Generalized gyrovector spaces and a Mazur–Ulam theorem

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Abstract. In this paper we introduce the notion of a generalized gyrovector space. The set of all positive invertible elements in a unital C^*-algebra is an example. We study surjective gyrometric preserving maps and show that such maps also preserve the algerbraic structures. As an application we exhibit a structure theorem for maps which preserve the metric of Thompson's type.

1. Introduction

In Einstein's theory of special relativity, the set of all admissible velocities is $\mathbb{R}_c^3 = \{\boldsymbol{u} \in \mathbb{R}^3 : \|\boldsymbol{u}\| < c\}$, where c is the speed of light in vacuum. The Einstein velocity addition $\oplus_E$ in $\mathbb{R}_c^3$ is given by

$$\boldsymbol{u} \oplus_E \boldsymbol{v} = \frac{1}{1 + \langle \boldsymbol{u}, \boldsymbol{v} \rangle/c} \left\{ \boldsymbol{u} + \frac{1}{\gamma_{\boldsymbol{u}}} \boldsymbol{v} + \frac{1}{c^2} \frac{\gamma_{\boldsymbol{u}}}{1 + \gamma_{\boldsymbol{u}}} \langle \boldsymbol{u}, \boldsymbol{v} \rangle \boldsymbol{u} \right\} \tag{1}$$

for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_c^3$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $\gamma_{\boldsymbol{u}}$ is the Lorentz factor given by

$$\gamma_{\boldsymbol{u}} = (1 - \|\boldsymbol{u}\|^2/c^2)^{-\frac{1}{2}}. \tag{2}$$

The Einstein velocity addition $\oplus_E$ is non-commutative and non-associative, hence $(\mathbb{R}_c^3, \oplus_E)$ does not have a group structure. However, $(\mathbb{R}_c^3, \oplus_E)$ has a gyrocommutative gyrogroup structure and is called the Einstein gyrogroup. The (gyrocommutative) gyrogroup is the generalization of the (commutative) group, which is not necessarily (commutative nor) associative.

Certain gyrocommutative gyrogroups admit scalar multiplication, giving rise to gyrovector spaces. UNGAR initiated the study on gyrogroups and gyrovector spaces (cf. [25]). He describes that the hyperbolic geometry of Bolyai and Lobachevsky is now effectively regulated by gyrovector spaces just as Euclidean geometry is regulated by vector spaces. Any gyrovector space is equipped with the gyrometric, which is a measurement of the distance while it needs not be the metric exactly. Any (positive definite) real inner product space is a gyrovector space and the gyrometric is the metric induced by its norm. On the other hand, a real normed space need not be a gyrovector space. In this paper, we introduce a notion of a generalized gyrovector space which is a common generalization of the notion of a real normed space and that of a gyrovector space. We exhibit a Mazur–Ulam Theorem for the generalized gyrovector spaces (cf. [1], [2]); a bijection between the generalized gyrovector spaces which preserves the gyrometric also preserves the algebraic structure automatically. The celebrated Mazur–Ulam theorem states that a surjective isometry from a normed vector space onto a possibly different normed vector space is a real linear isomorphism followed by a translation. A simple proof of the Mazur–Ulam Theorem was given by VÄISÄLÄ [26] by using the idea of VOGT [27].

2. A generalization of the gyrovector spaces

A gyrovector space is defined by UNGAR. It is a gyrocommutative gyrogroup G which is contained in a real inner product space $\mathbb{V}$ (the carrier of G) with exotic addition and scalar multiplication, and which admits the inner product inherited from the inner product on $\mathbb{V}$. For the precise definition see [25, Definition 6.2] or the comments after Definition 4.

We introduce a notion of generalized gyrovector spaces (GGV). The difference between the original gyrovector spaces and the generalized one is that the carrier of the generalized one need not be an inner product space but a normed space. A real normed space and a gyrovector space are simple examples of the generalized gyrovector spaces.

For the convenience of the readers we recall the definitions of the gyrocommutative gyrogroups. We essentially adopt the definition of UNGAR's book [25].

Definition 1. A groupoid $(S, +)$ is a nonempty set S and a binary operation $+$. An automorphism (on a groupoid) is a bijection between groupoid which preserves the groupoid operation. A groupoid $(G, \oplus)$ is called a gyrogroup if there

exists an element $\boldsymbol{e}$ such that the binary operation $\oplus$ satisfies the following (G1) to (G5).

(G1) The equation

$$\boldsymbol{e} \oplus \boldsymbol{a} = \boldsymbol{a}$$

for every $\boldsymbol{a} \in G$.

(G2) For every $\boldsymbol{a} \in G$ there exists an element $\ominus\boldsymbol{a}$ such that

$$\ominus\boldsymbol{a} \oplus \boldsymbol{a} = \boldsymbol{e}.$$

(G3) For any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in G$ there exists a unique element $\operatorname{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c} \in G$ such that

$$\boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}) = (\boldsymbol{a} \oplus \boldsymbol{b}) \oplus \operatorname{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c}.$$

(G4) For any $\boldsymbol{a}, \boldsymbol{b} \in G$ $\operatorname{gyr}[\boldsymbol{a}, \boldsymbol{b}]$ is an automorphism, which is called the gyroautomorphism of G generated by $\boldsymbol{a}, \boldsymbol{b} \in G$.

(G5) For any $\boldsymbol{a}, \boldsymbol{b} \in G$

$$\operatorname{gyr}[\boldsymbol{a} \oplus \boldsymbol{b}, \boldsymbol{b}] = \operatorname{gyr}[\boldsymbol{a}, \boldsymbol{b}].$$

A gyrogroup is gyrocommutative if the following (G6) is satisfied.

(G6) For any $\boldsymbol{a}, \boldsymbol{b} \in G$

$$\boldsymbol{a} \oplus \boldsymbol{b} = \operatorname{gyr}[\boldsymbol{a}, \boldsymbol{b}](\boldsymbol{b} \oplus \boldsymbol{a}).$$

Note that by [25, Theorem 2.10, (5),(6), (7), (8)] the element $\boldsymbol{e}$ which satisfies (G1) is unique, and the element $\ominus\boldsymbol{a}$ is unique for each $\boldsymbol{a}$. We also note that $\boldsymbol{a} \oplus \boldsymbol{e} = \boldsymbol{a}$ and $\boldsymbol{a} \oplus \ominus\boldsymbol{a} = \boldsymbol{e}$ for every $\boldsymbol{a} \in G$.

In the private communication in 2012 or 2013 Lajos Molnár told to the second author that the space of positive invertible elements $\mathscr{A}_+^{-1}$ in a unital C^*-algebra $\mathscr{A}$ is a gyrogroup. In fact Beneduci and Molnár [4] observe that the space of the positive invertible elements in a unital C^*-algebra is a standard K-loop. Note that Sabinin, Sabinina and Sbitneva [21] have proved that the K-loop is equivalent to the gyrocommutative gyrogroup (see also [4]).

One of the referees of the paper kindly recommended the following historical comments. For the theory of K-loops readers may consult with Kiechle's book [12]. Not unexpectedly, according to Kiechle [12, pp. 169–170], the term "K-loop" with K named after Karzel was coined by Ungar in 1985 [24] to describe the algebraic structure that later became known as a gyrocommutative gyrogroup. For different purposes, the term "K-loop" was already in use earlier by Sоĭkis, in 1970 [23] and later, but independently, by Basarab, in 1992 [3]. Unlike the term "K-loop" that Ungar coined, the "K" in each of the term "K-loop" coined

by Soĭkis and by Basarab does not refer to "Karzel". The early history of K-loops with "K" named after Karzel is unfolded in [22, p. 142].

We give a direct proof that $\mathscr{A}_+^{-1}$ is indeed a gyrocommutative gyrogroup.

Proposition 2. *Suppose that $\mathscr{A}$ is a unital C^*-algebra with the norm $\|\cdot\|$ and $\mathscr{A}_+^{-1}$ is the set of all positive invertible elements of $\mathscr{A}$. Let t be a positive real number. Put*

$$a \oplus_t b = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}}$$

for all $a, b \in \mathscr{A}_+^{-1}$. Then $(\mathscr{A}_+^{-1}, \oplus_t)$ is a gyrocommutative gyrogroup. The identity element e of $\mathscr{A}$ as the C^*-algebra is the identity element of the gyrogroup. The inverse element $\ominus a$ is a^{-1}, the inverse of a in $\mathscr{A}$. For $a, b \in \mathscr{A}_+^{-1}$ put

$$X = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}}.$$

Then X is a unitary element in $\mathscr{A}$ and

$$\operatorname{gyr}[a, b]c = XcX^*, \quad a, b, c \in \mathscr{A}_+^{-1}.$$

is the gyroautomorphism generated by a and b.

PROOF. Let e be the identity element of $\mathscr{A}$ as the C^*-algebra. Then e is in $\mathscr{A}_+^{-1}$ and satisfies $e \oplus_t a = a$ for every $a \in \mathscr{A}_+^{-1}$; (G1) holds.

