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On a generalized functional equation of Abel

By MACIEJ SABLIK (Katowice)

Abstract. We present some results concerning the following generalization of a functional equation of Abel

$$\psi\left(xf(y) + yg(x)\right) = \varphi(x) + \varphi(y).$$

With $f = g$ we get the original Abel's equation that was mentioned explixitly by D. Hilbert in the second part of his fifth problem. The present generalization implies many applications in the theory of functional equations, particularly those dealing with determination of parametrized subsemigroups. We solve the equation in the class of continuous real functions defined in an interval containing 0.

1. Introduction

In the second part of his fifth problem D. Hilbert (cf. [13]) dealt with functional equations, *usually investigated only under the assumption of the differentiability of functions involved*, and asked the following: *In how far are the assertions which we can make in the case of differentiable functions true under proper modifications without this assumption?* In particular, Hilbert mentioned explicitly the following equation

$$\psi\left(xf(y) + yf(x)\right) = \varphi(x) + \varphi(y) \tag{A}$$

which was considered by N. Abel (cf. [1]). Hilbert's question was recalled by J. Aczél during the Twenty-fifth International Symposium on Functional Equations in 1987 (see [2] and [3]). In our papers [15], [17]

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and [18] we gave a complete answer to Hilbert's question, as it was posed. Namely we determined all triples (ψ, f, φ) of real-valued continuous functions satisfying (1) for all x, y from a real interval. We have also shown that continuity assumption can be relaxed.

2. Generalized equation of Abel

In the present paper we shall consider the functional equation

$$\psi(xf(y) + yg(x)) = \varphi(x) + \varphi(y) \tag{2.1}$$

which at first glance is not that much different from the original equation of Abel but it has applications hardly available if we assume $f = g$.

All the functions ψ, f, g and φ are unknown. We assume that f, g and φ map a real interval I (also to be determined) into $\mathbb{R}$. We shall also admit that $0 \in I$. Let us define functions $A_{f,g} : I \times I \to \mathbb{R}$ and $B_{f,g} : I \to \mathbb{R}$ by

$$A_{f,g}(x, y) = xf(y) + yg(x)$$

and

$$B_{f,g}(x) = A_{f,g}(x, x).$$

Of course the function ψ is supposed to be defined in $A_{f,g}(I \times I)$. In the sequel we shall determine all quadruples (ψ, f, g, φ) of continuous functions which satisfy (2.1).

Let us begin with some simple observations which lead to a reduction of the problem. Suppose that (ψ, f, g, φ) is a quadruple of continuous functions satisfying (2.1). If we put $x = y = 0$ into (2.1) we see that $\psi(0) = 2\varphi(0)$. Consequently, putting $y = 0$ into (2.1) we get the following formula for φ

$$\varphi(x) = \psi(xf(0)) - \tfrac{1}{2}\psi(0).$$

If we define ψ_1 by $\psi_1(u) = \psi(u) - \psi(0)$ then we easily check that $\psi_1(0) = 0$ and

$$\psi_1(A_{f,g}(x, y)) = \psi_1(xf(0)) + \psi_1(yf(0)). \tag{2.2}$$

To solve (2.2) let us consider the case $f(0) = 0$ first. Then $\psi_1 = 0$, which means that ψ is constant. The functions f and g may be arbitrary (except that $f(0)=0$), and $\varphi=\frac{1}{2}\psi$. An analogous argument shows that if $g(0)=0$ then ψ is constant, f and g may be arbitrary (with $g(0)=0$), and $\varphi=\frac{1}{2}\psi$.

In what follows we shall assume therefore that $f(0)\neq 0\neq g(0)$. Let us define $F,G: I\to\mathbb{R}$ by

$$F(x)=\frac{f(x)}{f(0)},\quad \text{and}\quad G(x)=\frac{g(x)}{f(0)},$$

and $\Psi: A_{F,G}(I\times I)\to\mathbb{R}$ by

$$\Psi(u)=\psi_1(uf(0)).$$

Let us note that if $u=xF(y)+yG(x)$ for some $x,y\in I$ then $uf(0)\in A_{f,g}(I\times I)$ which means that the above definition of Ψ is correct. From (2.2) we derive the following equation

$$\Psi(xF(y)+yG(x))=\Psi(x)+\Psi(y) \tag{2.3}$$

for functions Ψ, F and G. Moreover, it is sufficient to look for solutions of (2.3) satisfying

$$\Psi(0)=0,\quad F(0)=1 \text{ and } G(0)\neq 0. \tag{$*$}$$

The above condition which implies in particular that

$$I=A_{F,G}(I\times\{0\})\subset A_{F,G}(I\times I),$$

will be assumed in sequel.

