),$$

and we have

$$\frac{a}{b} = s^{\nu}(\alpha_2 I + \beta_{c,2}), \tag{13}$$

where α_2 is a nonzero numerical constant, while $\beta_{c,2}$ is an operator from $\mathcal{F}_c$. Since $\nu > 0$, in view of the first member, the operator $\frac{a}{b}$ does not belong to $\mathcal{F}_c$, nor to $\mathcal{F}_I$. □

We can prove now

Theorem 1. *Suppose that in equation* (1) *it holds* $r = p + \nu$ *for some natural number* ν. *Then the solution of the homogeneous equation corresponding to the equation* (6) *has the form*

$$u_n = R \cdot s^{n\nu}\left(\alpha_3(n)I + \beta_{c,3}(n)\right), \quad n \neq 0, \quad n \in \mathbb{Z}, \tag{14}$$

where R *is given by* (2), $\alpha_3(n) \neq 0$ *is a numerical constant and* $\beta_{c,3}(n)$ *is an operator representing a continuous function.*

Moreover, if R *is from* $\mathcal{F}_I$, *then for* $n > 0$ *the solution* u_n *belongs neither to* $\mathcal{F}_c$ *nor to* $\mathcal{F}_I$.

If $n < 0$, *then* u_n *belongs to* $\mathcal{F}_c$ *provided that* R *is from* $\mathcal{F}_c$ *or from* $\mathcal{F}_I$.

Proof. Let us suppose that $n > 0$. From relation (13) (used in the proof of Lemma 2), it follows that the general solution of the homogeneous equation corresponding to (6) can be written as

$$u_n = R\left(-\frac{a}{b}\right)^n = (-1)^n R s^{n\nu}\left(\alpha_2 I + \beta_{c,2}\right)^n =$$

$$= (-1)^n R s^{n\nu}\left(\alpha_2^n I + \binom{n}{1}\alpha_2^{n-1}\beta_{c,2} + ... + \beta_{c,2}^n\right),$$

$\alpha_2 \neq 0$ is a numerical constant and $\beta_{c,2}$ is an operator from $\mathcal{F}_c$. From the last equality, relation (14) follows. Therefore, if R is from $\mathcal{F}_I$, then in this case u_n, $n \in \mathbb{N}$, is an operator which is neither from $\mathcal{F}_c$ nor from $\mathcal{F}_I$. However, if R is from $\mathcal{F}_c$, then for some n u_n might belong to $\mathcal{F}_c$.

If the integer $n < 0$, hence $n_1 = -n \in \mathbb{N}$, then from (13) it follows that

$$u_n = R(-\frac{a}{b})^{-n_1} = (-1)^{n_1} R \cdot \ell^{n_1\nu}\left(\frac{I}{\alpha_2 I + \beta_{c,2}}\right)^{n_1} =$$

$$= (-1)^{n_1} R \cdot \ell^{n_1 \nu} \left(\frac{1}{\alpha_2} \sum_{i=0}^{\infty} (-1)^i \left(\frac{\beta_{c,2}}{\alpha_2} \right)^i \right)^{n_1} =$$

$$= (-1)^{n_1} R \cdot s^{n \nu} \left(\frac{I}{\alpha_2} + \sum_{i=1}^{\infty} (-1)^i \left(\frac{\beta_{c,2}}{\alpha_2} \right)^i \right)^{n_1} .$$

(The last two series converge in the field $\mathcal{F}$ because $\beta_{c,2}$ is from $\mathcal{F}_c$.) So, again for $n < 0$, we obtain relation (14), wherefrom it follows that the solution u_n of equation (6) belongs to $\mathcal{F}_c$, provided that R is from $\mathcal{F}_c$, or from $\mathcal{F}_I$.

One can obtain the same result from Lemma 1, using the equality

$$u_n = R \left(-\frac{b}{a} \right)^{n_1}, \quad n = -n_1. \qquad \square$$

Remark. Notice that if in (14) $n = 0$, then $u_0 = R$.

