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Stone–Čech compactification with applications to evolution equations on Banach spaces

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Abstract. In this note we study the question of Stone–Čech compactification of Banach spaces. This is used to construct regular countably additive measure valued functions as solutions for evolution equations on Banach spaces where the standard notions of solutions fail. We show that if one admits finitely additive measure valued solutions one can define the notion of measure solutions in terms of the original state space.

1. Introduction

In the study of measure solutions of differential equations on Banach spaces having nonlinearities with polynomial or even exponential growth it is convenient to seek for a compact Hausdorff space containing a dense subspace which is homeomorphic with the given Banach space. This allows us to construct countably additive measure valued solutions for the system. In general the solutions are only finitely additive measure valued functions and by introducing compactification one obtains countably additive measure solutions. This is one of the motivation for seeking compactification. Further this prevents the measure solutions from leaving the compactified state space.

2. A general result

In this section we present a result on the Stone–Čech compactification of Banach spaces which, in addition to having independent interest, has

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application in the study of measure solutions for evolution equations on Banach spaces.

Theorem A. *Every Banach space X admits Stone–Čech compactification βX which is a compact Hausdorff space containing a dense subspace which is homeomorphic with X. Let X, Y be any two Banach spaces so that the embedding $X \hookrightarrow Y$ is continuous. Then their Stone–Čech compactifications satisfy the inclusions*

$$\beta Y \subseteq \beta X. \tag{2.1}$$

PROOF. Let $\mathcal{X}$ denote any one of the pair of Banach spaces $\{X, Y\}$ and $BC(\mathcal{X})$ the space of bounded continuous real valued functions on $\mathcal{X}$ with the topology induced by the norm

$$\|\phi\| \equiv \sup\{|\phi(x)|,\ x \in \mathcal{X}\}. \tag{2.2}$$

Clearly $BC(\mathcal{X})$ with this topology is a Banach space. For each $\phi \in BC(\mathcal{X})$, define its range by $I_\phi \equiv \phi(\mathcal{X})$. This is a closed bounded interval of R and hence compact. Since $\mathcal{X}$ is a Banach space it is a metric space and since every metric space is a Tychonoff space (a completely regular T_1 space, denoted by $T_{3\frac{1}{2}}$), the space $\mathcal{X}$ is a Tychonoff space. Thus the elements of $BC(\mathcal{X})$ separate points from closed sets in $\mathcal{X}$. Hence the evaluation map

$$\mathcal{X} \longrightarrow \prod\{I_\phi,\ \phi \in BC(\mathcal{X})\} \tag{2.3}$$

denoted by e is an embedding of $\mathcal{X}$ into $\prod\{I_\phi,\ \phi \in BC(\mathcal{X})\}$. Every Tychonoff space $\mathcal{X}$ has a Stone–Čech compactification $\beta\mathcal{X}$ which is a compact Hausdorff space [see 1, Theorem 8.3.1, p. 147]. In fact $\beta\mathcal{X}$ is unique (up to topological equivalence) given by the closure of $e(\mathcal{X})$ in the product topology. Clearly $\mathcal{X}$ is homeomorphic with $e(\mathcal{X})$ which is a dense subspace of $\beta\mathcal{X}$. This verifies the first assertion.

Now let us consider the pair of Banach spaces $\{X, Y\}$. By assumption the injection $X \hookrightarrow Y$ is continuous and hence the norm topology of X is stronger than the norm topology of Y. Thus if $\phi \in BC(Y)$ then its restriction $\phi|_X$, again denoted by ϕ, is also in $BC(X)$. Hence topologically $BC(Y) \subseteq BC(X)$. Then $e(Y) \subseteq e(X)$ and their closures in the respective product topologies satisfy the inclusion $\beta Y \subseteq \beta X$. This completes the proof. □

Remark. The question is, under what (additional) conditions, the inclusion given by (2.1) is actually an equality (up to topological equivalence). For this we recall the definition of C^*-embedding.

Definition 2.1. A subset K of a topological space T is said to be C^*-embedded in T if and only if every bounded continuous real valued function on K can be extended to T.