Further consideration is divided into two parts. The first one deals with the situation where Ψ does not vanish outside zero, the second one treats the remaining case in which some nondifferentiable solutions of (2.3), and hence of (2.1), come up.

A. Let us assume that $\Psi(u)\neq 0$ if $u\neq 0$. Under this assumption we get the following

Lemma 2.1. *If (Ψ, F, G) is a continuous solution of* (2.3) *then*

(i) $F(x) + G(x) \neq 1$ *for every* $x \in I$,

(ii) $F(x) + G(x) \neq 0$ *for every* $x \in I \setminus \{0\}$.

Proof. If $F(x) + G(x) = 1$ for some $x \in I$ then in view of (2.3) we have $\Psi(x) = 0$, and hence $x = 0$. But $F(0) + G(0) = 1 + G(0) \neq 1$. Next, if $F(x) + G(x) = 0$ for some $x \in I$ then (2.3) implies $\Psi(x) = 0$ and hence $x = 0$. $\square$

In view of the above lemma only the following three cases can occur if $I_+ \neq \emptyset$ ($I_- \neq \emptyset$)

α) $F(x) + G(x) \in (0, 1)$ for every $x \in I_+$ ($x \in I_-$);

β) $F(x) + G(x) < 0$ for every $x \in I_+$ ($x \in I_-$);

γ) $F(x) + G(x) > 1$ for every $x \in I_+$ ($x \in I_-$).

Let us show that neither α) nor β) is posssible. We will show it for I_+, the remaining case is analogous. If α) holds then

$$0 < B_{F,G}(x) < x, \quad x \in I_+. \tag{2.4}$$

From (2.3) we derive

$$\Psi(x) = \frac{1}{2}\Psi(B_{F,G}(x))$$

for every $x \in I_+$ whence by induction

$$\Psi(x) = \frac{1}{2^n}\Psi(B^n_{F,G}(x))$$

for every $n \in \mathbb{N}$ and $x \in I_+$. Hence by (2.4) and continuity of Ψ we get $\Psi(x) = 0$, contrary to our general assumption in the present case.

Similarly, if β) holds then for every $x \in I_+$ we get

$$A_{F,G}(x, 0) = x > 0 > A_{F,G}(x, x)$$

whence $A_{F,G}(x, y) = 0$ for a $y \in (0, x)$. Applying (2.3) we get

$$\Psi(x) + \Psi(y) = \Psi(0) = 0$$

which implies that Ψ changes the sign in I_+, again impossible in the case **A.**

Thus γ) holds in I_+ and I_- which together with continuity of F and G yields

$$F(x) + G(x) > 1 \tag{2.5}$$

for every $x \in I$ which in particular implies that

$$\frac{B_{F,G}(x)}{x} > 1 \quad \text{for every } x \neq 0. \tag{2.6}$$

Let us prove now

Lemma 2.2. *If* $I_+ \neq \emptyset \neq I_-$ *then* $\operatorname{sgn} \Psi \mid I_+ \neq \operatorname{sgn} \Psi \mid I_-$.

Proof. Since $A_{F,G}$ is continuous and vanishes at $(0,0)$ there exists an interval $J \subset I$ such that $0 \in J$ and $A_{F,G}(J \times J) \subset I$. Let $x \in J_-$ and $y \in J_+$. Suppose that $A_{F,G}(x,y) > 0$. Then in view of (2.6) we get $A_{F,G}(x,z) = 0$ for a $z \in (x,y)$. Similarly, if $A_{F,G}(x,y) < 0$ then $A_{F,G}(z,y) = 0$ for some $z \in (x,y)$. In other words, for every $u \in J \setminus \{0\}$ there exists a $z \in J$ such that $A_{F,G}(u,z) = 0$ or $A_{F,G}(z,u) = 0$. This in view of (2.3) means that for every $u \in J \setminus \{0\}$ there exists a $z \in J$ such that

$$\Psi(u) + \Psi(z) = 0$$

which means that Ψ cannot be of constant sign. Its continuity implies in the present case that $\operatorname{sgn} \Psi \mid J_+ \neq \operatorname{sgn} \Psi \mid J_-$. □

Now we can prove the invertibility of Ψ.