In the case when $r = p + \nu, \quad \nu > 1$, the solution of the homogeneous equation corresponding to the equation (1) in the field $\mathcal{F}$ does not exist (see relation (4)). However, in Theorem 1 we have proved that the solution of the difference equation (6) exists in the field $\mathcal{F}$. Since the equation (6) is the discrete analogue for the differential equation (1), it follows that relation (14) can be treated as the approximate solution of the problem (1), (2).

Theorem 2. *Assume that $p < r$ and that the operators a and b are given by* (7) *and* (8) *respectively and, moreover, assume that the right-hand side operators f_n, $n \in \mathbb{Z}$, from relation* (6) *either*

I) *satisfy the following equalities and estimates:*

$$f_n = F_n I, \quad |F_n| < F, \quad n \in \mathbb{Z}\,,$$

or else

II) *are from $\mathcal{F}_c$ and satisfy the estimates*

$$|f_n| \leq_T F_T \ell,$$

for some numerical constants $F_n, n \in \mathbb{Z}$ and some constant F, resp. F_T, independent from n.

Then there exists a solution of equation (6) *in the field of Mikusiński operators and it represents a continuous function.*

Proof. From relation (11) we have

$$\left(\frac{b}{a} \right)^n = \gamma_{c,n}, \quad n \in \mathbb{N},$$

where $\gamma_{c,n}$ is from $\mathcal{F}_c$. Using relation (10), we obtain that the operators G_{n-k} represent continuous functions only if $R = \frac{I}{a} \in \mathcal{F}_c$, and

$$G_{n-k} = \begin{cases} \frac{I}{a}(-1)^{k-n}\left(\frac{b}{a}\right)^{k-n}, & n-k \leq 0, \\ 0, & n-k \geq 1. \end{cases} \tag{15}$$

Now, the solution of (6) has the form

$$u_n = \frac{I}{a}\sum_{k=n}^{\infty}(-1)^{n-k}\gamma_{c,k-n}f_k. \tag{16}$$

Using conditions I or II, we easily get that the series (16) converges in the field of Mikusiński operators and represents a continuous function. Since the operator $\frac{I}{a}$ is from $\mathcal{F}_c$, the solution given by relation (16) represents a continuous function.

Since $\frac{a}{b}$ is neither an operator from $\mathcal{F}_c$, nor from $\mathcal{F}_I$, in the case when $R = 0$, the operator G_{n-k} will not represent a continuous function nor will it be from $\mathcal{F}_I$ and in the field $\mathcal{F}$ we do not consider such series. □

The case when $p = r$ was analyzed in the paper [5], so we shall give without proof only the statements on the character of the solution.

Lemma 3. *If in equation* (1) $p = r$ *and the operators* a *and* b *are given by relations* (7) *and* (8), *then the operators* $\frac{a}{b}, \frac{b}{a} \in \mathcal{F}$, *can be written as*

$$\frac{a}{b} = \alpha_1 I + \beta_{c,4}, \tag{17}$$

$$\frac{b}{a} = \alpha_2 I + \beta_{c,5}, \tag{18}$$

where α_1 *and* $\alpha_2 = \frac{I}{\alpha_1}$ *are nonzero numerical constants and* $\beta_{c,4}$ *and* $\beta_{c,5}$ *are operators from* $\mathcal{F}_c$.

Also, we have

Theorem 3. *Assume that* $|\alpha_1|$ *from Lemma 3 is nonequal to 1 and assume that the right-hand side operators* $f_n,\ n = 0, \pm 1, ..,$ *from relation* (6) *either*

I) *satisfy the following equalities and estimates:*

$$f_n = F_n I, \quad |F_n| < F,$$

or else

II) *are from* $\mathcal{F}_c$ *and satisfy the estimation*

$$|f_k| \leq_T F_T \ell,$$

for some numerical constants $F_n, n = 0, \pm 1, ..,$ *and some constants* F *and* F_T *independent from* n.

Then the solution u_n, $n \in \mathbb{Z}$, *of equation* (6) *exists in both cases (I and II) and represents a continuous function.*

If, however, $|\alpha_1| = 1$, *then equation* (6) *has no solution in* $\mathcal{F}_c$.

