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On the stability of a general gamma-type functional equation

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Abstract. We investigate the Hyers–Ulam stability of the functional equation

$$f(\varphi(x)) = \phi(x)f(x) + \psi(x)$$

and the stability in the sense of R. Ger of the functional equation

$$f(\varphi(x)) = \phi(x)f(x)$$

in the following two settings:

$$\|g(\varphi(x)) - \phi(x)g(x) - \psi(x)\| \le \varepsilon(x)$$

and

$$\left|\frac{g(\varphi(x))}{\phi(x)g(x)} - 1\right| \le \varepsilon(x).$$

1. Introduction

In this paper we deal with the functional equation

$$f(\varphi(x)) = \phi(x)f(x) + \psi(x). \tag{1.1}$$

Here φ, ϕ, and ψ are given functions, while f is the unknown function. The equation (1.1) has been extensively investigated by numerous authors (see [11, Ch. V–VIII] and the references therein). It includes as special cases the well known gamma functional equation

$$f(x+1) = xf(x), \tag{1.2}$$

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the generalized gamma functional equation

$$f(x+a) = \phi(x)f(x), \tag{1.3}$$

the Schröder functional equation

$$f(\varphi(x)) = sf(x), \tag{1.4}$$

as well as the generalized Schröder functional equation

$$f(\varphi(x)) = \phi(x)f(x). \tag{1.5}$$

S.-M. Jung [7]–[9] studied the stability of the gamma functional equation (1.2). His results have been recently generalized by G. H. Kim [10] to the framework of the generalized gamma functional equation (1.3) (see also [6]).

It is the main purpose of the present paper to study the stability of the more general functional equation (1.1). Our investigations are motivated by the fact that the above mentioned results of S.-M. Jung and G. H. Kim do not cover the important functional equations (1.4) and (1.5). Besides, the results proved here are generalizations of those established in [7]–[9] and [10].

In Section 2 of the paper, a general Hyers–Ulam–Rassias-type theorem concerning the stability of the functional equation (1.1) will be proved. Several applications to special cases of (1.1) are provided. Some of them are pointed out here for the first time. In Section 3, a modified Hyers–Ulam–Rassias stability of the functional equation (1.5) will be investigated in the spirit of R. Ger.

2. Hyers–Ulam stability of the functional equation (1.1)

Throughout this section $\mathbb{K}$ will be either the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers. The set of all nonnegative real numbers will be denoted by $\mathbb{R}_+$, while the set of all positive real numbers will be denoted by $\mathbb{R}_+^*$.

Given the nonempty set S and the function $\varphi : S \to S$, we put $\varphi_0(x) := x$ and $\varphi_n(x) := \varphi(\varphi_{n-1}(x))$ for all positive integers n and all points $x \in S$.

Theorem 2.1. *Let X be a Banach space over the field $\mathbb{K}$, let S be a nonempty set, and let $\varphi : S \to S$, $\psi : S \to X$, $\phi : S \to \mathbb{K} \setminus \{0\}$, and $\varepsilon : S \to \mathbb{R}_+$ be given functions such that*

$$\omega(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\phi(\varphi_j(x))|} < \infty$$

for all $x \in S$. If a function $g : S \to X$ satisfies

$$\|g(\varphi(x)) - \phi(x)g(x) - \psi(x)\| \le \varepsilon(x) \tag{2.1}$$

for all $x \in S$, then there exists a unique function $f : S \to X$ such that for each point x in S, f satisfies (1.1) *and*

$$\|g(x) - f(x)\| \le \omega(x). \tag{2.2}$$

Proof. For each positive integer n let $\omega_n : S \to \mathbb{R}_+$ and $f_n : S \to X$ be the functions defined by

$$\omega_n(x) := \sum_{k=0}^{n-1} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\phi(\varphi_j(x))|}$$

and

$$f_n(x) := \frac{g(\varphi_n(x))}{\prod_{j=0}^{n-1} \phi(\varphi_j(x))} - \sum_{k=0}^{n-1} \frac{\psi(\varphi_k(x))}{\prod_{j=0}^{k} \phi(\varphi_j(x))},$$

respectively.