Let $a \in \mathscr{A}_+^{-1}$. Then $a^{-1} \in \mathscr{A}_+^{-1}$, and $a^{-1} \oplus_t a = e$ is trivial; (G2) holds for $\ominus a = a^{-1}$.

For $a, b \in \mathscr{A}_+^{-1}$, put $X = \left(a^{\frac{t}{2}} b^t a^{\frac{t}{2}}\right)^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}}$. By a simple calculation we infer that X is a unitary element. Define $\operatorname{gyr}[a, b]c = XcX^*$, $c \in \mathscr{A}_+^{-1}$. By a simple calculation we see that

$$a \oplus_t (b \oplus_t c) = (a \oplus_t b) \oplus_t \operatorname{gyr}[a, b]c, \quad a, b, c \in \mathscr{A}_+^{-1}.$$

The uniqueness of $\operatorname{gyr}[a, b]c$ is obvious by the definition of $\oplus_t$. Thus we have that (G3) holds.

As $\operatorname{gyr}[a, b]$ is a unitary transform for every pair $a, b \in \mathscr{A}_+^{-1}$, we infer by a simple calculation that

$$\operatorname{gyr}[a, b](c \oplus_t d) = \operatorname{gyr}[a, b]c \oplus_t \operatorname{gyr}[a, b]d, \quad c, d \in \mathscr{A}_+^{-1}$$

for every pair $a, b \in \mathscr{A}_+^{-1}$; (G4) holds.

To prove that (G5) holds we first mention that by a simple calculation we have

$$\left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} a^{-\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}}\right)^2 = \left(a^{\frac{t}{2}} b^t a^{\frac{t}{2}}\right)^{\frac{1}{2}} b^t (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}}$$

for every pair $a, b \in \mathscr{A}_+^{-1}$. Hence we have

$$\left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} a^{-\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}}\right)^{-1} = \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} b^t (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}}$$

for every pair $a, b \in \mathscr{A}_+^{-1}$. It follows that

$$\left((a \oplus_t b)^{\frac{t}{2}} b^t (a \oplus_t b)^{\frac{t}{2}}\right)^{-\frac{1}{2}} (a \oplus_t b)^{\frac{t}{2}} b^{\frac{t}{2}} = \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} b^t (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} b^{\frac{t}{2}}$$

$$= \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} a^{-\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}}\right)^{-1} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} b^{\frac{t}{2}} = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}}$$

for every pair $a, b \in \mathscr{A}_+^{-1}$. Thus we obtain that $\mathrm{gyr}[a \oplus_t b, b] = \mathrm{gyr}[a, b]$ for every pair $a, b \in \mathscr{A}_+^{-1}$; (G5) holds.

As $\mathrm{gyr}[a, b]$ is a unitary transfomation, we have

$$\mathrm{gyr}[a, b](b \oplus_t a) = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}} (b^{\frac{t}{2}} a^t b^{\frac{t}{2}})^{\frac{1}{t}} b^{\frac{t}{2}} a^{\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}}$$

$$= \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}} (b^{\frac{t}{2}} a^t b^{\frac{t}{2}}) b^{\frac{t}{2}} a^{\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}}\right)^{\frac{1}{t}} = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}} = a \oplus_t b$$

for every pair $a, b \in \mathscr{A}_+^{-1}$; (G6) holds. $\qquad\square$

Definition 3. Let $(G, \oplus)$ be a gyrogroup. The gyrogroup coaddition $\boxplus$ is defined by

$$\boldsymbol{a} \oplus \mathrm{gyr}[\boldsymbol{a}, \ominus \boldsymbol{b}]\boldsymbol{b}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in G$.

Note that the gyrogroup coaddition $\boxplus$ is commutative if and only if the gyrogroup $(G, \oplus)$ is gyrocommutative (cf. [25, Theorem 3.4]).

Definition 4 (A generalized gyrovector space). Let $(G, \oplus)$ be a gyrocommutative gyrogroup with the map $\otimes : \mathbb{R} \times G \to G$. Let ϕ be an injection from G into a real normed space $(\mathbb{V}, \|\cdot\|)$. We say that $(G, \oplus, \otimes, \phi)$ (or $(G, \oplus, \otimes)$ just for a simple notation) is a generalized gyrovector space or a GGV in short if the following conditions (GGV0) to (GGV8) are fulfilled:

(GGV0) $\|\phi(\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a})\| = \|\phi(\boldsymbol{a})\|$ for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G$;

(GGV1) $1 \otimes \boldsymbol{a} = \boldsymbol{a}$ for every $\boldsymbol{a} \in G$;

(GGV2) $(r_1 + r_2) \otimes \boldsymbol{a} = (r_1 \otimes \boldsymbol{a}) \oplus (r_2 \otimes \boldsymbol{a})$ for any $\boldsymbol{a} \in G$, $r_1, r_2 \in \mathbb{R}$;

(GGV3) $(r_1 r_2) \otimes \boldsymbol{a} = r_1 \otimes (r_2 \otimes \boldsymbol{a})$ for any $\boldsymbol{a} \in G$, $r_1, r_2 \in \mathbb{R}$;

(GGV4) $(\phi(|r| \otimes \boldsymbol{a}))/\|\phi(r \otimes \boldsymbol{a})\| = \phi(\boldsymbol{a})/\|\phi(\boldsymbol{a})\|$ for any $\boldsymbol{a} \in G \setminus \{\boldsymbol{e}\}$, $r \in \mathbb{R} \setminus \{0\}$, where $\boldsymbol{e}$ denotes the identity element of the gyrogroup $(G, \oplus)$;

(GGV5) $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](r \otimes \boldsymbol{a}) = r \otimes \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a}$ for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G$, $r \in \mathbb{R}$;

(GGV6) $\mathrm{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = id_G$ for any $\boldsymbol{v} \in G$, $r_1, r_2 \in \mathbb{R}$;

(GGVV) $\|\phi(G)\| = \{\pm\|\phi(\boldsymbol{a})\| \in \mathbb{R} : \boldsymbol{a} \in G\}$ is a real one-dimensional vector space with vector addition $\oplus'$ and scalar multiplication $\otimes'$;

(GGV7) $\|\phi(r \otimes \boldsymbol{a})\| = |r| \otimes' \|\phi(\boldsymbol{a})\|$ for any $\boldsymbol{a} \in G$, $r \in \mathbb{R}$;

(GGV8) $\|\phi(\boldsymbol{a} \oplus \boldsymbol{b})\| \leq \|\phi(\boldsymbol{a})\| \oplus' \|\phi(\boldsymbol{b})\|$ for any $\boldsymbol{a}, \boldsymbol{b} \in G$.

Note that $\|\phi(\boldsymbol{e})\| = 0$ and $\ominus\boldsymbol{a} = (-1) \otimes \boldsymbol{a}$ are simple observations. In this paper we simply write $\alpha \otimes \boldsymbol{a} \oplus \beta \otimes \boldsymbol{b}$ instead of $(\alpha \otimes \boldsymbol{a}) \oplus (\beta \otimes \boldsymbol{b})$, and $\ominus\alpha \otimes \boldsymbol{a}$ instead of $\ominus(\alpha \otimes \boldsymbol{a})$ for any $\alpha, \beta \in \mathbb{R}$, $\boldsymbol{a}, \boldsymbol{b} \in G$.

A gyrovector space $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ with $G \subset \mathbb{W}$ for a (positive definite) real inner product space $\mathbb{W}$ such that the exotic scalar multiplication $\otimes : \mathbb{R} \times G \to G$ is defined and that the conditions through (GGV1) to (GGV8) with ϕ being the identity map, and the condition

(V0) $\langle \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a}, \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{b}\rangle = \langle \boldsymbol{a}, \boldsymbol{b}\rangle$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}, \boldsymbol{b} \in G$

instead of the condition (GGV0) are satisfied, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{W}$. In short a gyrovector space is a subset of an inner product space while a GGV is the inverse image of a normed vector space. Any gyrovector space is a GGV since the condition (V0) implies the condition (GGV0). Note that for any gyrovector space, ϕ is the identity map by the definition (cf. [25, Definition6.2]). In addition, any real normed space is a GGV. Let $\mathbb{V}$ be a real normed space with the addition $+$ and the scalar multiplication $\cdot$, then $(\mathbb{V}, +, \cdot)$ is a GGV (the map ϕ is the identity map of $\mathbb{V}$).

Definition 5. Let $(G, \oplus, \otimes)$ be a GGV. The gyromidpoint $\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})$ of $\boldsymbol{a}, \boldsymbol{b} \in (G, \oplus, \otimes)$ is defined as $\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b}) = \frac{1}{2} \otimes (\boldsymbol{a} \boxplus \boldsymbol{b})$, where $\boxplus$ is the gyrogroup coaddition of the gyrogroup $(G, \oplus)$.

Note that $\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{p}(\boldsymbol{b}, \boldsymbol{a})$ as $\boxplus$ is commutative. In addition we have

$$\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a} \oplus \frac{1}{2} \otimes (\ominus\boldsymbol{a} \oplus \boldsymbol{b}) \tag{3}$$

(cf. [25, Definition 6.32 and Theorem 6.34]). In particular, $\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b}) = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ if the GGV $(G, \oplus, \otimes)$ is indeed a real normed space $(\mathbb{V}, +, \cdot)$.