Proposition 2.3. *If* (Ψ, F, G) *is a continuous solution of* (2.3) *satisfying* $(*)$ *and* $\Psi(x) \neq 0$ *for* $x \neq 0$ *then* $\Psi \mid I$ *is strictly monotonic.*

Proof. Without loss of generality let us assume (cf. Lemma 2.2) that $\Psi \mid I_+ > 0 > \Psi \mid I_-$. Fix an $x \in I_+$ and let $z \in (x, B_{F,G}(x))$ (cf. (2.6)). Then

$$A_{F,G}(x,0) = x < z < A_{F,G}(x,x)$$

whence $z = A_{F,G}(x,y)$ for a $y \in (0,x)$. Hence by (2.3)

$$\Psi(z) = \Psi(x) + \Psi(y) > \Psi(x).$$

Continuity of Ψ implies now its strict monotonicity in I_+. Similarly one can prove that $\Psi \mid I_-$ is strictly increasing, which concludes the proof. □

Corollary 2.4. *Under the assumptions of Proposition 2.3, $G(0) = 1$.*

Proof. Let $y \in I \setminus \{0\}$ be such that $yG(0) \in I$. In view of (2.3) we get $\Psi(yG(0)) = \Psi(y)$, whence by Proposition 2.3 we infer that $yG(0) = y$ and thus $G(0) = 1$. □

The next step is to show cancellativity of the operation $A_{F,G}$. We have

Proposition 2.5. *If (Ψ, F, G) is a continuous solution of (2.3) satisfying $(*)$ and $\Psi(x) \neq 0$ for $x \neq 0$ then*

(i) *For every $x \in I$ the functions $A_{F,G}(x, \cdot)$ and $A_{F,G}(\cdot, x)$ are strictly increasing, and hence $B_{F,G}$ is strictly increasing;*

(ii) $A_{F,G}(I \times I) = B_{F,G}(I)$.

Proof. Fix an $x \in I$ and let $y, y' \in I$ be different. Then in view of Proposition 2.3 and (2.3) we get

$$\Psi(A_{F,G}(x, y)) \neq \Psi(A_{F,G}(x, y'))$$

whence $A_{F,G}(x, y) \neq A_{F,G}(x, y')$ which implies that $A_{F,G}(x, \cdot)$ is invertible. Since for every $x \in I_+$ we have $A_{F,G}(x, 0) = x < A_{F,G}(x, x)$, it follows that $A_{F,G}(x, \cdot)$ is strictly increasing. Similarly, using Corollary 2.4, we show that $A_{F,G}(\cdot, x)$ is strictly increasing.

To prove (ii) let us observe that obviously $B_{F,G}(I) \subset A_{F,G}(I \times I)$. On the other hand, just established monotonicity implies for every $x, y \in I$ that

$$B_{F,G}(\min(x, y)) \leq A_{F,G}(x, y) \leq B_{F,G}(\max(x, y)),$$

which shows that $A_{F,G}(I \times I) \subset B_{F,G}(I)$ and concludes the proof. □

Corollary 2.6. *Under the assumptions of Proposition 2.5, Ψ is invertible.*

Proof. Let $u, u' \in A_{F,G}(I \times I)$ be different. By Proposition 2.5 we can find $x, x' \in I$ such that $u = B_{F,G}(x)$ and $u' = B_{F,G}(x')$. Now, using (2.3) and Proposition 2.3 we get

$$\Psi(u) = 2\Psi(x) \neq 2\Psi(x') = \Psi(u'),$$

which proves invertibility of Ψ. □

The next step is to show that F and G cannot be quite independent, if they solve (2.3). Indeed, we have

Proposition 2.7. *If* (Ψ, F, G) *is a continuous solution of* (2.3) *which satisfies* $(*)$ *and* $\Psi(x) \neq 0$ *for* $x \neq 0$ *then there exists a* $c \in \mathbb{R}$ *such that*

$$G(x) = F(x) + cx \tag{2.7}$$

for every $x \in I$.

Proof. With the invertibility of Ψ (cf. Corollary 2.6) we get from (2.3)

$$A_{F,G}(x,y) = \Psi^{-1}(\Psi(x) + \Psi(y)) = A_{F,G}(y,x),$$

for every $x, y \in I$ which implies that

$$\frac{G(y) - F(y)}{y} = \frac{G(x) - F(x)}{x}$$

for every $x, y \in I \setminus \{0\}$. In other words, the function $I \setminus \{0\} \ni y \to \frac{G(y)-F(y)}{y}$ is constant, which together with $F(0) = G(0) = 1$ (cf. $(*)$ and Corollary 2.4) yields the assertion. $\square$

Let us observe that defining $H : I \to \mathbb{R}$ by

$$H(x) = F(x) + \frac{c}{2}x \tag{2.8}$$

we get in view of (2.7)

$$A_{F,G}(x,y) = xF(y) + y(F(x) + cx) = xH(y) + yH(x) = A_{H,H}(x,y).$$

Thus we get the following

Proposition 2.8. *If a triple* (Ψ, F, G) *is a continuous solution of* (2.3) *satisfying* $(*)$ *and* $\Psi(u) \neq 0$ *for* $u \neq 0$ *then* Ψ *is invertible and the couple* (Ψ, H), *where* H *is defined by* (2.8) *for some* $c \in \mathbb{R}$, *is a continuous solution of*

$$\Psi(xH(y) + yH(x)) = \Psi(x) + \Psi(y) \tag{2.9}$$

satisfying

$$H(0) = 1. \tag{2.10}$$

Conversely, if a couple (Ψ, H) *is a continuous solution of* (2.9) *satisfying* (2.10) *and* $\Psi(u) \neq 0$ *for* $u \neq 0$ *then the triple* (Ψ, F, G) *where* F *and* G *are given by*

$$F(x) = H(x) - \frac{c}{2}x \quad \text{and} \quad G(x) = H(x) + \frac{c}{2}x \tag{2.11}$$

for some $c \in \mathbb{R}$, *is a continuous solution of* (2.3), *satisfying* $(*)$.