First, we prove by induction on n that for all $x \in S$ it holds that

$$\|g(x) - f_n(x)\| \le \omega_n(x). \tag{2.3}$$

Dividing both sides of (2.1) by $|\phi(x)|$, we see that (2.3) holds for $n = 1$ and all $x \in S$. Now, assume that the inequality (2.3) holds for some positive integer n and all $x \in S$. Replacing x in (2.3) by $\varphi(x)$, and then dividing both sides of the obtained inequality by $|\phi(x)|$, we find that

$$\left\| \frac{g(\varphi(x))}{\phi(x)} - \frac{f_n(\varphi(x))}{\phi(x)} \right\| \le \frac{\omega_n(\varphi(x))}{|\phi(x)|} \quad \text{for all } x \in S.$$

Since

$$\frac{f_n(\varphi(x))}{\phi(x)} = \frac{\psi(x)}{\phi(x)} + f_{n+1}(x),$$

we get

$$\begin{aligned}\|g(x) - f_{n+1}(x)\| &\leq \left\| g(x) - \frac{g(\varphi(x))}{\phi(x)} + \frac{\psi(x)}{\phi(x)} \right\| \\ &\quad + \left\| \frac{g(\varphi(x))}{\phi(x)} - \frac{\psi(x)}{\phi(x)} - f_{n+1}(x) \right\| \\ &\leq \frac{\varepsilon(x)}{|\phi(x)|} + \frac{\omega_n(\varphi(x))}{|\phi(x)|} = \omega_{n+1}(x)\end{aligned}$$

for all $x \in S$. This completes the proof of (2.3).

Now, we claim that $(f_n(x))$ is a Cauchy sequence for all $x \in S$. Indeed, from (2.1) it follows that

$$\left\| \frac{g(\varphi(x))}{\phi(x)} - g(x) - \frac{\psi(x)}{\phi(x)} \right\| \leq \frac{\varepsilon(x)}{|\phi(x)|} \quad \text{for all } x \in S.$$

Replacing x in this inequality by $\varphi_n(x)$, and then dividing both sides of the obtained inequality by $\prod_{j=0}^{n-1} |\phi(\varphi_j(x))|$, we get

$$\|f_{n+1}(x) - f_n(x)\| \leq \frac{\varepsilon(\varphi_n(x))}{\prod_{j=0}^{n} |\phi(\varphi_j(x))|} \tag{2.4}$$

for each positive integer n and all $x \in S$.

Let m and n be arbitrary positive integers with $m < n$, and let x be any element in S. By virtue of (2.4) we have

$$\begin{aligned}\|f_n(x) - f_m(x)\| &\leq \sum_{k=m}^{n-1} \|f_{k+1}(x) - f_k(x)\| \leq \sum_{k=m}^{n-1} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\phi(\varphi_j(x))|} \\ &= \omega_n(x) - \omega_m(x).\end{aligned}$$

Since the sequence $(\omega_n(x))$ converges, this inequality implies that $(f_n(x))$ is a Cauchy sequence, as claimed. Consequently, we can define the function $f : S \to X$ by

$$f(x) := \lim_{n\to\infty} f_n(x).$$

Let x be any element in S. We have

$$f(\varphi(x)) = \lim_{n\to\infty} f_n(\varphi(x)) = \lim_{n\to\infty} [\phi(x) f_{n+1}(x) + \psi(x)] = \phi(x) f(x) + \psi(x),$$

hence f satisfies (1.1). Moreover, by passing to the limit in (2.3) when $n \to \infty$, we see that f satisfies also (2.2).

In order to prove the uniqueness of f, let $\tilde{f} : S \to X$ be any function satisfying

$$\tilde{f}(\varphi(x)) = \phi(x)\tilde{f}(x) + \psi(x) \quad \text{and} \quad \|g(x) - \tilde{f}(x)\| \le \omega(x)$$

for all $x \in S$. Then we have

$$\begin{aligned}\|f(x) - \tilde{f}(x)\| &= \frac{\|f(\varphi(x)) - \tilde{f}(\varphi(x))\|}{|\phi(x)|} = \frac{\|f(\varphi_n(x)) - \tilde{f}(\varphi_n(x))\|}{\prod_{j=0}^{n-1} |\phi(\varphi_j(x))|} \\ &\le \frac{\|f(\varphi_n(x)) - g(\varphi_n(x))\| + \|g(\varphi_n(x)) - \tilde{f}(\varphi_n(x))\|}{\prod_{j=0}^{n-1} |\phi(\varphi_j(x))|} \\ &\le 2\, \frac{\omega(\varphi_n(x))}{\prod_{j=0}^{n-1} |\phi(\varphi_j(x))|}\end{aligned}$$

for each positive integer n and all $x \in S$. Taking into account that

$$\frac{\omega(\varphi_n(x))}{\prod_{j=0}^{n-1} |\phi(\varphi_j(x))|} = \omega(x) - \omega_n(x),$$

we conclude that

$$\|f(x) - \tilde{f}(x)\| \le 2[\omega(x) - \omega_n(x)]$$

for all $n \in \mathbb{N}$ and all $x \in S$. Since $\omega_n(x) \to \omega(x)$ as $n \to \infty$, the last inequality ensures that $f(x) = \tilde{f}(x)$ for all $x \in S$. □

In what follows we give applications of Theorem 2.1 to some special cases of (1.1).