Definition 6. Let $(G, \oplus, \otimes)$ be a GGV. Let $\varrho(\boldsymbol{a}, \boldsymbol{b}) = \|\phi(\boldsymbol{a} \ominus \boldsymbol{b})\|$ for all $\boldsymbol{a}, \boldsymbol{b} \in G$. We call ϱ the gyrometric on G on a GGV.

The gyrometric ϱ satisfies the equation

$$\varrho(\boldsymbol{a}, \boldsymbol{b}) = \varrho(\ominus\boldsymbol{a}, \ominus\boldsymbol{b}) = \varrho(\boldsymbol{b}, \boldsymbol{a}) \tag{4}$$

as

$$\varrho(\boldsymbol{a}, \boldsymbol{b}) = \|\phi(\boldsymbol{a} \ominus \boldsymbol{b})\| = \|\phi(\ominus(\boldsymbol{a} \ominus \boldsymbol{b}))\| = \|\phi(\ominus \boldsymbol{a} \oplus \boldsymbol{b})\| = \varrho(\ominus \boldsymbol{a}, \ominus \boldsymbol{b})$$
$$= \|\phi(\text{gyr}[\ominus \boldsymbol{a}, \boldsymbol{b}](\boldsymbol{b} \ominus \boldsymbol{a}))\| = \|\phi(\boldsymbol{b} \ominus \boldsymbol{a})\| = \varrho(\boldsymbol{b}, \boldsymbol{a}).$$

In particular, if $(G, \oplus, \otimes)$ is a real normed space, then gyrometric is a metric induced by its norm.

We show a gyrocommutative gyrogroup of the positive invertible elements in a C^*-algebra is indeed a GGV.

Example 7. Suppose that $\mathscr{A}$ is a unital C^*-algebra with the norm $\| \cdot \|$ and $\mathscr{A}_+^{-1}$ is the set of all positive invertible elements of $\mathscr{A}$. Let t be a positive real number. Put
$$a \oplus_t b = \left(a^{\frac{t}{2}} b^t a^{\frac{t}{2}}\right)^{\frac{1}{t}}$$
for all $a, b \in \mathscr{A}_+^{-1}$. Then $(\mathscr{A}_+^{-1}, \oplus_t)$ is a gyrocommutative gyrogroup as is proved in Proposition 2.

Put $r \otimes a = a^r$ for every $a \in \mathscr{A}_+^{-1}$, $r \in \mathbb{R}$. Define $\phi = \log : \mathscr{A}_+^{-1} \to \mathscr{A}_S$. The vector space $(\| \log(\mathscr{A}_+^{-1})\|, \oplus', \otimes') = (\mathbb{R}, +, \times)$ is the usual 1 dimensional real vector space of the real line; $\oplus'$ is the addition of real numbers and $\otimes'$ is the scalar multiplication of real numbers. Then $(\mathscr{A}_+^{-1}, \oplus_t, \otimes, \log)$ is a GGV. In fact, (GGV0) holds since $\text{gyr}[a, b]$ is a unitary transform for every pair $a, b \in \mathscr{A}_+^{-1}$. Simple calculations confirm that the conditions from (GGV0) to (GGV6) and (GGV7) hold. The condition (GGVV) is trivial by the definition of $\oplus'$ and $\otimes'$. A proof that (GGV8) holds is as follows. Letting $c = e$ for [5, 1. of Remarks on p. 197] we have

$$\| \log(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}} \| \leq \frac{1}{t}(\| \log a^t \| + \| \log b^t \|) = \| \log a \| + \| \log b \|, \quad a, b \in \mathscr{A}_+^{-1}.$$

For the convenience of the readers we give an alternative elementary proof. Let $a, b \in \mathscr{A}_+^{-1}$. Put $s = \min\{\lambda : \lambda \in \sigma(a)\}$, $S = \max\{\lambda : \lambda \in \sigma(a)\}$, $s' = \min\{\lambda : \lambda \in \sigma(b)\}$, $S' = \max\{\lambda : \lambda \in \sigma(b)\}$. By the spectral mapping theorem we have that $\sigma(a^t) \subset [s^t, S^t]$, hence $s^t \leq a^t \leq S^t$. In the same way we see that $s'^t \leq b^t \leq S'^t$. Since $0 \leq b^t - s'^t e$ we have $0 \leq a^{\frac{t}{2}}(b^t - s'^t e)a^{\frac{t}{2}}$. Thus $s'^t a^t \leq a^{\frac{t}{2}} b^t a^{\frac{t}{2}}$. Thus $(s's)^t e \leq a^{\frac{t}{2}} b^t a^{\frac{t}{2}}$. In a way similar we have $a^{\frac{t}{2}} b^t a^{\frac{t}{2}} \leq (SS')^t e$. Hence we have $ss'e \leq (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}} \leq SS'e$. As $\log(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}}$ is self-adjoint, we have

$$\big\| \log(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}} \big\| = \max\{|\lambda| : \lambda \in \sigma(\log(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}})\}.$$

By the spectral mapping theorem we have

$$\max\{|\lambda| : \lambda \in \sigma(\log(a^{\frac{t}{2}}b^t a^{\frac{t}{2}})^{\frac{1}{t}})\} = \max\{|\lambda| : \lambda \in \log\sigma((a^{\frac{t}{2}}b^t a^{\frac{t}{2}})^{\frac{1}{t}})\}$$
$$\leq \max\{|\log SS'|, |\log ss'|\} \leq \max\{|\log S| + |\log S'|, |\log s| + |\log s'|\}$$
$$\leq \max\{|\log S|, |\log s|\} + \max\{|\log S'|, |\log s'|\} = \|\log a\| + \|\log b\|$$

The gyrometric ϱ is given by the equation

$$\varrho(a, b) = \|\log(a^{-\frac{t}{2}}b^t a^{-\frac{t}{2}})^{\frac{1}{t}}\|$$

for $a, b \in \mathscr{A}_+^{-1}$. Note that in the case where $t = 1$ the metric $\varrho(a, b)$ is the Thompson metric itself. The gyromidpoint of a and b is given by the equation

$$\boldsymbol{p}(a, b) = (a^{\frac{t}{2}}(a^{\frac{t}{2}}b^{-t}a^{\frac{t}{2}})^{-\frac{1}{2}}a^{\frac{t}{2}})^{\frac{1}{t}},$$

which coincides with the geometric mean of a and b for the case of $t = 1$.

Example 8. Let $\mathscr{A}_S$ be a real subspace of all self-adjoints elements of a unital C^*-algebra $\mathscr{A}$ and t a positive real number. Let

$$a \oplus_t b = \log\left(\exp\frac{ta}{2}\exp tb \exp\frac{ta}{2}\right)^{\frac{1}{t}}$$

for all $a, b \in \mathscr{A}_S$. Then $(\mathscr{A}_S, \oplus_t)$ is a gyrocommutative gyrogroup. The 0 is the identity element and $\ominus a = -a$ for every $a \in \mathscr{A}_S$. Define the scalar multiplication $\otimes$ by $r \otimes a = ra$ for every pair $r \in \mathbb{R}$ and $a \in \mathscr{A}_S$. Let $\phi : \mathscr{A}_S \to \mathscr{A}_S$ be the identity map. The vector space $(\|\mathscr{A}_S\|, \oplus', \otimes') = (\mathbb{R}, +, \times)$ is the usual 1 dimensional real vector space of the real line. Then $(\mathscr{A}_S, \oplus_t, \otimes, \mathrm{Id})$ is a GGV, where Id is the identity map. The difference of the structures between the $\mathscr{A}_S$ as a real normed subspace of the C^*-algebra $\mathscr{A}$ and the GGV $(\mathscr{A}_S, \oplus_t, \otimes, \mathrm{Id})$ is only the structure of addition. The GGV $(\mathscr{A}_S, \oplus_t, \otimes, \mathrm{Id})$ has the exotic additive structure. Note that $(\mathscr{A}_+^{-1}, \oplus_t, \otimes, \log)$ and $(\mathscr{A}_S, \oplus_t, \otimes, \mathrm{Id})$ are isomorphic as the GGV's. In fact the map $\log : \mathscr{A}_+^{-1} \to \mathscr{A}_S$ is a bijection and preserves the structure as the GGV.

Example 9. Suppose that A is a unital C^*-algebra on a Hilbert space of a finite dimension and A_+^{-1} is a set of all positive invertible elements in A. Let t be a positive real number and $\|\cdot\|$ a unitarily invariant norm. As same as in Example 7 $(A_+^{-1}, \oplus_t)$ is a gyrocommutative gyrogroup for the operation $\oplus_t$ defined by $a \oplus_t b = (a^{\frac{t}{2}}b^t a^{\frac{t}{2}})^{\frac{1}{t}}$ for $a, b \in A_+^{-1}$. Put $r \otimes a = a^r$ for every $a \in A_+^{-1}$, $r \in \mathbb{R}$. Define $\phi = \log : A_+^{-1} \to \mathscr{A}_S$. The vector space $(\|\log(A_+^{-1})\|, \oplus', \otimes') = (\mathbb{R}, +, \times)$ is the usual 1 dimensional real vector space of the real line; $\oplus'$ is the addition of real numbers and $\otimes'$ is the scalar multiplication of real numbers. As in the proof

in Example 7 we can prove that the conditions from (GGV0) through (GGV7) hold. As for the condition (GGV8), we have by [7, Theorem 5] that

$$\| \log(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}} \| \leq \| \log a \| + \| \log b \|$$

for every pair $a, b \in A_+^{-1}$. Thus $(A_+^{-1}, \oplus_t, \otimes, \log)$ is a GGV for any unitarily invariant norm.