Thus the problem of solving (2.3) (and hence (2.1)) in the present case is reduced to the problem considered in [15]. More exactly, by the above Proposition we get invertibility of Ψ and hence by (2.9) the operation $A_{H,H}$ is locally associative. Therefore (and because of (2.10)) we can apply [16; Proposition 1] to determine H. It suffices to determine Ψ. Inserting formulae given in [16; Proposition 1] to the equation (2.9) we obtain for Ψ different equations of Cauchy type on restricted domain which can be easily solved using the metods from previous chapters (cf. also [15]). Since we work under assumption that Ψ vanishes uniquely at 0 some additional restrictions have to be imposed on I. Summarizing we obtain the following

Proposition 2.9. *If a triple* (Ψ, F, G) *is a continuous solution of* (2.3) *satisfying* $(*)$ *and* $\Psi(u) \neq 0$ *for* $u \neq 0$ *then* F *and* G *are given by* (2.11) *and* (Ψ, H) *and* I *satisfy one of the following*

- (R_1) $H(x) = 1$, I *is arbitrary and* $\Psi(u) = Du$, *where* $D \neq 0$ *is an arbitrary constant;*
- (R_2) $H(x) = 1 + ax$, $I \subset \frac{1}{2a}(-1, \infty)$, $\Psi(u) = D\ln(2au+1)$, *where* a, D *are arbitrary constants different from* 0*;*
- (R_3) $H(x) = \gamma^{-1}(\frac{x}{E})$, $I \subset E[-\frac{1}{2\sqrt{2}}, \infty)$, $\Psi(u) = d\ln\gamma^{-1}(\frac{u}{E})$, *where* d, E *are arbitrary constants different from* 0 *and* $\gamma : [\frac{1}{e}, \infty) \to \mathbb{R}$ *is defined by* $\gamma(u) = u\ln u$*;*
- (R_4) $H(x) = g_\alpha^{-1}(ax) + ax$, $\Psi(u) = d\ln g_\alpha^{-1}(au)$ *where* $a \neq 0 \neq d$ *and* $\alpha \neq 1$ *are arbitrary constants,* $g_\alpha : K_\alpha \to \mathbb{R}$ *is given by* $g_\alpha(u) = \frac{u^\alpha - u}{2}$,

$$K_\alpha = \begin{cases} (0, \infty), & \text{if } \alpha \leq 0, \\ \left(\alpha^{1/(1-\alpha)}, \infty\right), & \text{if } \alpha \in (0, \infty) \setminus \{1\}, \end{cases}$$

and the interval I is contained in I_α where

$$I_\alpha = \begin{cases} \mathbb{R}, & \textit{if } \alpha < 0, \\ \dfrac{1}{a}\left(-\infty, \dfrac{1}{2}\right), & \textit{if } \alpha = 0, \\ \left(\dfrac{1}{a}\right) g_\alpha\left(\alpha^{1/2(1-\alpha)}\right)(-\infty, 1], & \textit{if } \alpha \in (0,\infty)\setminus\{1\}; \end{cases}$$

$$(R_5) \qquad H(x) = \begin{cases} ax/r_D^{-1}(x), & \textit{if } x \in I\setminus\{0\}, \\ 1, & \textit{if } x = 0, \end{cases}$$

$\Psi(u) = C \arctan r_D^{-1}(u)$, *if* $|(B_H \circ r_D)^{-1}(u)| \le 1$, *and*

$$\Psi(u) = C\left((\operatorname{sgn} L)\pi + \arctan l_D^{-1}(-u/\exp(|D|\pi)\right),$$

if $|(B_H \circ r_D)^{-1}(u)| > 1$. *Here* $a \ne 0 \ne C$ *and* $D \in \mathbb{R}$ *are some constants, and* $r_D = s_D \mid R_D$, $l_D = s_D \mid L_D$, $R_0 = L_0 = \mathbb{R}$ *and* $L_D = -((2/D) + R_D) = (1/D)(-\infty, 1]$ *and* $s_D : \mathbb{R} \to \mathbb{R}$ *is a function given by*