From general topology it is known [1] that if X is C^*-embedded in Y then $\beta X = \beta Y$, that is, these spaces are topologically equivalent. Clearly for arbitrary Banach spaces X and Y with continuous injection $X \hookrightarrow Y$, X is not necessarily C^*-embedded in Y. A simple example is $X \equiv H^1(R^n)$ and $Y = L_2(R^n)$ with the embedding $X \hookrightarrow Y$ being continuous and dense. Note that a continuous function on $H^1(R^n)$ need not have a continuous extension on to $L_2(R^n)$ and thus H^1 is not C^*-embedded in $L_2(R^n)$ and $\beta Y \subset \beta X$ only.

Here we have used what is known as the"standard analysis" for compactification. This is the classical approach. In recent years "nonstandard analysis" has emerged as a powerful, yet simple, tool whereby non Hausdorff compactification is easily obtained. Another step using quotient topology leads to compact Hausdorff. Stone–Čech compactification is obtained by a particular choice of an equivalence relation. For an excellent exposition of the nonstandard analysis and its application to point-set topology the reader is referred to the recent papers of Salbany and Todorov [14], [10].

Nonstandard analysis has been also used by Oberguggenberger [11] and Todorov [12], [13] to construct generalized solutions for ordinary and partial differential equations with smooth coefficients. Here we deal with abstract evolution equations on Banach spaces which cover many physical problems such as Navier–Stokes equation, MHD equations, Nonlinear Klein–Gordon equation, Reaction–Diffusion equation with bounded as well as unbounded coefficients [2], [4], [15], [16].

3. An application

Consider the semilinear evolution equation:

$$\begin{aligned} &\dot{x} + Ax = f(x), \quad t \in I \equiv [0,T], \qquad T < \infty, \\ &x(0) = x_0, \end{aligned} \tag{3.1}$$

in a Banach space E. Let $-A$ be the infinitesimal generator of an analytic semigroup on E and $\alpha \in (0,1]$ and let A^α denote the fractional power of A. Denote by $E_\alpha \equiv [D(A^\alpha)]$ the normed space induced by the graph norm

$$\|x\|_\alpha \equiv \|x\|_E + \|A^\alpha x\|_E, \quad \text{for } x \in D(A^\alpha). \tag{3.2}$$

Clearly $[D(A)] \subseteq E_\gamma \subset E_\beta \subset E_0 = E$ for $0 < \beta \le \gamma \le 1$. Since A and its fractional powers are closed operators, these are Banach spaces (see [3]) and further, the injections are continuous.

Let $\phi \in BC(E_\alpha)$ and $D\phi$ its Frechet derivative on E_α with values $D\phi(\xi) \in E_\alpha^*$ for $\xi \in E_\alpha$, where E_α^* is the dual of E_α. Since the embedding $E_\alpha \hookrightarrow E$ is continuous and dense we have $E^* \subset E_\alpha^*$. For study of measure solutions we introduce the following class of test functions denoted by $\mathcal{F}_\alpha$. This is given by

$$\mathcal{F}_\alpha \equiv \{\phi \in BC(E_\alpha) : D\phi \text{ exists, } D\phi(\xi) \in D(A^*) \subset E^*,\ \xi \in E_\alpha \\ \text{and } D\phi \in BC(E_\alpha, E^*)\}.$$

For each $\phi \in \mathcal{F}_\alpha$, let $\tilde{\phi}$ denote its continuous extension from E_α to βE_α so that $\tilde{\phi} \circ e_\alpha = \phi$ where e_α denotes the embedding of E_α into βE_α. Define the operator $\tilde{\mathcal{A}}$ with domain

$$\mathcal{D}(\tilde{\mathcal{A}}) \equiv \{\tilde{\phi} \in BC(\beta E_\alpha) : \mathcal{A}(\tilde{\phi} \circ e_\alpha) \in BC(E_\alpha)\}$$

by setting

$$\tilde{\mathcal{A}}(\psi) \equiv \mathcal{A}(\psi \circ e_\alpha) \quad \text{for} \quad \psi \in D(\tilde{\mathcal{A}}),$$

where the operator $\mathcal{A}$ is given by

$$(\mathcal{A}\phi)(\xi) = -\langle A^* D\phi(\xi), \xi \rangle_{E^*,E} + \langle D\phi(\xi), f(\xi) \rangle_{E^*,E}, \quad \xi \in E_\alpha, \tag{3.3}$$

for $\phi \in D(\mathcal{A}) \subset \mathcal{F}_\alpha$. As in Section 2, here βE_α is the Stone–Čech compactification of E_α. Note that $\mathcal{D}(\mathcal{A}) \neq \emptyset$. For example, for $\psi \in \mathcal{F}_\alpha$, the function φ defined by $\varphi(x) \equiv \psi(rR(r, -A^\alpha)x)$, belongs to $\mathcal{D}(\mathcal{A})$ for each $r \in \rho(-A^\alpha)$, the resolvent set of $-A^\alpha$.