Note that the set of all positive invertible complex matrics is an important example of A_+^{-1} in Example 9. Note also that not only the case of complex matrices but also the set of all positive invertible matrices plays an important role in several applications. See [6] and [15].

Example 10. Let $\mathbb{P}_n(\mathbb{R})$ be the set of all positive definite real-valued matrices of degree n. In the same way as Example 9 $(\mathbb{P}_n(\mathbb{R}), \oplus_t, \otimes)$ is a GGV.

Proposition 11. *Let G be a non-empty subset of a real normed space $\mathbb{V}$. Suppose that $(G, \oplus_1, \otimes_1, \phi_1)$ is a GGV such that $G \subset \mathbb{V}$ for a real normed space $\mathbb{V}$ with $\phi_1 : G \to \mathbb{V}$ being the identity map. Let $\|G\|_+ = \{\|\boldsymbol{a}\| \in \mathbb{R} : \boldsymbol{a} \in G\}$. Suppose that $h : \|G\|_+ \to \mathbb{R}$ is a strictly monotone increasing function with $h(0) = 0$, where we do not assume the continuity on h. Define $\varphi : G \to \mathbb{V}$ by*

$$\varphi(\boldsymbol{a}) = \frac{h(\|\boldsymbol{a}\|)}{\|\boldsymbol{a}\|} \boldsymbol{a}, \quad a \in G.$$

Then φ is an injection. Define $\oplus_2$ and $\otimes_2$ by

$$\varphi(\boldsymbol{a}) \oplus_2 \varphi(\boldsymbol{b}) = \varphi(\boldsymbol{a} \oplus_1 \boldsymbol{b}), \qquad \alpha \otimes_2 \varphi(\boldsymbol{a}) = \varphi(\alpha \otimes_1 \boldsymbol{a})$$

for any $\boldsymbol{a}, \boldsymbol{b} \in G, r \in \mathbb{R}$. Then $(\varphi(G), \oplus_2, \otimes_2, \phi_2)$ is a GGV such that $\phi_2 : \varphi(G) \to \mathbb{V}$ is the identity map.

By the definition, $(G, \oplus_1, \otimes_1)$ and $(\varphi(G), \oplus_2, \otimes_2)$ have a same algebraic structure. In contrast, $(G, \oplus_1, \otimes_1)$ and $(\varphi(G), \oplus_2, \otimes_2)$ have different gyrometrics: $\varrho_2 = h \circ \varrho_1$.

PROOF. Let $\phi_2 : \varphi(G) \to \mathbb{V}$ be the identity map. By the definition of the $(\varphi(G), \oplus_2, \otimes_2)$, it is trivial that $(\varphi(G), \oplus_2)$ is gyrocommutative gyrogroup with $G \subset \mathbb{V}$ and $(\varphi(G), \oplus_2, \otimes_2)$ satisfies the conditions through (GGV1) to (GGV3). By the definition of the φ, we have $\|\varphi(\boldsymbol{a})\| = h(\|\boldsymbol{a}\|)$ for any $\boldsymbol{a} \in G$. Thus

$$\frac{|r| \otimes_2 \varphi(\boldsymbol{a})}{\|r \otimes_2 \varphi(\boldsymbol{a})\|} = \frac{\varphi(|r| \otimes_1 \boldsymbol{a})}{\|\varphi(r \otimes_1 \boldsymbol{a})\|} = \frac{\frac{h(\||r| \otimes_1 \boldsymbol{a}\|)}{(\||r| \otimes_1 \boldsymbol{a})\|} (|r| \otimes_1 \boldsymbol{a})}{h(\|r \otimes_1 \boldsymbol{a}\|)}$$

$$= \frac{h(\||r| \otimes_1 \boldsymbol{a}\|)}{h(\|r \otimes_1 \boldsymbol{a}\|)} \frac{|r| \otimes_1 \boldsymbol{a}}{\||r| \otimes_1 \boldsymbol{a}\|} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

and

$$\frac{\varphi(\boldsymbol{a})}{\|\varphi(\boldsymbol{a})\|} = \frac{1}{h(\|\boldsymbol{a}\|)} \frac{h(\|\boldsymbol{a}\|)}{\|\boldsymbol{a}\|} \boldsymbol{a} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

for any $\boldsymbol{a} \in G$, $r \in \mathbb{R}$. The condition (GGV4) is satisfied. Note that

$$\mathrm{gyr}[\varphi(\boldsymbol{u}), \varphi(\boldsymbol{v})](\varphi(\boldsymbol{a})) = \varphi(\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](\boldsymbol{a}))$$

for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G$ ([25, Theorem2.26]). It follows that

$$\| \mathrm{gyr}[\varphi(\boldsymbol{u}), \varphi(\boldsymbol{v})](\varphi(\boldsymbol{a}))\| = h(\|(\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](\boldsymbol{a}))\|) = h(\|\boldsymbol{a}\|),$$

$$\mathrm{gyr}[\varphi(\boldsymbol{u}), \varphi(\boldsymbol{v})](r \otimes_2 \varphi(\boldsymbol{a})) = \mathrm{gyr}[\varphi(\boldsymbol{u}), \varphi(\boldsymbol{v})](\varphi(r \otimes_1 \boldsymbol{a}))$$
$$= \varphi(\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](r \otimes_1 \boldsymbol{a})) = r \otimes_2 \varphi(\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](\boldsymbol{a})),$$

$$\mathrm{gyr}[r_1 \otimes_2 \varphi(\boldsymbol{a}), r_2 \otimes_2 \varphi(\boldsymbol{a})](\varphi(\boldsymbol{b})) = \mathrm{gyr}[\varphi(r_1 \otimes_1 \boldsymbol{a}), \varphi(r_2 \otimes_1 \boldsymbol{a})](\varphi(\boldsymbol{b}))$$
$$= \varphi(\mathrm{gyr}[r_1 \otimes_1 \boldsymbol{a}, r_2 \otimes_1 \boldsymbol{a}](\boldsymbol{b})) = \varphi(\boldsymbol{b}).$$

Thus, $(\varphi(G), \oplus_2, \otimes_2)$ satisfies the conditions (GGV0), (GGV5) and (GGV6). Let $h' : \|G\| \to \|\varphi(G)\|$ be a surjection

$$h'(a) = \begin{cases} h(a) & \text{if } a \geq 0 \\ -h(-a) & \text{if } a < 0 \end{cases}$$

and define

$$h'(a) \oplus_2' h'(b) = h'(a \oplus_1' b), \qquad \alpha \otimes_2' h'(a) = h'(\alpha \otimes_1' a),$$

then $(\|\varphi(G)\|, \oplus_2', \otimes_2')$ is a one-dimensional real linear space. Moreover,

$$\|\varphi(\boldsymbol{a}) \oplus_2 \varphi(\boldsymbol{b})\| = \|\varphi(\boldsymbol{a} \oplus_1 \boldsymbol{b})\| = h(\|\boldsymbol{a} \oplus_1 \boldsymbol{b}\|) \leq h(\|\boldsymbol{a}\| \oplus_1' \|\boldsymbol{b}\|)$$
$$= h(\|\boldsymbol{a}\|) \oplus_2' h(\|\boldsymbol{b}\|) = \|\varphi(\boldsymbol{a})\| \oplus_2' \|\varphi(\boldsymbol{b})\|$$

since h is monotone increasing and

$$\|r \otimes_2 \varphi(\boldsymbol{a})\| = \|\varphi(r \otimes_1 \boldsymbol{a})\| = h(\|r \otimes_1 \boldsymbol{a}\|) = h(|r| \otimes_1' \|\boldsymbol{a}\|)$$
$$= |r| \otimes_2' h(\|\boldsymbol{a}\|) = |r| \otimes_2' \|\varphi(\boldsymbol{a})\| \qquad \qquad \square$$

Definition 12. Suppose that $(G_1, \oplus_1, \otimes_1)$ and $(G_2, \oplus_2, \otimes_2)$ are GGV's. Let ϱ_1 and ϱ_2 be gyrometrics of G_1 and G_2, respectively. We say that a map $T : G_1 \to G_2$ is gyrometric preserving if the equality

$$\varrho_2(T\boldsymbol{a}, T\boldsymbol{b}) = \varrho_1(\boldsymbol{a}, \boldsymbol{b})$$

holds for every pair $\boldsymbol{a}, \boldsymbol{b} \in G_1$.

Proposition 11 shows that the algebraic structure of GGV does not encode the gyrometric structure.