$$s_D(u) = \frac{1}{a}\frac{u}{\sqrt{1+u^2}}\exp(D\arctan u).$$

Moreover, I *is contained in* $r_D([D - \sqrt{1+D^2}, D + \sqrt{1+D^2}])$. *We adopt here the convention* $\arctan(\pm\infty) = \pm\frac{\pi}{2}$ *and*

$$r_D^{-1}(\pm(1/a)\exp(\pm D\pi/2)) = \pm\infty.$$

Let us procede now to the case

B. $\Psi(u) = 0$ for some $u \ne 0$.

Define the set Z by $Z = \Psi^{-1}(\{0\}) \cap I$. We have the following

Lemma 2.10. *If* $Z_+ \ne \emptyset$ *then there exists a* $b > 0$ *such that* $\Psi \mid [0,b] = 0$, *and if* $Z_- \ne \emptyset$ *then there exists an* $a < 0$ *such that* $\Psi|[a,0] = 0$.

Proof. We prove the first statement, the second one may be dealt with in an analogous way. Suppose that the first statement is false. Then without loss of generality we can admit that there exists a sequence $(y_n)_{n\in\mathbb{N}} \in (I_+)^{\mathbb{N}}$, converging to 0 and such that $\Psi(y_n) > 0$ for every $n \in \mathbb{N}$.

Let $x \in Z_+$, our supposition implies that the open set $[0, x] \setminus Z_+ \neq \emptyset$. Let (u, v) be its component such that $\Psi \mid (u, v) > 0$. Since $\Psi(u) = \Psi(v) = 0$, $\Psi|[u, v]$ admits its maximum at a $t \in (u, v)$. Continuity of F and G implies together with $(*)$ that

$$\lim_{n\to\infty} A_{F,G}(t, y_n) = \lim_{n\to\infty} (tF(y_n) + y_n G(t)) = t,$$

whence there exists an $N \in \mathbb{N}$ such that $A_{F,G}(t, y_N) \in (u, v)$. By choice of t, y_N and (2.3) we get

$$\Psi(t) \geq \Psi(A_{F,G}(t, y_N) = \Psi(t) + \Psi(y_N) > \Psi(t).$$

This contradiction concludes the proof. $\square$

Let us prove now

Lemma 2.11. *If $0 \in \operatorname{Int} I$ then there exist an $a < 0$ and $b > 0$ such that $\Psi \mid [a, b] = 0$.*

Proof. Assume that $Z_+ \neq \emptyset$, the other case may be treated analogously. By Lemma 2.9 we have $\Psi|[0, b] = 0$ for some $b > 0$. By continuity of F and G and because of $(*)$ we get

$$\lim_{y\to 0-} A_{F,G}\left(\frac{b}{2}, y\right) = \frac{b}{2} \in (0, b).$$

Thus there exists an $a < 0$ such that

$$A_{F,G}\left(\frac{b}{2}, y\right) \in [0, b],$$

for every $y \in [a, 0]$ which implies by (2.3)

$$\Psi(y) = \Psi\left(A_{F,G}\left(\frac{b}{2}, y\right)\right) - \Psi\left(\frac{b}{2}\right) = 0,$$

for every $y \in [a, 0]$. This concludes the proof. $\square$

Put $M := \sup\{b \in \overline{I}_+ : \Psi|[0, b] = 0\}$ and $m := \inf\{a \in \overline{I}_- : \Psi \mid [a, 0] = 0\}$. In view of Lemma 2.9 we have $M > 0$ if $I_+ \neq \emptyset$, and $m < 0$, if $I_- \neq \emptyset$. The following three cases are possible

(C1) $I \subset [m, M]$

(C2) $M < \sup \overline{I}_+$;

(C3) $m < \inf \overline{I}_-$.

It is clear in view of (2.3) that $\Psi = 0$ in the case (C1), and the functions F and G may be arbitrary.

In the case (C2) denote $I_1 := [m, M] \cap I$, and $L_x := A_{F,G}(\{x\} \times I_1)$, $R_x := A_{F,G}(I_1 \times \{x\})$, for every $x \in I$. Obviously, I_1, L_x and R_x are intervals. Taking into account definitions of m and M we get from (2.3)

$$\Psi(z) = \Psi(x) \tag{2.11}$$

for every $x \in I$ and $z \in L_x \cup R_x$. We also have $x = A_{F,G}(x, 0) \in L_x$. Let $\delta > 0$ be such that $[M, M+\delta] \subset I$. Then in particular