In finite dimensional spaces, if f is only continuous one can construct a classical solution possibly defined only locally with finite blow up time. But in infinite dimensional spaces, the situation is very different. If f is merely continuous and even bounded on bounded sets, the evolution equation

(3.1) may not possess any solution in any one of the classical senses such as *classical, strong, mild* or *weak*. For examples, see GODUNOV [8] and the references therein, where he shows the invalidity of Peano's theorem in Hilbert spaces. Similar counter examples can be easily constructed in the Banach space c_o, the space of infinite sequences converging to zero. Construction of such examples is possible since the unit balls in infinite dimensional spaces are not compact.

The concept of measure valued solutions generalizes the above notions and presents a wider horizon to look for existence of solutions. This subject has been studied by the author in several papers [2], [4], [6], [15], [16]. In [8] GODUNOV presented examples and proved the nonexistence of classical solutions, that is, $C^1(I, E)$. It is possible that a classical solution, even in the weakened form (see [8]), may not exist while a measure valued solution may. However, for the later, the author of this paper has given existence results under the assumption that $f : I \times E_\alpha \mapsto E$ is Borel measurable and that for almost all $t \in I$, $\xi \longrightarrow f(t, \xi)$ is continuous and bounded on bounded sets [2], [4], [15], [16]. If such conditions are not satisfied by the examples constructed in [8], even the question of existence of measure valued solutions can not be settled. This requires some verification. In order to discuss the measure solutions further, we need to consider finitely additive measures.

Let X be a normal topological space and $\Psi_c(X)$ the algebra generated by the class of closed subsets of X. Let $\Sigma_{\mathrm{rba}}(X)$ denote the space of regular bounded finitely additive measures defined on $\Psi_c(X)$ and $\Sigma_{\mathrm{rca}}(X)$ denote the class of regular countably additive measures defined on the sigma field $\sigma(\Psi_c(X))$ generated by the class of closed subsets of X. These spaces, furnished with the total variation norm, are Banach spaces. Let $\Pi_{\mathrm{rba}}(X) \subset \Sigma_{\mathrm{rba}}(X)$ denote the space of regular finitely additive probability measures on X. From the well known characterization results (see [7, Theorem 2, Theorem 3, p. 262–265–265]) the dual of $BC(X)$ is given by $\Sigma_{\mathrm{rba}}(X)$, and if X is a compact topological space then the dual of $BC(X)$ is given by $\Sigma_{\mathrm{rca}}(X)$. Thus for compact X, any $\mu \in \Sigma_{\mathrm{rba}}(X)$ has a countably additive extension from $\Psi_c(X)$ to $\sigma(\Psi_c(X))$.

The action of any measure $\nu \in \Sigma_{\mathrm{rba}}(X)$ on any element $\phi \in BC(X)$ will be denoted by

$$\nu(\phi) \equiv \int_X \phi(\xi)\ \nu(d\xi).$$

For any interval finite $I = [0, \tau]$, let $L_1(I, BC(X))$ denote the Lebesgue–Bochner space of integrable functions on I taking values from the Banach space $BC(X)$. Since $BC(X)$ does not satisfy the Radon–Nikodym property, the dual of $L_1(I, BC(X))$ is not given by $L_\infty(I, \Sigma_{\rm rba}(X))$. However, by the theory of "lifting" [9], its dual is given by $L^w_\infty(I, \Sigma_{\rm rba}(X))$ which is the space of *weak** measurable functions on I with values in $\Sigma_{\rm rba}(X)$ with the norm given by

$$\|\lambda\| \equiv \inf\{k \geq 0 : \operatorname*{ess-sup}_{t\in I} \lambda_t(\phi) \leq k,\ \forall \phi \in BC(X) : \|\phi\|_{BC(X)} = 1\},$$

for $\lambda \in L^w_\infty(I, \Sigma_{\rm rba}(X))$.

