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Intrinsic characterization of completely ruled hypersurfaces

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Abstract. The aim of the present paper is to give an intrinsic characterization of completely ruled (immersed) hypersurfaces in Euclidean spaces through the induced Riemannian metrics. The "number" of locally non-isometric hypersurfaces is also calculated in each dimension.

There are many recent results devoted to, or related to, ruled submanifolds, see e.g. [1], [2], [5]. Let us recall [5] that an isometric immersion ϕ of a Riemannian manifold N^n in the Euclidean space R^{n+1} is *ruled* if N^n admits a continuous codimension one foliation such that ϕ maps each leaf ("ruling") onto an open subset of an affine subspace of R^{n+1}. A ruled map $\phi : N^n \to R^{n+1}$ is *completely ruled* if all rulings are complete (and thus isometric to R^{n-1}). In [5] the following remark was made (with a short indication of the proof). Because this fact is essential for our purpose, we shall formulate it as

Theorem A. *If $\phi : N^n \to R^{n+1}$ $(n \geq 3)$ is completely ruled then the scalar curvature s of N^n is constant along each leaf of the nullity foliation.*

Here the nullity foliation means the integral foliation of the $(n-2)$-dimensional distribution of N^n consisting of the nullity spaces of the curvature tensor. In particular, N^n must be a Riemannian manifold of conullity two [4] unless it is flat in some domain. Also, let us remark that each

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ruling of N^n admits a codimension one foliation composed of the nullity leaves of N^n.

Now, the following nontrivial theorem is crucial for our goal (see [3], Theorem 5.1, in a slightly modified notation):

Theorem B. *Let (N^n, g), $n \geq 3$, be a locally irreducible Riemannian manifold admitting the $(n-2)$-dimensional nullity foliation and such that its scalar curvature is constant along each nullity leaf. Then, in a neighborhood of each point p of a dense open subset $U \subset N^n$, there exist local coordinates $w, x^1, \dots, x^{n-1}$ and an orthonormal coframe of the form*

$$(1) \qquad \begin{cases} \omega^0 = f(w, x^1)\mathrm{d}w, \\ \omega^i = \mathrm{d}x^i + \left(\sum_{j=1}^{n-1} A^i_j(w)x^j\right)\mathrm{d}w \quad (i = 1, \dots, n-1), \end{cases}$$

where $f \neq 0$ and $A^i_j(w) + A^j_i(w) = 0$ The scalar curvature of this metric is given by

$$(2) \qquad s = -2f^{-1}f_{x^1x^1}.$$

Let us add that $f \neq 0$ and $A^i_j = -A^j_i$ are arbitrary smooth functions where the second partial derivative $f_{x^1x^1}$ is nonzero on an open dense subset. Under these conditions the converse of Theorem B also holds (cf. [6]), i.e., the formulas (1) and (2) determine a Riemannian metric of conullity two. Here the nullity foliation is given by the relations $w = \text{const.}$, $x^1 = \text{const.}$ In [6] some criteria of local irreducibility and completeness are also given.

Now, Theorem A says that the immersed completely ruled hypersurfaces must belong, as Riemannian manifolds, to the class described in Theorem B. (The rulings are then characterized by $w = \text{const.}$) Thus, our next goal is to characterize all Riemannian manifolds from Theorem B which admit (locally) an isometric immersion in R^{n+1}. The following calculations generalize and extend those from [6], Section 12.

Let V_p be a simply connected neighborhood of a point $p \in U$ in which the metric g is described through the local coordinates $w, x^1, \dots, x^{n-1}$ and the orthonormal coframe (1). We are looking for a (1,1) tensor field S (the shape operator) satisfying the Gauss equation

$$(3) \qquad R_{XY}Z = g(SX, Z)SY - g(SY, Z)SX$$

and the Codazzi equation

$$(D_X S)Y = (D_Y S)X. \tag{4}$$

(Here D denotes the Levi–Civita connection, and we use the sign convention $R_{XY} = D_{[X,Y]} - [D_X, D_Y]$ for the curvature transformations.)

Let $(E_0, E_1, \dots, E_{n-1})$ be the orthonormal moving frame which is dual to $(\omega^0, \omega^1, \dots \omega^{n-1})$. We have

$$\begin{cases} E_0 = f^{-1}(w, x^1)\Big(\dfrac{\partial}{\partial w} - \sum\limits_{i,j=1}^{n-1} A^i_j x^j \dfrac{\partial}{\partial x^i}\Big), \\ E_i = \dfrac{\partial}{\partial x^i}, \qquad i = 1, \dots, n-1. \end{cases} \tag{5}$$