$$(M, M+\delta) \subset \bigcup_{x \in (M, M+\delta)} L_x.$$

By the definition of M, $\Psi((M, M+\delta))$ is a nondegenerate interval. Therefore there exists an $x \in (M, M+\delta)$ such that $L_x = \{x\}$ for otherwise $L_x \cap (M, M+\delta)$ would be a nondegenerate interval on which Ψ is constant for every $x \in (M, M+\delta)$ (cf. (2.11)). This however would imply that $\Psi((M, M+\delta))$ is a countable set, and therefore $\Psi \mid (M, M+\delta)$ would be constant, a contradiction. It follows from the above that we can pick up a sequence $(x_n)_{n \in \mathbb{N}}$ of numbers in (M, ∞) which converges to M and satisfies

$$A_{F,G}(\{x_n\} \times I_1) = \{x_n\}, \quad n \in \mathbb{N}.$$

This means that for every $n \in \mathbb{N}$ and $y \in I_1$ the equality

$$x_n F(y) + y G(x_n) = x_n$$

holds, or

$$F(y) = -\frac{G(x_n)}{x_n} y + 1,$$

for every $n \in \mathbb{N}$ and $y \in I_1$. Letting $n \to \infty$ we obtain

$$F(y) = -\frac{G(M)}{M} y + 1 \tag{2.12}$$

for every $y \in I_1$ whence in particular

$$F(M) = -G(M) + 1. \tag{2.13}$$

Let us show now that

$$G(0) = 1. \tag{2.14}$$

Suppose that (2.14) is not true. Let us consider three possible cases.

(a) $G(0) \leq 0$. Then $A_{F,G}(0, y) = G(0)y \leq 0$ for every $y \in I_+$. From (2.13) we get $A_{F,G}(M, M) = M > 0$, whence there exists a $\delta > 0$ such that $A_{F,G}(M, y) > 0$ for every $y \in (M, M + \delta)$. Thus for every $y \in (M, M + \delta)$ there exists a $t(y) \in [0, M)$ such that $A_{F,G}(t(y), y) = 0$, whence

$$\Psi(y) = \Psi(0) - \Psi(t(y)) = 0,$$

which contradicts the definition of M.

(b) $0 < G(0) < 1$. Then for every $y \in (M, \frac{M}{G(0)}) \cap I$ we get $G(0)y \in (0, M)$ whence $\Psi(y) = \Psi(G(0)y) - \Psi(0) = 0$ by (2.3), which again contradicts the definition of M.

(c) $1 < G(0)$. Then for every $u \in (M, G(0)M) \cap I$ we have $u = G(0)y$ for some $y \in (0, M)$ which in view of (2.3) implies $\Psi(u) = \Psi(G(0)y) = \Psi(0) + \Psi(y) = 0$, again a contradiction.

We have dismissed other possibilities and hence (2.14) holds. We can now repeat the argument we used before to get (2.12) but reversing the roles of F and G. We obtain

$$G(y) = -\frac{F(M)}{M}y + 1 \tag{2.15}$$

for every $y \in I_1$. In view (2.12) and (2.15) we obtain for every $x \in (M, \infty) \cap I$

$$L_x = \{x\} \Leftrightarrow G(x) = \frac{G(M)}{M}x \quad \text{and} \quad R_x = \{x\} \Leftrightarrow F(x) = \frac{F(M)}{M}x.$$

We are going to show that actually $L_x = R_x = \{x\}$ for every $x \in (M, \infty) \cap I$. Let us define

$$W = \Big\{x \in [M, \infty) \cap I : F(x) = \frac{F(M)}{M}x \text{ and } G(x) = \frac{G(M)}{M}x\Big\}.$$

W is a closed subset of $[M, \infty) \cap I$. Suppose that $W' = [M, \infty) \cap I \setminus W \neq \emptyset$ and let (s, t) be a component of W'. Then for every $x \in (s, t)$ either L_x or R_x is nondegenerate. It follows that for every $x \in (s, t)$ there exists a nondegenerate interval J_x with $x \in J_x$ such that $\Psi \mid J_x$ is constant. In view of continuity of Ψ this is possible only if $\Psi \mid (s,t)$ is constant. Thus Ψ is constant on each component of W'. Let (s,t) be a fixed component of W'. Then $s,t \in W$. For every $y \in [M, \frac{t}{s}M] \cap W$ we have in view of (2.13) and the definition of W

$$A_{F,G}(s,y) = \frac{sy}{M} \in [s,t].$$

Hence, because $\Psi|[s,t]$ is constant, we obtain by (2.3)

$$\Psi(y) = \Psi(A_{F,G}(s,y)) - \Psi(s) = \Psi\left(\frac{sy}{M}\right) - \Psi(s) = 0.$$

Thus $\Psi \mid [M, \frac{s}{t}M] \cap W = 0$, and since Ψ is constant on each component of W' we get $\Psi \mid [M, \frac{s}{t}M] = 0$, contrary to the definition of M. It follows that $W' = \emptyset$. Summarizing we have

$$F(x) = \begin{cases} -\dfrac{1-C}{M}x + 1, & \text{if } x \in [m,M] \cap I \\ \dfrac{C}{M}x, & \text{if } x \in (M,\infty) \cap I, \end{cases} \tag{2.16}$$

and

$$G(x) = \begin{cases} -\dfrac{C}{M}x + 1, & \text{if } x \in [m,M] \cap I \\ \dfrac{1-C}{M}x, & \text{if } x \in (M,\infty) \cap I, \end{cases} \tag{2.17}$$

where C is a real constant.