Now we return to the system (3.1) and assume that f is a general nonlinear map from E_α to E which is continuous and bounded on bounded sets. The following definition for measure solutions was introduced in [2], [4] generalizing a similar notion proposed in [5] which was further extended to stochastic differential equations on Hilbert spaces [6]. In the stochastic case the differential operator $\mathcal{A}$ is second order while in the deterministic case it is a first order differential operator given by (3.3). Let the initial state x_0 be given by either a fixed element of E_α, in which case it is considered as a Dirac measure on E_α concentrated at the point $\{x_0\}$, or, in general, an arbitrary measure $\mu_0 \in \Pi_{\rm rba}(E_\alpha) \subset \Sigma_{\rm rba}(E_\alpha)$. Define $\lambda_0 = \mu_0 \cdot e_\alpha^{-1}$.

Definition 3.1. A measure valued function $\lambda \in L^w_\infty(I, \Sigma_{\rm rba}(\beta E_\alpha)) = L^w_\infty(I, \Sigma_{\rm rca}(\beta E_\alpha))$ is said to be a generalized solution of equation (3.1) corresponding to the initial state μ_0 if, for every $\tilde{\phi} \in \mathcal{D}(\tilde{\mathcal{A}})$ given by $\tilde{\phi} = \phi \circ e_\alpha^{-1}$ for some $\phi \in \mathcal{F}_\alpha$ with $D\phi$ having bounded supports on E_α, the following equality holds

$$\lambda_t(\tilde{\phi}) = \lambda_0(\tilde{\phi}) + \int_0^t \lambda_s(\tilde{\mathcal{A}}\tilde{\phi})ds, \quad t \in I, \tag{3.4}$$

where

$$\lambda_t(\psi) \equiv \int_{\beta E_\alpha} \psi(\xi)\lambda_t(d\xi), \quad t \in I \text{ and } \psi \in BC(\beta E_\alpha).$$

The following result was proved in [2].

Theorem B. *Let $-A$ be the infinitesimal generator of an analytic semigroup in the Banach space E and $f : E_\alpha \mapsto E$ continuous and bounded on bounded subsets of E_α, for some $\alpha \in (0,1)$. Then for each $x_0 \in E_\alpha$, or more generally, for each $\lambda_0 \in \Pi_{\mathrm{rba}}(\beta E_\alpha)$, the evolution equation* (3.1) *has at least one measure solution $\lambda \in L^w_\infty(I, \Sigma_{\mathrm{rba}}(\beta E_\alpha))$ in the sense of Definition 3.1. Further $t \to \lambda_t$ is w^* continuous with values in $\Pi_{\mathrm{rba}}(\beta E_\alpha)$.*

Proof. For detailed proof see [2] and more general results see [16]. □

Remark. We note that if the problem (3.1) has a solution in the sense of any one of the standard notions of solutions like the classical, strong, mild, weak, then these solutions are also measure valued solutions satisfying the identity (3.4) [2], [16]. However the reverse is not true. In general the (locally convex) topological vector space $C(I, E^w_\alpha)$, the space of vector valued functions on I which are weakly continuous, is embedded in $L^w_\infty(I, \Sigma_{\mathrm{rba}}(E_\alpha))$.

Since βE_α is a compact Hausdorff space we have $\Sigma_{\mathrm{rba}}(\beta E_\alpha) = \Sigma_{\mathrm{rca}}(\beta E_\alpha)$. According to Theorem B, for each $t \in I$, $\lambda_t \in \Pi_{\mathrm{rba}}(\beta E_\alpha)$ and hence $\lambda_t \in \Pi_{\mathrm{rca}}(\beta E_\alpha)$. By Theorem A of the previous section we have $\beta E \subset \beta E_\alpha$, and clearly this means that the measure solutions are supported on a rather very large space. Our concern in this section is to investigate the properties of their restrictions to smaller spaces. Let e denote the embedding of E into $\prod\{I_\phi,\ \phi \in BC(E)\}$ and e_α the embedding of E_α into $\prod\{I_\phi,\ \phi \in BC(E_\alpha)\}$. Associated with any measure solution λ of (3.1), we define the following pair of measure functions $\{\mu^E, \mu^\alpha\}$ given by:

$$\begin{aligned} &(1) : \mu^E_t \equiv \nu^E_t \circ e, \quad \text{where } \nu^E_t \equiv \lambda_t|_{e(E)} \\ &(2) : \mu^\alpha_t \equiv \nu^\alpha_t \circ e_\alpha, \quad \text{where } \nu^\alpha_t \equiv \lambda_t|_{e_\alpha(E_\alpha)} \end{aligned} \qquad \text{for } t \in I.$$