As in [6] we see easily that the shape operator S must be of the form

$$\begin{cases} SE_0 = aE_0 + bE_1, \\ SE_1 = bE_0 + cE_1, \\ SE_i = 0 \quad \text{for } i = 2, \dots, n-1, \end{cases} \tag{6}$$

where the functions a, b and c satisfy $ac - b^2 \neq 0$. The Gauss equation is then equivalent to

$$s = 2(ac - b^2). \tag{7}$$

The Codazzi equation (4) is equivalent to a system of equations

$$(D_{E_i} S)E_j = (D_{E_j} S)E_i, \qquad i, j = 0,\, 1, \dots, n-1. \tag{8}$$

For $i = 0$ and $j = 1$ we get, as in [6], the explicit equations

$$\begin{cases} E_0(b) + (c - a)f^{-1} f_{x^1} = E_1(a), \\ E_0(c) - 2bf^{-1} f_{x^1} = E_1(b), \\ cf^{-1} A^k_1 = 0 \quad \text{for } k = 2, \dots, n-1. \end{cases} \tag{9}$$

For $i = 0$, $j \geq 2$ we obtain

$$\begin{cases} -f^{-1} b A^1_j = E_j(a), \\ -f^{-1} c A^1_j = E_j(b). \end{cases} \tag{10}$$

Next, for $i = 1$, $j \geq 2$ we compute

$$E_j(b) = E_j(c) = 0. \tag{11}$$

Finally, for $i, j \geq 2$ the equations (8) are satisfied identically.

Now, let us suppose that the metric g on V_p is locally irreducible and that $A^1_j = 0$ for $j = 2, \ldots, n-1$ in an open domain of V_p. Then (1) takes on the form

$$\begin{cases} \omega^0 = f(w, x^1)\mathrm{d}w, \\ \omega^1 = \mathrm{d}x^1, \\ \omega^i = \mathrm{d}x^i + \left(\sum\limits_{j=2}^{n-1} A^i_j(w)x^j\right)\mathrm{d}w \quad (i = 2, \ldots, n-1). \end{cases} \tag{12}$$

Using the idea of the proof of Proposition 11.2 in [6] we can introduce new local coordinates $u^1, \ldots, u^{n-1}$ in such a way that $u^1 = x^1$ and $\sum_{i=2}^{n-1}(\omega^i)^2 = \sum_{i=2}^{n-1}(\mathrm{d}u^i)^2$. (Here $u^2, \ldots, u^{n-1}$ are certain linear combinations of $x^2, \ldots, x^{n-1}$ with the coefficients which are functions of w. These coefficients come out as solutions of a specific system of linear differential equations.) Hence we see that

$$g = f^2(w, u^1)\mathrm{d}w^2 + \sum_{i=1}^{n-1}(\mathrm{d}u^i)^2 \qquad (n \geq 3)$$

is a product metric, which is a contradiction.

Thus, under the assumption of irreducibility, the last equations of (9) imply $c = 0$ on a dense subset of V_p and hence on the whole V_p. According to (11), the function b depends on w and x^1 only. The second equation of (9) gives

$$b = \overline{b}(w)f^{-2} \tag{13}$$

where $\overline{b}(w) \neq 0$. Further, the first part of (10) implies

$$a = -\sum_{j=2}^{n-1} f^{-1}bA^1_j x^j + \overline{a}(w, x^1), \tag{14}$$

where $\overline{a}(w, x^1)$ satisfies, due to the first equation of (9),

$$(\overline{a}f)_{x^1} = (\overline{b}f^{-2})_w. \tag{15}$$

Now we check easily that the formulas (13)–(15) together with $c = 0$ determine a shape operator S satisfying the Codazzi equation (4). The Gauss equation (7) then means, due to (2) and (13)

$$f^3 f_{x^1x^1} = \overline{b}(w)^2. \tag{16}$$

Differentiating this equation with respect to x^1, we obtain

$$(f^2)_{x^1x^1x^1} = 0. \tag{17}$$

We get a general solution in the form

$$f^2 = f_1(w)(x^1)^2 + f_2(w)x^1 + f_3(w) \tag{18}$$

and the solvability condition $f^3 f_{x^1x^1} > 0$ for (16) is equivalent to

$$4f_1f_3 - (f_2)^2 > 0. \tag{19}$$

Then (18) makes sense if and only if

$$f_1(w) > 0, \quad f_3(w) > 0. \tag{20}$$

We can summarize:

Proposition 1. *In the locally irreducible case, an isometric immersion of (V_p, g) into R^{n+1} exists if and only if the function $f(w, x^1)$ satisfies* (18)–(20). *If this is the case, then all such isometric immersions depend on two arbitrary functions of one variable w.*

(The last statement is obvious from (13)–(15).)

Now we formulate our basic theorem:

Theorem 1. *Let a locally irreducible Riemannian manifold N^n admit a completely ruled isometric immersion $\varphi : N^n \to R^{n+1}$. Then there is an open dense subset $U \subset N^n$ such that, in a neighborhood of each point $p \in U$, there exists a local coordinate system $w, x^1, \ldots, x^{n-1}$ and an orthonormal coframe of the form* (1) *where the function $f(w, x^1)$ satisfies* (18)–(20).