To end the study of the case (C2) let us prove that $m = \inf \overline{I}_-$. Suppose that $m > \inf \overline{I}_-$. This can only happen if $I_- \neq \emptyset$ and then $m < 0$. In view of (2.16) and (2.17) we get

$$A_{F,G}(m,m) = m(F(m) + G(m)) = m\left(2 - \frac{m}{M}\right) < 2m < m.$$

Hence there exist some $s,t \in \mathbb{R}$ such that $s < m < 0 \le t$ and

$$A_{F,G}([m,0] \times [m,0]) = [s,t].$$

Take any $z \in [s,0]$. Then $z = A_{F,G}(x,y)$ for some $x,y \in [m,0]$ which implies by (2.3) and the definition of m that $\Psi(z) = 0$. Thus Ψ vanishes in $[s,0]$ which contradicts the definition of m. In other words, we have proved that in the case (C2) function F and G are given by

$$F(x) = \begin{cases} -\dfrac{1-C}{M}x+1, & \text{if } x \in (-\infty, M] \cap I \\ \dfrac{C}{M}x, & \text{if } x \in (M,\infty) \cap I, \end{cases} \tag{2.18}$$

and

$$G(x) = \begin{cases} -\dfrac{C}{M}x+1, & \text{if } x \in (-\infty, M] \cap I \\ \dfrac{1-C}{M}x, & \text{if } x \in (M,\infty) \cap I, \end{cases} \tag{2.19}$$

where C is a real constant. Let us observe that $A_{F,G}$ is now given by formulae

$$\begin{aligned} &A_{F,G}(x,y) \\ &= \begin{cases} M\left(1-\left(1-\dfrac{x}{M}\right)\left(1-\dfrac{y}{M}\right)\right), & \text{if } x,y \in (-\infty, M] \cap I, \\ \dfrac{xy}{M}, & \text{if } x,y \in (M,\infty) \cap I, \\ \max(x,y), & \text{otherwise.} \end{cases} \end{aligned} \tag{2.20}$$

By the definition of M we know that $\Psi \mid (-\infty, M] \cap I = 0$, and hence it easily follows from (2.20) that $\Psi \mid (-\infty, M] \cap A_{F,G}(I \times I) = 0$. Using (2.20) and (2.3) we get also

$$\Psi\left(\frac{xy}{M}\right) = \Psi(x) + \Psi(y)$$

for every $x, y \in (M, \infty) \cap I$, whence it follows that for every $u \in (M, \infty)$

$$\Psi(u) = d \ln \frac{u}{M}, \tag{2.21}$$

where $d \neq 0$ is a constant. Using (2.20) again, we infer that (2.21) holds for every $u \in (M, \infty) \cap I \times I$.

To manage the case (C3), let us observe that a triple (Ψ, F, G) solves (2.3) for all $x, y \in I$ if and only if the triple (Ψ^*, F^*, G^*) given by

$$\Psi^*(u) = \Psi(-u), \quad F^*(x) = F(-x), \quad G^*(x) = G(-x)$$

solves (2.3) for all $x, y \in -I$. It is also obvious that the case (C3) for the triple (Ψ, F, G) is equivalent to the case (C2) for (Ψ^*, F^*, G^*). Thus in the case (C3) functions F and G are given by

$$F(x) = \begin{cases} -\dfrac{1-C}{M}x + 1, & \text{if } x \in [m, \infty) \cap I \\ \dfrac{C}{M}x, & \text{if } x \in (-\infty, m) \cap I, \end{cases} \tag{2.22}$$

and

$$G(x) = \begin{cases} -\dfrac{C}{M}x + 1, & \text{if } x \in [m, \infty) \cap I \\ \dfrac{1-C}{M}x, & \text{if } x \in (-\infty, m) \cap I, \end{cases} \tag{2.23}$$

where C is a real constant. Ψ is now given by

$$\Psi(u) = \begin{cases} d \ln \dfrac{u}{m}, & \text{if } u \in A_{F,G}(I \times I) \cap (-\infty, m) \\ 0, & \text{if } u \in A_{F,G}(I \times I) \cap [m, \infty) \end{cases} \tag{2.24}$$

where $d \neq 0$ is a real constant.