Since λ is weak star continuous in t, for each $t \in I$ these are well defined functions on the field of sets $\Psi_c(E)$ and $\Psi_c(E_\alpha)$ respectively. In general the measure solution $\{\lambda_t, t \in I\}$ is supported on βE_α. The measure ν^E_t (and hence μ^E_t) is nontrivial only if the

$$\mathrm{spt}(\lambda_t) \subset \beta E \subset \beta E_\alpha.$$

The measures $\{\nu_t^E, \nu_t^\alpha\}$ are defined on $\Psi_c(e(E))$ and $\Psi_c(e_\alpha(E_\alpha))$ respectively and their countably additive extensions to the σ-algebras generated by the fields $\Psi_c(e(E))$ and $\Psi_c(e_\alpha(E_\alpha))$ respectively are $\lambda_t|_{\beta E}$ and λ_t respectively. Clearly, for almost all $t \in I$,

$$\mu_t^E \in \Sigma_{\text{rba}}(E) \quad \text{and} \quad \mu_t^\alpha \in \Sigma_{\text{rba}}(E_\alpha),$$

and further

$$\mu_t^E \in \Pi_{\text{rba}}(E) \quad \text{and} \quad \mu_t^\alpha \in \Pi_{\text{rba}}(E_\alpha),$$

if and only if

$$\nu_t^E \in \Pi_{\text{rba}}(\beta E), \quad \text{and} \quad \nu_t^\alpha \in \Pi_{\text{rba}}(\beta E_\alpha)$$

respectively.

The following result is useful for defining measure solutions in terms of the basic spaces E and E_α.

Theorem C. *For each $t \in I$, the measure μ_t^E is μ_t^α continuous on $\Psi_c(E_\alpha)$ in the sense that*

$$\lim_{\mu_t^\alpha(\Gamma)\to 0} \mu_t^E(\Gamma) = 0, \quad \textit{for } \Gamma \in \Psi_c(E_\alpha).$$

Further μ^α, an element of $L_\infty^w(I, \Sigma_{\text{rba}}(E_\alpha))$, is a measure solution of equation (3.1) *in the sense that*

$$\mu_t^\alpha(\phi) = \mu_0^\alpha(\phi) + \int_0^t \mu_s^\alpha(\mathcal{A}\phi)ds, \quad \forall t \in I, \tag{3.5}$$

for all $\phi \in D(\mathcal{A})$ having Frechet derivatives with bounded supports.

PROOF. For convenience of presentation, we accept a slight abuse of notation. For any measure $\nu \in \Sigma_{\text{rba}}(E_\alpha)$ and any set $\Gamma \in \Psi_c(E_\alpha)$, we write $\nu(\Gamma)$ for $\nu(\xi_\Gamma)$ where ξ_Γ denotes the characteristic function of the set Γ. Though $\xi_\Gamma \notin BC(E_\alpha)$, $\nu(\xi_\Gamma)$ is well defined which can be demonstrated by simple regularisation and taking limits. We show that μ_t^E is μ_t^α continuous. This will follow if we show that for each set $\Gamma \in \Psi_c(E_\alpha)$

$$\mu_t^E(\Gamma) \le \mu_t^\alpha(\Gamma). \tag{3.6}$$

Let $\tilde{\Gamma} \in \Psi_c(E)$ and define $\Gamma \in \Psi_c(E_\alpha)$ as being the largest set contained in $\tilde{\Gamma}$. Clearly the set $\Gamma \neq \emptyset$ whenever $\tilde{\Gamma} \neq \emptyset$ and the map $\tilde{\Gamma} \longrightarrow \Gamma$

exhausts $\Psi_c(E_\alpha)$. Indeed, this map can be characterized by the expression $\Gamma = c\ell_{E_\alpha}(\tilde{\Gamma} \cap E_\alpha)$. Clearly, $\Gamma \subset \tilde{\Gamma}$ and we have

$$\mu_t^E(\Gamma) \le \mu_t^E(\tilde{\Gamma}). \tag{3.7}$$

Since $BC(E) \subset BC(E_\alpha)$, it follows from similar arguments as used in Theorem A that $e_\alpha(\Gamma) \supseteq e(\tilde{\Gamma})$. Thus

$$(\lambda_t \circ e_\alpha)(\Gamma) \ge (\lambda_t \circ e)(\tilde{\Gamma}). \tag{3.8}$$