At the end of this section, we shall analyze the case $p > r$.

Lemma 4. *If in equation* (1) $p > r$ *and* a *and* b *are given by relations* (7) *and* (8), *then the operator* $\frac{b}{a} \in \mathcal{F}$ *can be written as*

$$\frac{b}{a} = -I + \beta_{c,6}, \tag{19}$$

where $\beta_{c,6}$ *is an operator from* $\mathcal{F}_c$.

Proof. We have

$$\frac{b}{a} = \frac{\frac{I}{h}\sum\limits_{m=0}^{p} k_m s^m}{\sum\limits_{j=0}^{r} q_j s^j - \frac{I}{h}\sum\limits_{m=0}^{p} k_m s^m} = \frac{\sum\limits_{m=0}^{p} k_m \ell^{p-m}}{h\sum\limits_{j=0}^{r} q_j \ell^{p-j} - \sum\limits_{m=0}^{p} k_m \ell^{p-m}} =$$

$$= \frac{k_p I + \sum\limits_{m=0}^{p-1} k_m \ell^{p-m}}{-k_p I + h\sum\limits_{j=0}^{r} q_j \ell^{p-j} - \sum\limits_{m=0}^{p-1} k_m \ell^{p-m}}.$$

Using the notations given by (3), we have

$$\frac{b}{a} = -I + \frac{-P}{k_p} + \frac{-k_p I - P}{k_p} \sum_{i=1}^{\infty} \left(\frac{hQ - P}{k_p}\right)^i.$$

Since the operators P and Q from (3) and $hQ - P$ are from $\mathcal{F}_c$, the last infinite series converges and represents a continuous function. This implies the decomposition (19). $\square$

In a similar manner we can prove

Lemma 5. *If in equation* (1) $p > r$ *and* a *and* b *are given by relations* (7) *and* (8), *then the operator* $\frac{a}{b}$ *can be written as*

$$\frac{a}{b} = -I + \beta_{c,7}, \tag{20}$$

where $\beta_{c,7}$ is an operator from $\mathcal{F}_c$.

Proof. Using notation (3), we can write

$$\frac{a}{b}=\frac{\sum\limits_{j=0}^{r}q_js^j-\frac{I}{h}\sum\limits_{m=0}^{p}k_ms^m}{\frac{I}{h}\sum\limits_{m=0}^{p}k_ms^m}=\frac{-k_pI+hQ-P}{k_pI+P}=$$

$$=-I+\frac{hQ-P}{k_p}+\frac{-k_pI+hQ-P}{k_p}\sum_{i=1}^{\infty}(-1)^i\left(\frac{P}{k_p}\right)^i.$$

and the decomposition (20) follows. □

Analogously as Theorem 1, we can prove

Theorem 4. *If in equation (1) $r\le p$, then the solution of the homogeneous equation corresponding to the equation (6) has the form*

$$u_n=R(\alpha_3(n)I+\beta_{c,8}(n)), \tag{21}$$

where $\alpha_3(n)$ is a numerical constant and $\beta_{c,8}(n)$ is an operator representing a continuous function.

Also, we can prove now

Theorem 5. *Assume that in equation (1) $p>r$ and the operators a and b are given by relations (7) and (8) respectively and assume that the right-hand side operators $f_n,\ n=0,\pm1,..,$ from relation (6) either*

I) *satisfy the following equalities and estimates:*

$$f_n=F_nI,\quad |F_n|<F,$$

or else

II) *are from $\mathcal{F}_c$ and satisfy the estimates*

$$|f_n|\le_T F_T\ell$$

for some numerical constants $F_n, n=0,\pm1,..,$ and some constants F and F_T independent from n.

Then, there is no solution of equation (6) in the field of Mikusiński operators.

Proof. From relations (19) and (20), for $n\in\mathbb{N}$, it follows that $(\frac{a}{b})^n$ and $(\frac{b}{a})^n$ can be written in the form

$$\left(\frac{a}{b}\right)^n=(-1)^nI+\delta_{c,n},$$

$$\left(\frac{b}{a}\right)^n = (-1)^n I + \delta^1_{c,n},$$

where $\delta_{c,n}$ and $\delta^1_{c,n}$ are operators from $\mathcal{F}_c$, for every $n \in \mathbb{N}$.

