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Petrov's law of the iterated logarithm on simply connected nilpotent Lie groups

By DANIEL NEUENSCHWANDER (Bern)

Abstract. We carry over Petrov's law of the iterated logarithm (for symmetric random variables) to all simply connected positively graded nilpotent Lie groups.

1. Introduction

In 1971, Petrov proved the following theorem (see Petrov (1971), Theorem and remark at the end of his paper; furthermore, denote by $\Phi(x)$ the standard normal distribution function):

Theorem 1. *Let $\{X_n\}_{n\geq 1}$ be independent (real-valued) random variables, $S_n := \sum_{k=1}^n X_k$ and assume $\{B_n\}_{n\geq 1}$ are positive numbers such that $B_n \to \infty$ and $B_{n+1}/B_n \to 1$ $(n \to \infty)$. Assume that*

$$\sup_{x\in\mathbb{R}} |P(S_n < B_n^{1/2}x) - \Phi(x)| = O(\log B_n (\log B_n)^{-1-\delta})$$

for some $\delta > 0$. Then

$$\limsup_{n\to\infty} \frac{S_n}{(2B_n \log\log B_n)^{1/2}} \overset{\text{a.s.}}{=} 1.$$

The same theorem, but under the additional assumption that the X_n are symmetric, was already proved before by Petrov (1968). The aim of this note will be to carry over Theorem 1 for symmetric random variables to all simply connected nilpotent positively graded Lie groups. Simply

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connected nilpotent Lie groups are the groups arising as follows: Consider a skew-symmetric bilinear map $[\,.\,,\,.\,] : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying the Jacobi identity

$$\big[[x,y],z\big] + \big[[y,z],x\big] + \big[[z,x],y\big] = 0 \quad (x,y,z \in \mathbb{R}^d).$$

Then $G \cong \mathbb{R}^d$, equipped with the multiplication given by the Campbell–Hausdorff-formula

$$x \cdot y := x + y + \frac{1}{2}[x,y] + \frac{1}{12}\big\{\big[[x,y],y\big] + \big[[y,x],x\big]\big\} + \ldots \quad (x,y \in G)$$

is a simply connected nilpotent Lie group. If $[\,.\,,\,.\,],\,.\,] \equiv 0$, then G is even called step 2-nilpotent. Of course, the neutral element e is 0 and the inverse element x^{-1} of x is $-x$. The simplest (non-commutative) simply connected (step 2-)nilpotent Lie group is the Heisenberg group $\mathbb{H}$. See Neuenschwander (1996) for an account of probabilistic results on $\mathbb{H}$ and (p. 9) the general form of the Campbell–Hausdorff formula for higher steps of nilpotency. The Heisenberg group has its origin in quantum mechanics, where it can be interpreted as the Lie algebra generated by the location operator, the momentum operator, and the identity operator. Another representation of $\mathbb{H}$ is as group of upper triangular 3×3-matrices with 1's in the diagonal. Also Heisenberg groups of higher order and all groups of type H (introduced by Kaplan (1980) in the context of composition of quadratic forms) are simply connected step 2-nilpotent Lie groups.

2. Petrov's LIL

Let $G \cong \mathbb{R}^d$ be a simply connected step r-nilpotent Lie group. A gradation of G is a vector space decomposition

$$G \cong \mathbb{R}^d \cong \bigoplus_{i=1}^{r} V_i \tag{1}$$

such that

$$[V_i, V_j] \subset V_{i+j} \quad (i,j \in \{1,2,\ldots,r\})$$

(where $V_s := \{0\}$ $(s > r)$). W.l.o.g. we may assume that the underlying basis of $\mathbb{R}^d$ is a so-called Jordan–Hölder basis, i.e. a basis $E = \bigcup_{i=1}^{r} E_i$, where $E_i = \{e_{i,1}, e_{1,2}, \ldots, e_{i,d(i)}\}$ is a basis of V_i ($d(i)$ thus denoting the dimension of V_i). For $x \in G \cong \mathbb{R}^d$ we will write $x = (x_1, x_2, \ldots, x_d)$ with respect to a Jordan–Hölder basis. Denote, for $a > 0$, the dilatation δ_a of G by

$$\begin{aligned}\delta_a(x) &:= (ax_1, ax_2, \ldots, ax_{d(1)}, a^2 x_{d(1)+1}, a^2 x_{d(1)+2}, \ldots, a^2 x_{d(1)+d(2)}, \ldots, \\ &\quad a^r x_{d(1)+d(2)+\cdots+d(r-1)+1}, a^r x_{d(1)+d(2)+\cdots+d(r-1)+2}, \ldots, a^r x_d). \qquad (2)\end{aligned}$$