Using these identities and the definition of the measures μ_t^E and μ_t^α, it follows that

$$\mu_t^E(\Gamma) \le \mu_t^E(\tilde{\Gamma}) \le \mu_t^\alpha(\Gamma). \tag{3.9}$$

This gives (3.6) thereby proving the first assertion. In fact this also proves that μ_t^E is μ_t^α null. For the second part we use the fact that λ is a generalized solution of equation (3.1) in the sense of Definition 3.1. We use the expression (3.4) for this. Let $\phi \in D(\mathcal{A})$ with $D\phi$ having bounded support and let $\tilde{\phi}$ denote the extension of ϕ from E_α to βE_α so that $\tilde{\phi} \circ e_\alpha = \phi$. Then

$$\begin{aligned} \lambda_t(\tilde{\phi}) &\equiv \int_{\beta E_\alpha} \tilde{\phi}(\xi)\lambda_t(d\xi) = \int_{E_\alpha} (\tilde{\phi} \circ e_\alpha)(\eta)(\lambda_t \circ e_\alpha)(d\eta) \\ &= \int_{E_\alpha} \phi(\eta)\mu_t^\alpha(d\eta). \end{aligned} \tag{3.10}$$

Similarly we have

$$\lambda_0(\tilde{\phi}) = \int_{E_\alpha} \phi(\eta)\ \mu_0^\alpha(d\eta) = \mu_0^\alpha(\phi). \tag{3.11}$$

Again for any $t \in I$,

$$\begin{aligned} \lambda_t(\tilde{\mathcal{A}}\tilde{\phi}) &\equiv \int_{\beta E_\alpha} (\tilde{\mathcal{A}}\tilde{\phi})(\xi)\lambda_t(d\xi) \\ &= \int_{E_\alpha} \mathcal{A}(\tilde{\phi} \circ e_\alpha)(\eta)\ (\lambda_t \circ e_\alpha)(d\eta) \\ &= \int_{E_\alpha} \mathcal{A}\phi(\eta)\mu_t^\alpha(d\eta). \end{aligned} \tag{3.12}$$

Thus it follows from (3.4) and (3.10)–(3.12) that

$$\mu_t^\alpha(\phi) = \mu_0^\alpha(\phi) + \int_0^t \mu_s^\alpha(\mathcal{A}\phi)ds, \quad \forall t \in I.$$

This proves the second part. □

From this result it is clear that the Definition 3.1 for measure solutions of the evolution equation (3.1) can be rephrased directly in terms of the basic space E_α as follows.

Definition 3.2. A measure valued function $\mu \in L_\infty^w(I, \Sigma_{\mathrm{rba}}(E_\alpha))$ is said to be a generalized solution of equation (3.1) corresponding to the initial state μ_0 if, for every $\phi \in D(\mathcal{A})$ with Frechet derivative $D\phi$ having bounded supports on E_α, the following equality holds

$$\mu_t(\phi) = \mu_0(\phi) + \int_0^t \mu_s(\mathcal{A}\phi)ds, \quad t \in I. \tag{3.13}$$

This is obtained at the cost of countable additivity, though its extension to βE_α is countably additive.

Remark. The measure function $\{\mu_t^E,\ t \in I\}$ cannot be a generalized solution of equation (3.1) unless f is defined on all of E. Here f is a much more general operator mapping $E_\alpha \subset E$ into E.

It is evident from the integral expression (3.13) that one could consider μ as being the weak solution of the abstract linear differential equation

$$(d/dt)\mu = \mathcal{A}^*\mu, \quad \mu(0) = \mu_0, \tag{3.14}$$

on the state space $\Sigma_{\mathrm{rba}}(E_\alpha)$ which is a Banach space with respect to the total variation norm. The interesting fact is, looked at it from this point of view, it is always a linear evolution equation with unbounded operator $\mathcal{A}^*$ with both domain $D(\mathcal{A}^*)$ and range $R(\mathcal{A}^*)$ in $\Sigma_{\mathrm{rba}}(E_\alpha)$. Here $\mathcal{A}^*$ is the dual of the operator $\mathcal{A}$. Since by Theorem B this equation has a w^*-continuous solution, there exists a w^*-continuous semigroup $U(t), t \geq 0$, on $\Sigma_{\mathrm{rba}}(E_\alpha)$ so that the solution is given by

$$\mu_t = U(t)\mu_0, \quad t \geq 0. \tag{3.15}$$

For the time varying case, it is given by $\mu_t = U(t,0)\mu_0$, where $\{U(t,s), 0 \leq s \leq t \leq T\}$ is a weak star continuous evolution operator on $\Sigma_{\mathrm{rba}}(E_\alpha)$.