3. A Mazur–Ulam theorem for GGV

The main result of the paper is the following. The theorem asserts that a surjective gyrometric preserving map preserves gyromidpoints. This is a generalization of the Mazur–Ulam theorem for the real normed spaces.

Theorem 13. *Let $(G_1, \oplus_1, \otimes_1)$ and $(G_2, \oplus_2, \otimes_2)$ be GGV's. Let ϱ_1 and ϱ_2 be gyrometrics of G_1 and G_2, respectively. Suppose that $T : G_1 \to G_2$ is a gyrometric preserving surjection. Then T preserves the gyromidpoints;*

$$\boldsymbol{p}(T\boldsymbol{a}, T\boldsymbol{b}) = T\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})$$

for any pair $\boldsymbol{a}, \boldsymbol{b} \in G_1$.

The following corollary asserts that a surjective gyrometric preserving map preserves the algebraic structure followed by the left gyrotranslations. It follows that two GGV's which have the same gyrometric structure have the same GGV structure essentially.

Corollary 14. *Let $(G_1, \oplus_1, \otimes_1)$ and $(G_2, \oplus_2, \otimes_2)$ be GGV's. Let ϱ_1 and ϱ_2 be gyrometrics of G_1 and G_2, respectively. Suppose that a surjection $T : G_1 \to G_2$ satisfies*

$$\varrho_2(T\boldsymbol{a}, T\boldsymbol{b}) = \varrho_1(\boldsymbol{a}, \boldsymbol{b})$$

for any pair $\boldsymbol{a}, \boldsymbol{b} \in G_1$. Then T is of the form $T = T(\boldsymbol{e}) \oplus_2 T_0$, where T_0 is an isometrical isomorphism in the sense that the equalities

$$T_0(\boldsymbol{a} \oplus_1 \boldsymbol{b}) = T_0(\boldsymbol{a}) \oplus_2 T_0(\boldsymbol{b}); \tag{5}$$

$$T_0(\alpha \otimes_1 \boldsymbol{a}) = \alpha \otimes_2 T_0(\boldsymbol{a}); \tag{6}$$

$$\varrho_2(T_0\boldsymbol{a}, T_0\boldsymbol{b}) = \varrho_1(\boldsymbol{a}, \boldsymbol{b}). \tag{7}$$

for every $\boldsymbol{a}, \boldsymbol{b} \in G_1$ and $\alpha \in \mathbb{R}$ hold.

4. Preparations for the proof

The following proposition is proved when $(G, \oplus, \otimes)$ is a gyrovector space ([25, Theorem 6.12 and Theorem 6.33]). A proof of Proposition 15 is same as the proofs of [25, Theorem 6.12] and [25, Theorem 6.33].

Proposition 15. *Let ϱ be the gyrometric of a GGV $(G, \oplus, \otimes)$. Then*

$$\varrho(\boldsymbol{a}, \boldsymbol{b}) = \varrho(\boldsymbol{x} \oplus \boldsymbol{a}, \boldsymbol{x} \oplus \boldsymbol{b}) \tag{8}$$

for every triple $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x} \in G$ and

$$\varrho(\boldsymbol{a}, \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})) = \varrho(\boldsymbol{b}, \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})) = \frac{1}{2} \otimes' \varrho(\boldsymbol{a}, \boldsymbol{b}) \qquad (9)$$

for every pair $\boldsymbol{a}, \boldsymbol{b} \in G$.

PROOF. By [25, Theorem 3.13], the equation

$$(\boldsymbol{x} \oplus \boldsymbol{a}) \ominus (\boldsymbol{x} \oplus \boldsymbol{b}) = \mathrm{gyr}[\boldsymbol{x}, \boldsymbol{a}](\boldsymbol{a} \ominus \boldsymbol{b})$$

holds. Therefore

$$\varrho(\boldsymbol{x} \oplus \boldsymbol{a}, \boldsymbol{x} \oplus \boldsymbol{b}) = \|\phi((\boldsymbol{x} \oplus \boldsymbol{a}) \ominus (\boldsymbol{x} \oplus \boldsymbol{b}))| = \|\phi(\mathrm{gyr}[\boldsymbol{x}, \boldsymbol{a}](\boldsymbol{a} \ominus \boldsymbol{b}))\|$$
$$= \|\phi(\boldsymbol{a} \ominus \boldsymbol{b})\| = \varrho(\boldsymbol{a}, \boldsymbol{b}),$$

thus verifying (8).

By the equation (3) and the left cancellation law, we have

$$\ominus \boldsymbol{a} \oplus \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b}) = \frac{1}{2} \otimes (\ominus \boldsymbol{a} \oplus \boldsymbol{b}).$$

Therefore, we have

$$\varrho(\boldsymbol{a}, \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})) = \varrho(\ominus \boldsymbol{a}, \ominus \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})) = \|\phi(\ominus \boldsymbol{a} \oplus \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b}))\|$$
$$= \|\phi(\frac{1}{2} \otimes (\ominus \boldsymbol{a} \oplus \boldsymbol{b}))\| = \frac{1}{2} \otimes' \|\phi(\ominus \boldsymbol{a} \oplus \boldsymbol{b})\| = \frac{1}{2} \otimes' \varrho(\ominus \boldsymbol{a}, \ominus \boldsymbol{b}) = \frac{1}{2} \otimes' \varrho(\boldsymbol{a}, \boldsymbol{b}).$$

Moreover,

$$\varrho(\boldsymbol{b}, \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})) = \varrho(\boldsymbol{b}, \boldsymbol{p}(\boldsymbol{b}, \boldsymbol{a})) = \frac{1}{2} \otimes' \varrho(\boldsymbol{b}, \boldsymbol{a}) = \frac{1}{2} \otimes' \varrho(\boldsymbol{a}, \boldsymbol{b}). \qquad \square$$

A simple proof of the Mazur–Ulam Theorem by Väisälä makes use of the reflection. For a point $z \in E$, the reflection of the normed vector space E in z is the map $\psi_z : E \to E$ defined by $\psi_z(x) = 2z - x$. In order to prove our main theorem, we consider the map similar to the reflection.

Proposition 16. *Let $(G, \oplus, \otimes)$ be a GGV. For $\boldsymbol{z} \in G$, a bijective self map $\psi_{\boldsymbol{z}} : G \to G$ defined by $\psi_{\boldsymbol{z}}(\boldsymbol{x}) = 2 \otimes \boldsymbol{z} \ominus \boldsymbol{x}$ satisfies the following properties.*

(p1) $\psi_{\boldsymbol{z}}^{-1} = \psi_{\boldsymbol{z}}$,

(p2) $\varrho(\psi_{\boldsymbol{z}}(\boldsymbol{a}), \psi_{\boldsymbol{z}}(\boldsymbol{b})) = \varrho(\boldsymbol{a}, \boldsymbol{b})$,

(p3) $\psi_{\boldsymbol{z}}(\boldsymbol{a}) = \boldsymbol{a}$ *if and only if $\boldsymbol{z} = \boldsymbol{a}$*,

(p4) $\psi_{\boldsymbol{z}}(\boldsymbol{a}) = \boldsymbol{b}$ *and $\psi_{\boldsymbol{z}}(\boldsymbol{b}) = \boldsymbol{a}$ if $\boldsymbol{z} = \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})$*,

(p5) $\varrho(\psi_{\boldsymbol{z}}(\boldsymbol{a}), \boldsymbol{a}) = 2 \otimes' \varrho(\boldsymbol{a}, \boldsymbol{z})$

for any $\boldsymbol{a}, \boldsymbol{b} \in G$.

PROOF. (p1) By the gyroautomorphic inverse property ([25, Definition 3.1, Theorem 3.2]) and the left cancellation low, we have

$$\phi_{\boldsymbol{z}}(\phi_{\boldsymbol{z}}(\boldsymbol{x})) = 2 \otimes \boldsymbol{z} \ominus (2 \otimes \boldsymbol{z} \ominus \boldsymbol{x}) = 2 \otimes \boldsymbol{z} \oplus \{\ominus(2 \otimes \boldsymbol{z}) \oplus \boldsymbol{x}\} = \boldsymbol{x}$$

(p2) By the equation (8), we have

$$\varrho(\psi_{\boldsymbol{z}}(\boldsymbol{a}), \psi_{\boldsymbol{z}}(\boldsymbol{b})) = \varrho(2 \otimes \boldsymbol{z} \ominus \boldsymbol{a}, 2 \otimes \boldsymbol{z} \ominus \boldsymbol{b}) = \varrho(\ominus \boldsymbol{a}, \ominus \boldsymbol{b}) = \varrho(\boldsymbol{a}, \boldsymbol{b})$$

(p3) First, we have

$$\boldsymbol{z} \oplus (\boldsymbol{z} \ominus \boldsymbol{a}) = (\boldsymbol{z} \oplus \boldsymbol{z}) \oplus \mathrm{gyr}[\boldsymbol{z}, \boldsymbol{z}](\ominus \boldsymbol{a}) = \psi_{\boldsymbol{z}}(\boldsymbol{a})$$

and hence

$$\ominus \boldsymbol{z} \oplus \psi_{\boldsymbol{z}}(\boldsymbol{a}) = \ominus \boldsymbol{z} \oplus (\boldsymbol{z} \oplus (\boldsymbol{z} \ominus \boldsymbol{a})) = \boldsymbol{z} \ominus \boldsymbol{a}.$$