Corollary 2.2 ([10]). *Let $a > 0$, let S be an unbounded subinterval of $\mathbb{R}_+^*$, and let $\phi : S \to \mathbb{R} \setminus \{0\}$, $\varepsilon : S \to \mathbb{R}_+$ be given functions such that*

$$\omega(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(x + ka)}{\prod_{j=0}^{k} |\phi(x + ja)|} < \infty \tag{2.5}$$

for all $x \in S$. If a function $g : S \to \mathbb{R}$ satisfies

$$|g(x+a) - \phi(x)g(x)| \leq \varepsilon(x) \quad \text{for all } x \in S,$$

then there exists a unique function $f : S \to \mathbb{R}$ such that for each point x in S, f satisfies (1.3) *and* $|g(x) - f(x)| \leq \omega(x)$.

Proof. Follows from Theorem 2.1 for $X = \mathbb{K} = \mathbb{R}$, $\psi(x) = 0$, and $\varphi(x) = x + a$. □

Corollary 2.3 ([7])**.** *Let $\delta > 0$ and let S be an unbounded subinterval of $\mathbb{R}_+^*$. If a function $g : S \to \mathbb{R}$ satisfies*

$$|g(x+1) - xg(x)| \leq \delta \quad \text{for all } x \in S,$$

then there exists a unique function $f : S \to \mathbb{R}$ such that for each point x in S, f satisfies (1.2) *and*

$$|g(x) - f(x)| \leq \frac{e\delta}{x}.$$

Proof. Follows from Corollary 2.2 for $\phi(x) = x$ and $\varepsilon(x) = \delta$, taking into account that

$$\omega(x) = \delta \sum_{k=0}^{\infty} \frac{1}{x(x+1)\cdots(x+k)} < \frac{\delta}{x} \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{e\delta}{x}$$

for all $x \in S$. □

Next, we prove a q-analogue of Corollary 2.3. Given $q \in \,]0,1[$, the q-factorials are defined by

$$k!_q := \frac{(1-q)(1-q^2)\cdots(1-q^k)}{(1-q)^k} \quad \text{if } k \geq 1$$

$$0!_q := 1,$$

while the q-exponential function is defined by

$$E_q(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!_q} \qquad |x| < \frac{1}{1-q}.$$

For definitions and properties of the q-hypergeometric functions the reader is referred to [2].

Corollary 2.4. *Let $0 < q < 1$, let $\delta > 0$, and let S be an unbounded subinterval of $\mathbb{R}_+^*$. If a function $g : S \to \mathbb{R}$ satisfies*

$$\left| g(x+1) - \frac{1-q^x}{1-q} g(x) \right| \le \delta \quad \text{for all } x \in S,$$

then there exists a unique function $f : S \to \mathbb{R}$ such that

$$f(x+1) = \frac{1-q^x}{1-q} f(x) \quad \text{and} \quad |g(x) - f(x)| \le \delta E_q(1) \frac{1-q}{1-q^x}$$

for all $x \in S$.

Proof. Follows from Corollary 2.2 for $\phi(x) = \frac{1-q^x}{1-q}$ and $\varepsilon(x) = \delta$, taking into account that

$$\begin{aligned} \omega(x) &= \delta \sum_{k=0}^{\infty} \frac{(1-q)^{k+1}}{(1-q^x)(1-q^{x+1}) \cdots (1-q^{x+k})} \\ &< \delta \frac{1-q}{1-q^x} \sum_{k=0}^{\infty} \frac{1}{k!_q} = \delta E_q(1) \frac{1-q}{1-q^x} \end{aligned}$$

for all $x \in S$. □

Assume now that a and p are positive real numbers, while $b \ne c$ are nonnegative real numbers. We are concerned with the functional equation

$$f(x+a) = \left(\frac{x+b}{x+c} \right)^p f(x), \tag{2.6}$$

another special case of (1.3) which occurs (sometimes implicitly) in the literature. For instance, I. B. Lazarević and A. Lupaş [12] considered the functional equation

$$f(x+1) = \frac{x+1}{x+\theta} f(x), \tag{2.7}$$

where $\theta \in]0,1[$ is given. It is immediately seen that the so-called Wallis function $W(\,\cdot\,,\theta)$, defined by $W(x,\theta) := \Gamma(x+1)/\Gamma(x+\theta)$ satisfies (2.7) for all $x > 0$. Moreover, in [12, Theorem 1] it was proved that the only eventually convex function $f : \mathbb{R}_+^* \to \mathbb{R}$, satisfying (2.7) for all $x > 0$ and the initial condition $f(1) = 1$, is given by $f(x) := \Gamma(\theta+1) W(x,\theta)$. Recall that f is said to be *eventually convex* on $\mathbb{R}_+^*$ if there is an unbounded subinterval of $\mathbb{R}_+^*$ on which the restriction of f is convex.