Then from relation (10) we have

$$G_{n-k} = \begin{cases} R(I + (-1)^{n-k}\delta^1_{c,k-n}), & n-k \leq 0, \\ (R - \frac{I}{a})(I + (-1)^{n-k}\delta_{c,n-k}), & n-k \geq 1. \end{cases} \tag{22}$$

So, the solution of equation (6) should have the form

$$u_n = \left(R - \frac{I}{a}\right) \sum_{k=-\infty}^{n-1} (I + (-1)^{n-k}\delta_{c,n-k}) f_k + R \sum_{k=n}^{\infty} (I + (-1)^{k-n}\delta^1_{c,k-n}) f_k.$$

Hence,

$$\begin{aligned} u_n = \left(R - \frac{I}{a}\right) &\sum_{k=-\infty}^{n-1} f_k + R \sum_{k=n}^{\infty} f_k + \\ + \sum_{k=-\infty}^{n-1} (-1)^{n-k}\delta_{c,n-k} f_k &+ \sum_{k=n}^{\infty} (-1)^{k-n}\delta^1_{c,k-n} f_k. \end{aligned} \tag{23}$$

If the f_k satisfy either the relation (I) or the relation (II), then neither of the two series from relation (23) converges in the field of Mikusiński operators.

So, even if in the previous cases we have either $R = 0$ or $R = \frac{I}{a}$ we still do not have a solution of the equation (6) via formula (9). □

However, even if neither of the conditions (I) or (II) from Theorem 5 is fulfilled, still there might exist a solution of equation (6) for some $f_n, n \in \mathbb{Z}$.

Theorem 6. *If in equation* (6) *the right-hand side operators* f_n, $n \in \mathbb{N}$ *are of the form* $f_n = \ell^n$ *and zero otherwise, then its solution belongs to* $\mathcal{F}_c$.

Proof. We shall prove only the case when in equation (1) $p > r$, the other two cases ($p = r$ and $p < r$) being similar.

If G_{n-k} has the form (22), then for $R = \frac{I}{a}$ the solution has the form

$$u_n = \frac{I}{a} \sum_{k=n}^{\infty} (I + (-1)^{k-n}\delta^1_{c,k-n})\ell^k.$$

Clearly, the last series converges to an operator from $\mathcal{F}_c$. □

4. The error of approximation

In view of Theorem 2, we shall give the error of approximation for the solution of the problem (1), (2) for $r > p$. This solution is approximated by the solution u_n of the difference equation (6).

Theorem 7. *Let us suppose that the first and the second derivative of the function $u(x)$ (which is the solution of equation (1)) are continuous operational functions.*

If we denote by $u(x_n)$ the value of the exact solution of equation (1) at the point $x = x_n$, then the error of approximation for its approximate solution obtained as the solution of equation (6) u_n, can be estimated by

$$|u(x_n) - u_n| \leq_T h\mathcal{R}(T)M_2(X,T)\ell \tag{24}$$

where $\mathcal{R}(T)$ and

$$M_2(X,T) = \max_{(0\leq t\leq T)\times(-X\leq x\leq X)} \left| \frac{\partial^2 u(x,t)}{\partial x^2} \right| .$$

are positive numerical constants.

Proof. From the difference between the following equations

$$\sum_{m=0}^{p} k_m s^m u'(x_n) + \sum_{j=0}^{r} q_j s^j u(x_n) = f(x_n),$$

$$\sum_{m=0}^{p} k_m s^m \frac{u_{n+1} - u_n}{h} + \sum_{j=0}^{r} q_j s^j u_n = f(x_n),$$

we obtain

$$\sum_{j=0}^{r} q_j s^j (u(x_n) - u_n) = -\sum_{m=0}^{p} k_m s^m \left(u'(x_n) - \frac{u_{n+1} - u_n}{h} \right).$$