Conversely, let $I \subset R[w]$ be an interval and let $f_1(w)$, $f_2(w)$, $f_3(w)$, $A^i_j(w)$ be smooth functions on I satisfying (19), (20) *and the skew-symmetry conditions $A^i_j + A^j_i = 0$ $(i, j = 1, \dots, n-1)$. Then the Riemannian manifold $\bigl(I[w] \times R^{n-1}[x^1, \dots, x^{n-1}], g\bigr)$, where*

$$
\begin{aligned}
g &= [f_1(w)(x^1)^2 + f_2(w)x^1 + f_3(w)]\mathrm{d}w^2 \\
&\quad + \sum_{i=1}^{n-1} \left(\mathrm{d}x^i + \left(\sum_{j=1}^{n-1} A^i_j(w)x^j \right) \mathrm{d}w \right)^2,
\end{aligned}
\tag{21}
$$

admits an isometric immersion in R^{n+1} as a completely ruled hypersurface.

Proof. It suffices to prove the second part of the Theorem. First, because $I \times R^{n-1}$ is simply connected and the metric g is globally defined, an isometric immersion always exists according to Proposition 1. Here the condition $c = 0$ for the shape operator (6) is enforced if the metric g is locally irreducible but we can assume this condition in the general case, too. But $c = 0$ means that the second fundamental form of the immersion is identically zero on each tangent $(n-1)$-plane generated by $E_1, \dots, E_{n-1}$, i.e., along each hypersurface $w = \text{const}$. Moreover, these hypersurfaces are known to be totally geodesic (cf. formulas (6.13) in [6]). Thus, each hypersurface $w = w_0 \in I[w]$ of $(I \times R^{n-1}, g)$ is embedded as an affine $(n-1)$-space in R^{n+1}. Hence we get a completely ruled immersion. $\square$

Let us add that, if the Riemannian manifold in question is irreducible, then every isometric immersion in R^{n+1} is completely ruled. Also, if $I = R[w]$ and $0 < A < f_i(w) < B$ for some A, B and $i = 1, 2, 3$, the corresponding Riemannian manifold is complete (see [6], Corollary 11.1).

According to Proposition 11.2 from [6], each metric of the form (1) can be expressed, using new local coordinates, in a more transparent form

$$
g = \sum_{i=1}^{n-1} (\mathrm{d}u^i)^2 + f^2\left(w, \sum_{j=1}^{n-1} b_j(w)u^j \right) \mathrm{d}w^2,
\tag{22}
$$

where $b_j(w)$ are smooth functions such that

$$
\sum_{j=1}^{n-1} [b_j(w)]^2 = 1.
\tag{23}
$$

If a completely ruled immersion exists and if g is locally irreducible, then

$$f^2(w,u) = f_1(w)u^2 + f_2(w)u + f_3(w), \tag{24}$$

where (19) and (20) hold and $b_i(w) \neq 0$ for $i = 1, \dots, n-1$ on a dense open subset.

Nevertheless, the form (22) of the metric is less convenient for the computations made in the previous pages. We shall use (22)–(24) for another purpose, namely for calculating the number of locally irreducible completely ruled hypersurfaces *up to the local isometries.* Due to (22)–(24), just $n+1$ arbitrary functions of one variable are involved. It remains to evaluate how big is each local isometry class.

Let us have two Riemannian manifolds (M,g), $(\overline{M},\overline{g})$ characterized locally by the formulas (22)–(24) and suppose that $F : (M,g) \to (\overline{M},\overline{g})$ is a local isometry. (Here the corresponding variables and functions connected with the second manifold will be marked by bars.) Because $w =$ const. an $\overline{w} =$ const. define the rulings on the corresponding manifolds, we must have (locally)

$$\overline{w} = \varphi(w), \tag{25}$$

where φ is a smooth function. We can assume (up to a possible change of the sign) that, via F,

$$\overline{f}\Big(\overline{w}, \sum_{j=1}^{n-1} b_j(\overline{w})\overline{u}^j\Big)\mathrm{d}\overline{w} = f\Big(w, \sum_{j=1}^{n-1} b_j(w)u^j\Big)\mathrm{d}w \tag{26}$$

and also

$$\sum_{i=1}^{n-1}(\mathrm{d}\overline{u}^i)^2 = \sum_{i=1}^{n-1}(\mathrm{d}u^i)^2. \tag{27}$$

From the last equation we get

$$\overline{u}^i = \sum_{j=1}^{n-1} a^i_j u^j + c^i \qquad (i = 1, \dots, n-1) \tag{28}$$

where (a^i_j) is a constant orthogonal matrix.

Now, we write down explicitly the equation (26) using the formula (24) and its analog. After almost routine calculations (which uses also (23) and its analog) we conclude that the functions $\overline{f}_i(\overline{w})$ and $\overline{b}_j(\overline{w})$ can be calculated from $f_i(w)$ and $b_j(w)$ using the arbitrary function $\varphi(w)$, its first derivative and the parameters from (28) (which are negligible). This means that each local isometry class depends on one arbitrary function of one variable. We obtain hence:

Theorem 2. *The local isometry classes of locally irreducible Riemannian manifolds N^n which admit completely ruled isometric immersion in R^{n+1} are parametrized by n arbitrary functions of one variable.*

Remark. The problem of intrinsic characterization of ruled hypersurfaces in R^{n+1} which are not locally isometric to completely ruled ones is much more difficult. For dimension $n = 3$, such a classification has been done in [4], Chapter 10.

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