Therefore, we have

$$\begin{aligned}
\boldsymbol{a} = \psi_{\boldsymbol{z}}(\boldsymbol{a}) &\iff \ominus \boldsymbol{z} \oplus \boldsymbol{a} = \ominus \boldsymbol{z} \oplus \psi_{\boldsymbol{z}}(\boldsymbol{a}) \\
&\iff \ominus \boldsymbol{z} \oplus \boldsymbol{a} = \ominus(\ominus \boldsymbol{z} \oplus \boldsymbol{a}) \\
&\iff \ominus \boldsymbol{z} \oplus \boldsymbol{a} = \boldsymbol{0} \\
&\iff \boldsymbol{z} = \boldsymbol{a}.
\end{aligned}$$

(p4) Suppose that $\boldsymbol{z} = \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})$. By the (second) right cancellation low, we have

$$\psi_{\boldsymbol{z}}(\boldsymbol{b}) = 2 \otimes \left\{\frac{1}{2} \otimes (\boldsymbol{a} \boxplus \boldsymbol{b})\right\} \ominus \boldsymbol{b} = (\boldsymbol{a} \boxplus \boldsymbol{b}) \ominus \boldsymbol{b} = \boldsymbol{a}.$$

Since $\boxplus$ is commutative, we also have

$$\psi_{\boldsymbol{z}}(\boldsymbol{a}) = (\boldsymbol{a} \boxplus \boldsymbol{b}) \ominus \boldsymbol{a} = (\boldsymbol{b} \boxplus \boldsymbol{a}) \ominus \boldsymbol{a} = \boldsymbol{b}.$$

(p5) By the (first) right cancellation low, we have

$$\frac{1}{2} \otimes (\psi_{\boldsymbol{z}}(\boldsymbol{a}) \boxplus \boldsymbol{a}) = \frac{1}{2} \otimes \{(2 \otimes \boldsymbol{z} \ominus \boldsymbol{a}) \boxplus \boldsymbol{a}\} = \boldsymbol{z}.$$

It implies that $\boldsymbol{z}$ is the gyromidpoint of $\boldsymbol{a}$ and $\psi_{\boldsymbol{z}}(\boldsymbol{a})$. Thus (9) follows that

$$\|\boldsymbol{a} \ominus \boldsymbol{z}\| = \frac{1}{2} \otimes' \|\boldsymbol{a} \ominus \psi_{\boldsymbol{z}}(\boldsymbol{a})\|$$

as desired.   □

A gyrogroup $(G, \oplus)$ and its associated cogyrogroup $(G, \boxplus)$ have the same automorphisms ([25, Theorem 2.28]),

$$\mathrm{Aut}(G, \oplus) = \mathrm{Aut}(G, \boxplus).$$

We have the following proposition. A proof is easy and is omitted.

Proposition 17. *Let $(G_1, \oplus_1)$ and $(G_2, \oplus_2)$ be gyrogroups. Suppose that $T : G_1 \to G_2$ is a bijection. Then the following (I1) and (I2) are equivalent to each other.*

(I1) $T(\boldsymbol{a} \oplus_1 \boldsymbol{b}) = T(\boldsymbol{a}) \oplus_2 T(\boldsymbol{b})$ *for any $\boldsymbol{a}, \boldsymbol{b} \in G_1$,*

(I2) $T(\boldsymbol{a} \boxplus_1 \boldsymbol{b}) = T(\boldsymbol{a}) \boxplus_2 T(\boldsymbol{b})$ *for any $\boldsymbol{a}, \boldsymbol{b} \in G_1$.*

Let $(G, \oplus, \otimes)$ be a GGV. Suppose that $\boldsymbol{a} \in G$ and $0 \le \alpha \le \beta$. Then

$$
\begin{aligned}
\alpha \otimes' \|\phi(\boldsymbol{a})\| = \|\phi(\alpha \otimes \boldsymbol{a})\| &= \left\| \phi\left(\left(\frac{\beta + \alpha}{2} - \frac{\beta - \alpha}{2} \right) \otimes \boldsymbol{a} \right) \right\| \\
&= \left\| \phi\left(\left(\frac{\beta + \alpha}{2} \right) \otimes \boldsymbol{a} \oplus \left(-\frac{\beta - \alpha}{2} \right) \otimes \boldsymbol{a} \right) \right\| \\
&\le \left(\frac{\beta + \alpha}{2} \right) \otimes' \|\phi(\boldsymbol{a})\| \oplus' \left(\frac{\beta - \alpha}{2} \right) \otimes' \|\phi(\boldsymbol{a})\| \\
&= \left(\frac{\beta + \alpha}{2} + \frac{\beta - \alpha}{2} \right) \otimes' \|\phi(\boldsymbol{a})\| = \beta \otimes' \|\phi(\boldsymbol{a})\|
\end{aligned}
$$

Since $(\|\phi(G)\|, \oplus', \otimes')$ is a real linear space, $\alpha \otimes' \|\phi(\boldsymbol{a})\| = \beta \otimes' \|\phi(\boldsymbol{a})\|$ if and only if $\alpha = \beta$ or $\boldsymbol{a} = \boldsymbol{e}$. Hence,

$$0 < \alpha < \beta \iff 0 < \alpha \otimes' \|\phi(\boldsymbol{a})\| < \beta \otimes' \|\phi(\boldsymbol{a})\| \tag{10}$$

for any $\boldsymbol{a} \in G \setminus \{\boldsymbol{e}\}$.

Proposition 18. *Let $(G, \oplus, \otimes)$ be a GGV. Then there exists a bijection $f : \|\phi(G)\| \to \mathbb{R}$ that satisfies the following conditions;*

(F1) $f(a \oplus' b) = f(a) + f(b)$ *and* $f(r \otimes' a) = rf(a)$ *for any $a, b \in \|\phi(G)\|$, $r \in \mathbb{R}$*

(F2) $0 < a < b$ *if and only if* $0 < f(a) < f(b)$ *for $a, b \in \|\phi(G)\|$.*

PROOF. By the condition (GGVV), there exists a bijection $f : \|\phi(G)\| \to \mathbb{R}$ that satisfies the condition (F1). Needless to say, $-f$ also satisfies the condition (F1). Hence, we may assume that $f(\|\phi(\boldsymbol{x}_0)\|) > 0$ for some $\boldsymbol{x}_0 \in G$.

For $0 < a, b \in \|\phi(G)\|$, put $x = \|\phi(\boldsymbol{x}_0)\|$ and $r_1 = f(a)/f(x)$, $r_2 = f(b)/f(x)$. Then $r_1 \otimes' x = a$, $r_2 \otimes' x = b$ and $r_1, r_2 > 0$. Therefore,

$$0 < a < b \iff 0 < r_1 < r_2$$

as (10). Obviously, $0 < f(a) < f(b) \iff 0 < r_1 < r_2$. The map f satisfies the condition (F2). $\square$

5. A proof of Theorem 13

A proof of Theorem 13 is given by modifying the proof of the Mazur–Ulam theorem due to Väisälä [26].

PROOF OF THEOREM 13. Let $\boldsymbol{a}, \boldsymbol{b} \in G_1$ and $\boldsymbol{z}$ be the gyromidpoint of $\boldsymbol{a}$ and $\boldsymbol{b}$. Let W be the family of all bijective gyrometric preserving maps $S : G_1 \to G_1$ keeping the points $\boldsymbol{a}$ and $\boldsymbol{b}$ fixed, and set

$$\lambda = \sup\{f(\varrho(S\boldsymbol{z}, \boldsymbol{z})) : S \in W\} \in [0, \infty], \tag{11}$$

where f is the bijection in Proposition 18. For $S \in W$ we have $\varrho(S\boldsymbol{z}, \boldsymbol{a}) = \varrho(S\boldsymbol{z}, S\boldsymbol{a}) = \varrho(\boldsymbol{z}, \boldsymbol{a})$, hence

$$\varrho(S\boldsymbol{z}, \boldsymbol{z}) \leq \varrho(S\boldsymbol{z}, \boldsymbol{a}) \oplus_1' \varrho(\boldsymbol{a}, \boldsymbol{z}) = 2 \otimes_1' \varrho(\boldsymbol{a}, \boldsymbol{z}), \tag{12}$$

so $f(\varrho(Sz, z)) \leq 2f(\varrho(a, z))$, which yields $\lambda < \infty$.