A simple calculation shows that formulae for cases (C2) and (C3) may be written jointly, and we get the following

Proposition 2.12. *A triple* (Ψ, F, G) *is a continuous solution of* (2.3) *satisfying* $(*)$ *and* $\Psi(u) = 0$ *for some* $u \neq 0$ *if and only if either* $\Psi = 0$ *and* F *and* G *are arbitrary or there exist a* $p \in I \setminus \{0\}$ *and real constants* C *and* $d \neq 0$ *such that*

$$F(x) = \max\left(-\frac{1-C}{p}x + 1, \frac{C}{p}x\right), \tag{2.25}$$

$$G(x) = \max\left(-\frac{C}{p}x + 1, \frac{1-C}{p}x\right), \tag{2.26}$$

and

$$\Psi(u) = \begin{cases} d \ln \dfrac{u}{p}, & \textit{if } u \in A_{F,G}(I \times I) \cap p(1, \infty) \\ 0, & \textit{if } u \in A_{F,G}(I \times I) \cap p(-\infty, 1]. \end{cases} \tag{2.27}$$

Let us summarize the results of the present section in the following theorem which is the main result of the present paper.

Theorem 2.13. *A quadruple* (ψ, f, g, φ) *of continuous functions* f, g, φ: $I \to \mathbb{R}$, $\psi : A_{f,g}(I \times I) \to \mathbb{R}$, $0 \in I$, *is a solution of* (2.1) *if and only if one of the following cases occurs:*

I. I *is arbitrary interval containing* 0, $\psi = \text{const}$, $\varphi = \frac{1}{2}\psi$, f *and* g *are arbitrary.*

II. *There exist real constants* $a \neq 0$ *and* b *such that*

$$\psi(v) = \Psi\left(\frac{v}{a}\right) + b \qquad f(x) = aF(x)$$

$$g(x) = aG(x) \qquad \varphi(x) = \Psi(x) + \frac{1}{2}b$$

where the triple (Ψ, F, G) *is defined by* (2.11) *and* $(R_1), \dots, (R_5)$ *from Proposition 2.9, or by* (2.25), (2.26) *and* (2.27) *from Proposition 2.12.*

3. Final remarks

Results of the previous sections may be used now to deal with some other functional equations that have been known in the literature but not associated with the original equation of Abel from 1827, i.e.

$$\psi(xf(y) + yf(x)) = \varphi(x) + \varphi(y). \tag{3.1}$$

(Of course, the above equation is a particular case ($f = g$) of the equation (2.1).) One of them is the Gołąb–Schinzel functional equation

$$f(x + yf(x)) = f(x)f(y). \tag{3.2}$$

Another equation motivated by the search for one-parameter subgroups is

$$f(xf(y) + yf(x)) = tf(x)f(y), \tag{3.3}$$

where t is a parameter.

Also some generalizations of (3.3) have been investigated (cf. e.g. [4]–[10], [14], [20]), namely

$$f(xf(y)^k + yf(x)^n) = tf(x)f(y), \tag{3.4}$$

where k and n are nonnegative integers. Recently a further generalization of (3.4) has been considered by J. Chudziak (see [11])

$$f\big(x\phi[f(y)] + y\psi[f(x)]\big) = f(x)f(y). \tag{3.5}$$

The equation (3.4) with $k = n = t = 1$ becomes the equation

$$f(xf(y) + yf(x)) = f(x)f(y), \tag{3.6}$$

which is the equation defining *antiderivations* in the case where f is a bounded linear operator defined in an algebra. Together with the equation

$$f(xf(y) + yf(x) - xy) = f(x)f(y), \tag{3.7}$$

they are special cases of Baxter equation

$$f(xf(y) + yf(x) - cxy) = f(x)f(y). \tag{3.8}$$

Let us observe that (3.2)–(3.8) are special cases of

$$\Phi(xF(y) + yG(x)) = \Gamma(x)\Gamma(y). \tag{3.9}$$

Now, the equation (3.9) is more general than our equation (2.1) in the case where the unknown functions are defined in a real domain because usually you cannot take a logarithm of both sides of (3.9). However, in some cases it helps that we have solved (2.1) in *local case*. This means that if we know that Γ in (3.9) admits on I positive values only, where I is a proper subinterval of $\mathbb{R}$ then for this interval we can consider (2.1) instead of (3.9) and then try to extend the solution in the class of continuous functions using the equation. This is what we have done in [19] where the continuous solution of a local version of Gołąb–Schinzel equation (3.2) is given.

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MACIEJ SABLIK
INSTYTUT MATEMATYKI
UNIWERSYTET ŚLĄSKI
UL. BANKOWA 14
PL–40–007 KATOWICE
POLAND

E-mail: mssablik@us.edu.pl

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