Hence, the difference between the exact and the approximate solution is

$$u(x_n) - u_n = \frac{-\sum\limits_{m=0}^{p} k_m s^m}{\sum\limits_{j=0}^{r} q_j s^j} \left(u'(x_n) - \frac{u_{n+1} - u_n}{h} \right).$$

Using the properties of the absolute value in the field of Mikusiński operators, we get

$$|u(x_n) - u_n| \leq \left| \frac{-\sum\limits_{m=0}^{p} k_m \ell^{r-m}}{\sum\limits_{j=0}^{r} q_j \ell^{r-j}} \right| \cdot \left| u'(x_n) - \frac{u_{n+1} - u_n}{h} \right| . \tag{25}$$

The first factor on the right hand side represents a continuous function and can be estimated as follows:

$$\left|\frac{-\sum\limits_{m=0}^{p} k_m \ell^{r-m}}{\sum\limits_{j=0}^{r} q_j \ell^{r-j}}\right| \leq \left|\frac{P_1}{q_r}\right| + \left|\frac{P_1}{q_r}\right| \sum_{i=1}^{\infty} \left|\frac{Q_1}{q_r}\right|^i.$$

If we denote by $\mathcal{P}_1(T)$ and $\mathcal{Q}_1(T)$ the numerical constants obtained from the estimates:

$$\left|\frac{P_1}{q_r}\right| = \left\{\frac{\left|\sum\limits_{m=0}^{p} k_m \frac{t^{r-m-1}}{(r-m-1)!}\right|}{|q_r|}\right\} \leq_T \left\{\frac{\sum\limits_{m=0}^{p} |k_m| \frac{T^{r-m-1}}{(r-m-1)!}}{|q_r|}\right\} \ell \leq \mathcal{P}_1(T)\ell,$$

and

$$\left|\frac{Q_1}{q_r}\right| = \left\{\frac{\left|\sum\limits_{j=0}^{r-1} q_j \frac{t^{r-j-1}}{(r-j-1)!}\right|}{|q_r|}\right\} \leq_T \left\{\frac{\sum\limits_{j=0}^{r-1} |q_j| \frac{T^{r-j-1}}{(r-j-1)!}}{|q_r|}\right\} \ell \leq \mathcal{Q}_1(T)\ell,$$

then we have

$$(26)\qquad \left|\frac{-\sum\limits_{m=0}^{p} k_m \ell^{r-m}}{\sum\limits_{j=0}^{r} q_j \ell^{r-j}}\right| \leq_T \mathcal{P}_1(T)\ell + \mathcal{P}_1(T)\ell \sum_{i=1}^{\infty} \mathcal{Q}_1^i(T)\ell^i \leq_T$$

$$\leq_T \mathcal{P}_1(T)\ell + \mathcal{P}_1(T) \sum_{i=1}^{\infty} \mathcal{Q}_1^i(T) \frac{T^{i-1}}{(i-1)!} \ell^2 \leq_T$$

$$\leq_T \mathcal{P}_1(T)(\ell + \mathcal{Q}_1(T)\exp(T\mathcal{Q}_1(T))\ell^2).$$

From the assumption on the second derivative of the function $u(x)$ (which is the solution of equation (1)) it follows that the partial derivatives by x are continuous functions of two variables on their domains. Then using the Taylor series we obtain

$$(27)\qquad \left|u'(x_n) - \frac{u_{n+1} - u_n}{h}\right| \leq_T M_2(X,T)\ell h, \qquad x_n \in [-X, X] \quad X > 0.$$

From relations (25), (26) and (27) we obtain

$$|u(x_n) - u_n| \leq_T h\mathcal{P}_1(T)\left(1 + \frac{\mathcal{Q}_1 T}{2}\exp(T\mathcal{Q}_1(T))\right) M_2(X,T)T\ell \leq_T$$

$$\leq_T h\mathcal{R}(T)M_2(X,T)\ell. \qquad \square$$

Finally let us remark that the error of approximation for the solution of equation (6), which is treated as the approximate solution of equation (1), is $\mathcal{O}(h)$, $\quad h \to 0$ (as in "classical" numerical analysis).

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