Let $\psi(\boldsymbol{x}) = 2 \otimes_1 \boldsymbol{z} \ominus_1 \boldsymbol{x}$ on G_1. If $S \in W$, then so also is $S^* = \psi S^{-1} \psi S$, and therefore $\varrho(S^*\boldsymbol{z}, \boldsymbol{z}) \leq \lambda$. Since S^{-1} is a gyrometric preserving map, this fact and (p5) imply that

$$\lambda \geq f(\varrho(S^*\boldsymbol{z}, \boldsymbol{z})) = f(\varrho(\psi S^{-1} \psi S\boldsymbol{z}, \boldsymbol{z})) = F(\varrho(S^{-1} \psi S\boldsymbol{z}, \boldsymbol{z}))$$
$$= f(\varrho(\psi S\boldsymbol{z}, S\boldsymbol{z})) = f(2 \otimes_1' \varrho(S\boldsymbol{z}, \boldsymbol{z})) = 2f(\varrho(Sz, z)) \tag{13}$$

for all $S \in W$, showing that $\lambda \geq 2\lambda$. Thus $\lambda = 0$, which means that $S\boldsymbol{z} = \boldsymbol{z}$ for all $S \in W$.

Let $T : G_1 \to G_2$ be a bijective gyrometric preserving map. Let $\boldsymbol{z}'$ be the gyromidpoint of $T(\boldsymbol{a})$ and $T(\boldsymbol{b})$. To prove the theorem we must show that $T(\boldsymbol{z}) = \boldsymbol{z}'$. Let $\psi'(\boldsymbol{y}) = 2 \otimes_2 \boldsymbol{z}' \ominus_2 \boldsymbol{y}$ on G_2. Then the map $\psi T^{-1} \psi' T$ is in W, whence $\psi T^{-1} \psi' T(\boldsymbol{z}) = \boldsymbol{z}$. This implies that $\psi'(T(\boldsymbol{z})) = T(\boldsymbol{z})$. Since $\boldsymbol{z}'$ is the only fixed point of ψ', we obtain $T(\boldsymbol{z}) = \boldsymbol{z}'$. $\square$

PROOF OF COROLLARY 14. Let $T_0 = \ominus_2 T(\boldsymbol{e}_1) \oplus_2 T$. Indeed, $T_0 : G_1 \to G_2$ is surjective and $T_0(\boldsymbol{e}_1) = \boldsymbol{e}_2$. By the left cancellation law, we have $T = T(\boldsymbol{e}_1) \oplus_2 T_0$. By the equation (8), T_0 is a gyrometric preserving map because T is so. Applying Theorem 13 to T_0, we have

$$T_0\left(\frac{1}{2} \otimes_1 (\boldsymbol{a} \boxplus_1 \boldsymbol{b})\right) = \frac{1}{2} \otimes_2 (T_0(\boldsymbol{a}) \boxplus_2 T_0(\boldsymbol{b})) \tag{14}$$

for any $\boldsymbol{a}, \boldsymbol{b} \in G_1$. Since $T_0(\boldsymbol{e}_1) = \boldsymbol{e}_2$, we have

$$T_0\left(\frac{1}{2} \otimes_1 \boldsymbol{x}\right) = T_0\left(\frac{1}{2} \otimes_1 (\boldsymbol{x} \boxplus_1 \boldsymbol{e}_1)\right)$$
$$= \frac{1}{2} \otimes_2 (T_0(\boldsymbol{x}) \boxplus_2 T_0(\boldsymbol{e}_1)) = \frac{1}{2} \otimes_2 T_0(\boldsymbol{x}) \tag{15}$$

for any $\boldsymbol{x} \in G_1$. It follows that

$$T_0(\boldsymbol{a} \boxplus_1 \boldsymbol{b}) = T_0(\boldsymbol{a}) \boxplus_2 T_0(\boldsymbol{b}) \tag{16}$$

for any $\boldsymbol{a}, \boldsymbol{b} \in G_1$. Since T_0 is bijective, we have the equation (5) by Proposition 17.

Note that $\|\phi_1(G_1)\| = \|\phi_2(G_2)\|$ since T_0 is a bijection and

$$\|\phi_2(T_0(\boldsymbol{a}))\| = \varrho_2(T_0(\boldsymbol{a}), \boldsymbol{e}_2) = \varrho_1(\boldsymbol{a}, \boldsymbol{e}_1) = \|\phi_1(\boldsymbol{a})\|$$

for any $\boldsymbol{a} \in G_1$. Furthermore, $\frac{1}{2} \otimes_1' a = \frac{1}{2} \otimes_2' a$ for any $0 \le a \in \|\phi_1(G_1)\| = \|\phi_2(G_2)\|$ as

$$\frac{1}{2} \otimes_1' \|\phi_1(\boldsymbol{a})\| = \|\phi_1(\frac{1}{2} \otimes_1 \boldsymbol{a})\| = \|\phi_2(T_0(\frac{1}{2} \otimes_1 \boldsymbol{a}))\| = \|\phi_2(\frac{1}{2} \otimes_2 T_0(\boldsymbol{a}))\|$$
$$= \frac{1}{2} \otimes_2' \|\phi_2(T_0(\boldsymbol{a}))\| = \frac{1}{2} \otimes_2' \|\phi_1(\boldsymbol{a})\|.$$

Next we show that $T_0(\alpha \otimes_1 \boldsymbol{a}) = \alpha \otimes_2 T_0(\boldsymbol{a})$ for any $\boldsymbol{a} \in G_1$ and $\alpha \in \mathbb{R}$. For any $\boldsymbol{a} \in G_1$ and for any integer m, $T_0(m \otimes_1 \boldsymbol{a}) = m \otimes_2 T_0(\boldsymbol{a})$ is satisfied by the equation (5). By the equation (15), we have

$$T_0\left(\frac{m}{2^n} \otimes_1 \boldsymbol{a}\right) = \frac{m}{2^n} \otimes_2 T_0(\boldsymbol{a}) \tag{17}$$

for any integer n, m. Let $\alpha \in \mathbb{R}$ and $r_k \to \alpha$, where $\{r_k\}$ be a sequence in $\{m/2^n \mid m, n : \text{integer}\}$. By Proposition 18, there is a bijection $f_i : \|\phi_i(G_i)\| \to \mathbb{R}$ satisfying the conditions (F1) and (F2) $(i = 1, 2)$. For any $\boldsymbol{b} \in G_1 \setminus \{\boldsymbol{e}_1\}$, a number k_0 can be found such that $k > k_0$ implies that

$$2|\alpha - r_k| f_1(\|\phi_1(\boldsymbol{a})\|) < f_1(\|\phi_1(\boldsymbol{b})\|)$$

and

$$2|\alpha - r_k| f_2(\|\phi_2(T_0(\boldsymbol{a}))\|) < f_2(\|\phi_2(T_0(\boldsymbol{b}))\|).$$

Altogether, $k > k_0$ implies that

$$|\alpha - r_k| \otimes_1' \|\phi_1(\boldsymbol{a})\| < \frac{1}{2} \otimes_1' \|\phi_1(\boldsymbol{b})\| = \frac{1}{2} \otimes_2' \|\phi_2(T_0(\boldsymbol{b}))\|$$

and

$$|\alpha - r_k| \otimes_2' \|\phi_2(T_0(\boldsymbol{a}))\| < \frac{1}{2} \otimes_2' \|\phi_2(T_0(\boldsymbol{b}))\|.$$

By the equations (8) and (17), we have

$$\varrho_2(T_0(\alpha \otimes_1 \boldsymbol{a}), \alpha \otimes_2 T_0(\boldsymbol{a}))$$
$$\leq \varrho_2(T_0(r_k \otimes_1 \boldsymbol{a}), T_0(\alpha \otimes_1 \boldsymbol{a})) \oplus_2' \varrho_2(r_k \otimes_2 T_0(\boldsymbol{a}), \alpha \otimes_2 T_0(\boldsymbol{a}))$$
$$= \varrho_1((r_k \otimes_1 \boldsymbol{a}), \alpha \otimes_1 \boldsymbol{a}) \oplus_2' \|\phi_2(r_k \otimes_2 T_0(\boldsymbol{a}) \ominus_2 \alpha \otimes_2 T_0(\boldsymbol{a}))\|$$
$$= \|\phi_1(r_k \otimes_1 \boldsymbol{a} \ominus_1 \alpha \otimes_1 \boldsymbol{a})\| \oplus_2' \|\phi_2((r_k - \alpha) \otimes_2 T_0(\boldsymbol{a}))\|$$
$$= \|\phi_1((r_k - \alpha) \otimes_1 \boldsymbol{a})\| \oplus_2' |r_k - \alpha| \otimes_2' \|\phi_2(T_0(\boldsymbol{a}))\|$$
$$= |r_k - \alpha| \otimes_1' \|\phi_1(\boldsymbol{a})\| \oplus_2' |r_k - \alpha| \otimes_2' \|\phi_2(T_0(\boldsymbol{a}))\|$$
$$< \frac{1}{2} \otimes_2' \|\phi_2(T_0(\boldsymbol{b}))\| \oplus_2' \frac{1}{2} \otimes_2' \|\phi_2(T_0(\boldsymbol{b}))\| = \|\phi_2(T_0(\boldsymbol{b}))\|.$$

It implies that $T_0(\alpha \otimes_1 \boldsymbol{a}) \ominus_2 \alpha \otimes_2 T_0(\boldsymbol{a}) \neq T_0(\boldsymbol{b})$ for any $\boldsymbol{b} \in G_1 \setminus \{\boldsymbol{e}_1\}$. Since T_0 is bijective and $T_0(\boldsymbol{e}_1) = \boldsymbol{e}_2$, we have $T_0(\alpha \otimes_1 \boldsymbol{a}) = \alpha \otimes_2 T_0(\boldsymbol{a})$.