On the other hand, we define the gauge $|\,.\,|$ on G by

$$\begin{aligned}|x| := \max\{&|x_1|, |x_2|, \ldots, |x_{d(1)}|, |x_{d(1)+1}|^{1/2}, |x_{d(1)+2}|^{1/2}, \ldots, |x_{d(1)+d(2)}|^{1/2}, \\ &\ldots, |x_{d(1)+d(2)+\cdots+d(r-1)+1}|^{1/r}, |x_{d(1)+d(2)+\cdots+d(r-1)+2}|^{1/r}, \ldots, |x_d|^{1/r}\} \\ &(x \in G).\end{aligned}$$

Clearly, $|\delta_a(x)| = a|x|$, so $|\,.\,|$ is what is called a homogeneous gauge. The symbol $|\,.\,|$ will be used for the ordinary modulus function on $\mathbb{R}$, too; it will be clear from the context which one is meant. Now we say what a standard gaussian law on G is. (See Neuenschwander (1996) for complements and details.) A continuous convolution semigroup (c.c.s. for short) on G is a weakly continuous one-parameter family $\{\mu_t\}_{t\geq 0}$ of probability measures on G with the property $\mu_t * \mu_s = \mu_{s+t}$ $(s, t \geq 0)$. It follows automatically that μ_0 is the degenerate probabiliy measure at $0 \in G$. It can be shown that the so-called generating distribution (note that here the word "distribution" is used in the sense of a linear functional, not in the sense of a probability measure)

$$\mathcal{A}(f) := \lim_{t \to 0+} \int_G \big(f(x) - f(0)\big)\mu_t(dx)$$

exists for all bounded real-valued C^∞-functions f on G. In general, such generating distributions on Lie groups have a decomposition corresponding to the classical Lévy–Hinčin formula: They consist of a translation term, a second-order differential operator (which corresponds to the gaussian part), and a term which arises as limit of composed Poisson laws. Now standard Brownian motion on G can be defined as the c.c.s. $\{\gamma_t\}_{t\geq 0}$ whose generating distribution is given by the Kohn Laplacian

$$\sum_{i=1}^{d(1)} \frac{\partial^2}{\partial x_i^2} f(0).$$

For the Heisenberg group $\mathbb{H}$, a more explicit description is possible: Let $\{B_1(t)\}_{t\geq 0}$, $\{B_2(t)\}_{t\geq 0}$ be independent standard real-valued Brownian motions and let $\{A(t)\}_{t\geq 0}$ be Lévy's area process:

$$A(t) := \frac{1}{2}\int_0^t (B_2 dB_1 - B_1 dB_2).$$

In other words, $A(t)$ is the area enclosed by the curve $\{(B_1(s), B_2(s)\}_{0\leq s\leq t}$ and the chord joining $(B_1(t), B_2(t))$ to the origin. Now standard Brownian motion on $\mathbb{H}$ is given by the process

$$\big\{(B_1(t), B_2(t), A(t))\big\}_{t\geq 0}.$$

A G-valued random variable X is called symmetric, if X and $-X$ have the same law. Denote by $\mathcal{J}$ the set of all intervals in $\mathbb{R}^d$. Let $\{\gamma_t\}_{t\geq 0}$ be the standard gaussian c.c.s. on G defined above. Now we may formulate our result:

Theorem 2. *Let G be a simply connected positively graded nilpotent Lie group. Let $\{X_n\}_{n\geq 1}$ be independent symmetric G-valued random variables, $P_n := \prod_{k=1}^n X_k$ and assume $\{a_n\}_{n\geq 1}$ are positive numbers such that $a_n \to \infty$ and $a_{n+1}/a_n \to 1$ $(n\to\infty)$. Assume that*

$$\sup_{J\in\mathcal{J}} |P(\delta_{a_n^{-1/2}}(P_n)\in J) - \gamma_1(J)| = O\big(\log a_n (\log a_n)^{-1-\delta}\big)$$

for some $\delta > 0$. Then

$$\limsup_{n\to\infty} |\delta_{(2a_n \log\log a_n)^{-1/2}}(P_n)| \overset{\text{a.s.}}{=} 1.$$

Proof. The proof is based on the proof of the Theorem in Petrov (1971). We show what has to be changed, generalized and adapted. Let $\{W(t)\}_{t\geq 0}$ be the G-valued process with independent and stationary increments (in the sense of the group multiplication $\cdot$ on G) corresponding to $\{\gamma_t\}_{t\geq 0}$. The main idea is to use estimations for the stochastic intergrals occurring in the development of $W(t)$ (by analogy to the Lévy area integral on $\mathbb{H}$) and on the other hand to observe that (for symmetric random variables X_n on G) from the reflection principle, an analogue of the result of the Lemma in Petrov (1971) holds. In the following, let C

denote a generic constant. Let us explain the details. First, we look for a generalization of line 3 of the proof of the Theorem in PETROV (1971), which tells that

$$1 - \Phi(t) \sim \frac{1}{t}\exp(-t^2/2) \qquad (t \to \infty). \tag{3}$$

Put

$$\chi(n) := (2a_n \log\log a_n)^{1/2}.$$

Thus we have, by the symmetry of the X_n and the exponential upper tail estimate which follows from Theorem 1.2 in BALDI (1986) and Section 4 of the same paper (see also SCHOTT (1981), (1983) for special cases), an analogue of (5) in PETROV (1971), namely

$$P(|\delta_{\chi(n)^{-1}}(P_n) \geq b) = P(|P_n| \geq b\chi(n)) \leq C(\log a_n)^{-b^2} \tag{4}$$

for $0 < b < (1+\delta)^{1/2}$. As in PETROV (1971) it follows that there exists a subsequence $\{n_k\}_{k\geq 1}$ of $\{n\}_{n\geq 1}$ and a $\tau > 0$ such that

$$P(|\delta_{\chi(n)^{-1}}(P_n)| \geq b) \leq C(k\log(1+\tau))^{-b^2} \tag{5}$$

for $0 < b < (1+\delta)^{1/2}$ and k large enough. As in the proof of BERTHUET (1979), Lemma 2 ("maximal lemma") one obtains, by the reflection principle and due to the symmetry of the random variables X_n, for any projection q of $G \cong \mathbb{R}^d$ onto some one-dimensional coordinate subspace,

$$P\Big(\max_{1\leq k\leq n} |q(P_k)| \geq t\Big) \leq CP(|P_n| \geq t) \quad (t \in \mathbb{R}). \tag{6}$$

Now (5) and (6) yield

$$\sum_{k=1}^{\infty} P\Big(\max_{1\leq n\leq n_k} |\delta_{\chi(n)^{-1}}(P_n)| \geq 1+\gamma\Big) < \infty \quad (\gamma > 0). \tag{7}$$

Now by a Borel–Cantelli argument similar to the one on p. 702 in PETROV (1971) it follows that for every $\varepsilon > 0$

$$|\delta_{\chi(n)^{-1}}(P_n)| \leq 1+\varepsilon \quad \text{a.s. for almost all } n. \tag{8}$$

The other direction, namely

$$|\delta_{\chi(n)^{-1}}(P_n)| \geq 1-\varepsilon \quad \text{a.s. for infinitely many } n, \tag{9}$$

follows already by using the classical result on the x_1-component. □

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DANIEL NEUENSCHWANDER
INSTITUT DES SCIENCES ACTUARIELLES
UNIVERSITÉ DE LAUSANNE
CH–1015 LAUSANNE
AND
INSTITUT FÜR MATHEMATISCHE STATISTIK UND VERSICHERUNGSLEHRE
UNIVERSITÄT BERN
SIDLERSTRASSE 5
CH–3012 BERN
AND
FEDERAL SECTION OF CRYPTOLOGY
MINISTERY OF DEFENSE
CH–3003 BERN
SWITZERLAND

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