4. Utility of measure solutions

In view of the remark following Theorem B, one may question, how a measure valued solution can be useful in physical problems. One simple answer is: the measure solutions can be used in the same spirit as the solutions of stochastic differential equations. There one may solve the associated Kolmogorov equation which yields probability measure valued functions as solutions. Another possibility is to construct vector valued, in this case, E_α-valued, trajectories from the measure valued solution $\{\mu_t, t \in I\}$ as follows. Suppose the measure solution has a bounded support in E_α and let $e^* \in E_\alpha^*$. Let E_α^w denote the B-space E_α furnished with the weak topology. Define the process $x \in C(I, E_\alpha^w)$ as follows:

$$(e^*, x(t)) \equiv \int_{E_\alpha} (e^*, \xi)\ \mu_t(d\xi), \quad t \in I,$$

with the initial value $x(0)$ as given in (3.1). Since the map $t \longrightarrow \mu_t$ is only weak star continuous with values in the space of regular bounded finitely additive measures on E_α, the process x is only weakly continuous. In control problems where f depends also on a control variable, the measure valued process is a controlled process [4] and if one wishes to find a control to capture a target, one may force the process x itself to hit the target. By the phrase, capturing a target, here means

$$\bigcup_{t\in I} \{\operatorname{spt} \mu_t \cap \mathcal{T}(t)\} \neq \emptyset,$$

where $\mathcal{T}(t)$ is a moving target (multi function) with values from the class of nonempty closed subsets of the Banach space E_α, continuous on I with respect to the Hausdorff metric. Readers interested in detailed application of measure solutions to control problems are referred to [4], [15].

In case the support of the measure process $\{\mu_t,\ t \in I\}$ is unbounded, one can construct a family of solution trajectories as follows. Take any element $e^* \in E_\alpha^*$ and the open ball $B_r \in E_\alpha$ of radius $r > 0$ with center at the origin and define

$$\varphi_r(\xi) = \begin{cases} (e^*, \xi), & \text{if } \xi \in B_r; \\ (r/\|\xi\|)(e^*, \xi), & \text{otherwise.} \end{cases}$$

Using the function φ_r we may define a sequence of mean trajectories $\{x_r(t),\ t \in I\}$ through the identity,

$$(e^*, x_r(t)) \equiv \int_{E_\alpha} \varphi_r(\xi)\mu_t(d\xi), \quad t \in I.$$

If $x_0 \in B_r \subset E_\alpha$, the process x_r is well defined and lies in the ball $\bar{B}_r$ for all $t \in I$. However the limit process $\lim_{r\to\infty} x_r(t)$, $t \in I$, may leave E_α and escape to βE_α. In any case these trajectories may play the role of vector valued solutions in place of the measure valued solutions.

For application to systems and control problems, one is generally interested in evaluating functionals of the form:

$$\begin{aligned} &F(\mu) \equiv \int_0^\tau G(\mu_t(\phi_1), \mu_t(\phi_2), \ldots, \mu_t(\phi_n))dt, \\ &F(\mu) \equiv G(\mu_\tau(\phi_1), \mu_\tau(\phi_2), \ldots, \mu_\tau(\phi_n)), \\ &F(\mu) \equiv \int_0^\tau \int_{E_\alpha} g(t,\xi)\mu_t(d\xi)dt, \end{aligned} \tag{3.16}$$

where $G \in C(R^n)$, $g \in L_1(I, BC(E_\alpha))$ and the functions $\phi_k \in BC(E_\alpha)$, $k = 1, 2, \ldots, n$, are given. It is the action of the measure solutions on certain observables that matter, not the measure itself. These quantities have physical significance though not easy to compute. One must use some approximation techniques, similar to the Galerkin method, to solve the differential equation (3.14) to compute such functionals. This is left as an open problem for future work.

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