Finally by Proposition 15 we observe

$$\varrho_2(T_0(\boldsymbol{a}), T_0(\boldsymbol{b})) = \varrho_2(T(\boldsymbol{e}) \oplus_2 T_0(\boldsymbol{a}), T(\boldsymbol{e}) \oplus_2 T_0(\boldsymbol{b})) = \varrho_1(\boldsymbol{a}, \boldsymbol{b}), \quad \boldsymbol{a}, \boldsymbol{b} \in G_1. \quad \square$$

MOLNÁR proved a Mazur–Ulam theorem [16, Theorem 3] for the metric spaces equipped with the binary operations with which they form the point-reflection geometries (cf. [14]). It asserts that under some additional assumptions an isometry between such spaces locally preseves the inverted Jordan products (cf. [8, Theorem 2.4, Corollaries 3.9 and 3.10], [19, Proposition 11]). If a generalized gyrovector space is torsion-free ([25, Definition 3.32]) and two-divisible ([25, Definition 3.33]), we can apply a Mazur–Ulam theorem of Molnár to give an alternative proof of Theorem 13 although the proof is far from beeing trivial.

6. Applications

Theorem 19 was proved by HONMA and NOGAWA [10, Theorem 8] and the case of $t = 1$ is exhibited as Theorem 9 in [9]. The proofs in [10, Theorem 8] and

[9, Theorem 9] employ a non-commutative Mazur–Ulam theorem (cf. [8]). In this section we give a simple proof of Theorem 19 as an application of Corollary 14. Recall that a Jordan *-isomorphism from a C^*-algebra onto another one is a complex linear bijection which preserves * and the square of the elements.

Theorem 19. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital C^*-algebras and t a positive real number. Put $d_{\mathscr{A}}(a,b)$(resp. $d_{\mathscr{B}}(a,b)$) $= \| \log(a^{-\frac{t}{2}} b^t a^{-\frac{t}{2}})^{\frac{1}{t}} \|$, $a,b \in \mathscr{A}_+^{-1}$(resp. $\mathscr{B}_+^{-1}$). Suppose that $T : \mathscr{A}_+^{-1} \to \mathscr{B}_+^{-1}$ is a surjective isometry; $\| \log(a^{-\frac{t}{2}} b^t a^{-\frac{t}{2}})^{\frac{1}{t}} \| = \| \log(T(a)^{-\frac{t}{2}} T(b)^t T(a)^{-\frac{t}{2}})^{\frac{1}{t}} \|$, $a,b \in \mathscr{A}_+^{-1}$. Then there exists a Jordan *-isomorphism and a central projection $p \in \mathscr{B}$ such that T has the form*

$$T(a) = (T(e)^{\frac{t}{2}}(pJ(a) + (e-p)J(a^{-1}))^t T(e)^{\frac{t}{2}})^{\frac{1}{t}}, \quad a \in \mathscr{A}_+^{-1}.$$

PROOF. By Corollary 14 we have that $T(a) = T(e) \oplus_t T_0(a)$, $a \in \mathscr{A}_+^{-1}$ for an isometrical isomorphism T_0:

$$T_0\big((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}}\big) = \big(T_0(a)^{\frac{t}{2}} T_0(b)^t T_0(a)^{\frac{t}{2}}\big)^{\frac{1}{t}}, \quad a,b \in \mathscr{A}_+^{-1}; \qquad (18)$$

and

$$\| \log(a^{-\frac{t}{2}} b^t a^{-\frac{t}{2}})^{\frac{1}{t}} \| = \| \log(T_0(a)^{-\frac{t}{2}} T_0(b)^t T_0(a)^{-\frac{t}{2}})^{\frac{1}{t}} \|, \quad a,b \in \mathscr{A}_+^{-1}. \qquad (19)$$

By (6) in Corollary 14 we have

$$T_0(a^{\frac{1}{n}}) = T_0(a)^{\frac{1}{n}} \qquad (20)$$

for every positive integer n. The rest of the proof is similar to that of [9, Theorem 9], but for the convenience of the readers we give a sketch of it. Let us consider the bijective transform S_0 from $\mathscr{A}_S$ onto $\mathscr{B}_S$ defined by

$$S_0(x) = \log T_0(\exp x), \quad x \in \mathscr{A}_S.$$

By (20) we obtain $S_0(\frac{x}{n}) = \frac{S_0(x)}{n}$ for every $a \in \mathscr{A}_+^{-1}$ and for every positive integer n. Letting $n \to \infty$ in the equation

$$n d_{\mathscr{A}}\Big(\exp \frac{x}{n}, \exp \frac{y}{n}\Big) = n d_{\mathscr{B}}\Big(T_0\Big(\exp \frac{x}{n}\Big), T_0\Big(\exp \frac{y}{n}\Big)\Big) = n d_{\mathscr{B}}\Big(\exp \frac{S_0(x)}{n}, \exp \frac{S_0(y)}{n}\Big),$$

$$\|y - x\| = \|S_0(y) - S_0(x)\|, \quad x,y \in \mathscr{A}_S;$$

S_0 is a bijective isometry and $S_0(0) = 0$. By the Mazur–Ulam theorem and the result [11, Theorem 2] of KADISON there is a central projection $p \in \mathscr{B}$ and

a Jordan *-isomorphism $J : \mathscr{A} \to \mathscr{B}$ such that $S_0(e) = 2p - e$ and $S_0(x) = S_0(e)J(x)$ for every $x \in \mathscr{A}_S$. We compute

$$T_0(\exp x) = pJ(\exp x) + (e - p)J(\exp(-x)), \quad x \in \mathscr{A}_S.$$

It follows that

$$T(a) = (T(e)^{\frac{t}{2}}(pJ(a) + (e - p)J(a^{-1}))^t T(e)^{\frac{t}{2}})^{\frac{1}{t}}$$

for every $a \in \mathscr{A}_+^{-1}$. $\qquad\square$

Corollary 20. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital C^*-algebras and t a positive real number. Suppose that $T : \mathscr{A}_+^{-1} \to \mathscr{B}_+^{-1}$ is an isomorphism between the gyro-commutative gyrogroups $(\mathscr{A}_+^{-1}, \oplus_t)$ and $(\mathscr{B}_+^{-1}, \oplus_t)$. Suppose that T preserves the spectrum; $\sigma(a) = \sigma(T(a))$ for every $a \in \mathscr{A}_+^{-1}$, where $\sigma(\cdot)$ denotes the spectrum. Then T is extended to a Jordan $*$-isomorphism from $\mathscr{A}$ onto $\mathscr{B}$.*

PROOF. Since an isomorphism preserves the inverse, we have $T(b^{-1}) = T(b)^{-1}$ for every $b \in \mathscr{A}_+^{-1}$ since b^{-1} (resp. $T(b)^{-1}$) is the inverse of b (resp. $T(b)$) in the gyrogroup $\mathscr{A}_+^{-1}$ (resp. $\mathscr{B}_+^{-1}$). By the assumption the isomorphism T preserves the spectrum we have

$$\sigma(a^{\frac{t}{2}}b^{-t}a^{\frac{t}{2}}) = \sigma(T(a^{\frac{t}{2}}b^{-t}a^{\frac{t}{2}})) = \sigma(T(a)^{\frac{t}{2}}T(b)^{-t}T(a)^{\frac{t}{2}}), \quad a, b \in \mathscr{A}_+^{-1}.$$

By the spectrum mapping theorem we infer that

$$\sigma(\log(a^{\frac{t}{2}}b^{-t}a^{\frac{t}{2}})) = \sigma(\log(T(a)^{\frac{t}{2}}T(b)^{-t}T(a)^{\frac{t}{2}}))$$

for every pair $a, b \in \mathscr{A}_+^{-1}$. Hence

$$\|\log(a^{\frac{t}{2}}b^{-t}a^{\frac{t}{2}})\| = \|\log(T(a)^{\frac{t}{2}}T(b)^{-t}T(a)^{\frac{t}{2}})\|$$

for every pair $a, b \in \mathscr{A}_+^{-1}$. As $\sigma(T(e)) = \sigma(e) = \{1\}$ and $T(e)$ is a positive element, we have $T(e) = e$. By Theorem 19 there exists a Jordan *-isomorphism J from $\mathscr{A}$ onto $\mathscr{B}$ and a central projection $p \in \mathscr{B}$ with

$$T(a) = pJ(a) + (e - p)J(a^{-1}), \quad a \in \mathscr{A}_+^{-1}.$$

Letting $a = e/2$ we have $T(e/2) = pJ(e/2) + (e - p)J(2e) = pe/2 + 2(e - p)e$. As $\sigma(T(e/2)) = \sigma(e/2) = \{1/2\}$ we infer that $p = e$. Therefore $T(a) = J(a)$ for every $a \in \mathscr{A}_+^{-1}$. $\qquad\square$

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