On the other hand, H. P. Thielman [15] investigated the functional equation

$$\frac{1}{f(x+a)} = x^p f(x). \tag{2.8}$$

This generalizes the functional equation

$$\frac{1}{f(x+1)} = xf(x),$$

studied by A. E. Mayer [13]. In [15, Theorem 1] it was proved that the only eventually convex function $f : \mathbb{R}_+^* \to \mathbb{R}$, satisfying (2.8) for all $x > 0$, is given by

$$f(x) := \left[\frac{\Gamma\left(\frac{x}{2a}\right)}{\sqrt{2a}\,\Gamma\left(\frac{x+a}{2a}\right)}\right]^p.$$

It is immediately seen that if f satisfies (2.8), then it is a solution to the functional equation

$$f(x+2a) = \left(\frac{x}{x+a}\right)^p f(x),$$

a special case of (2.6).

By a slight modification of Theorem 3.2 in the very interesting paper by R. Webster [16], we can deduce that the function $f : \mathbb{R}_+^* \to \mathbb{R}$, defined by

$$f(x) := \left[\frac{\Gamma\left(1+\frac{c}{a}\right)\Gamma\left(\frac{x+b}{a}\right)}{\Gamma\left(1+\frac{b}{a}\right)\Gamma\left(\frac{x+c}{a}\right)}\right]^p,$$

is the unique eventually convex function satisfying (2.6) for all $x > 0$ and the initial condition $f(a) = 1$.

If δ is a positive real number and the functions ε and ϕ are defined by $\varepsilon(x) := \delta$ and $\phi(x) := \left(\frac{x+b}{x+c}\right)^p$, respectively, then the convergence of the series (2.5) is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty}\prod_{j=0}^{k-1}\left(\frac{x+c+ja}{x+b+ja}\right)^p. \tag{2.9}$$

Letting $u_k := \prod_{j=0}^{k-1} \left(\frac{x+c+ja}{x+b+ja}\right)^p$, for $x \in \mathbb{R}_+^*$ fixed we have

$$\begin{aligned}
\frac{u_k}{u_{k+1}} &= \left(1 + \frac{b-c}{x+c+ka}\right)^p = 1 + p \cdot \frac{b-c}{x+c+ka} + O\left(\left(\frac{b-c}{x+c+ka}\right)^2\right) \\
&= 1 + \frac{p(b-c)}{ka} \cdot \frac{1}{1+\frac{x+c}{ka}} + O\left(\frac{1}{k^2}\right) \\
&= 1 + \frac{p(b-c)}{ka}\left(1 + O\left(\frac{x+c}{ka}\right)\right) + O\left(\frac{1}{k^2}\right) \\
&= 1 + \frac{p(b-c)}{a} \cdot \frac{1}{k} + O\left(\frac{1}{k^2}\right).
\end{aligned}$$

By virtue of the Gauss test, we conclude that the series (2.9) converges if $p(b-c) > a$ and diverges if $p(b-c) \le a$. Therefore, as an application of Corollary 2.2 we can derive the following corollary, concerning the Hyers–Ulam stability of the functional equation (2.6).

Corollary 2.5. *Let a, δ, and p be positive real numbers, and let $b \neq c$ be nonnegative real numbers such that $p(b-c) > a$. Further, let S be an unbounded subinterval of $\mathbb{R}_+^*$ and let*

$$\omega(x) := \delta \sum_{k=1}^{\infty} \prod_{j=0}^{k-1} \left(\frac{x+c+ja}{x+b+ja}\right)^p \quad \text{for all } x \in S.$$

If a function $g : S \to \mathbb{R}$ satisfies

$$\left| g(x+a) - \left(\frac{x+b}{x+c}\right)^p g(x) \right| \le \delta \quad \text{for all } x \in S,$$

then there exists a unique function $f : S \to \mathbb{R}$ such that for each point x in S, f satisfies (2.6) and $|g(x) - f(x)| \le \omega(x)$.

Now, let us consider the hypergeometric series

$$F(\alpha, \beta, \gamma; z) := 1 + \sum_{k=1}^{\infty} z^k \prod_{j=0}^{k-1} \frac{(\alpha+j)(\beta+j)}{(\gamma+j)(1+j)}.$$

It is well-known that if $\alpha \ge 0$, $\beta \ge 0$, and $\gamma > \alpha + \beta$, then

$$F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}.$$

Consequently, if $\alpha \geq 0$ and $\gamma > \alpha + 1$, then

$$F(\alpha, 1, \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - 1)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - 1)} = \frac{\gamma - 1}{\gamma - \alpha - 1}.$$

Taking this into account, for $p = 1$ and $b - c > a$, the function ω, defined in Corollary 2.5, has the simple form

$$\omega(x) = \delta \sum_{k=1}^{\infty} \prod_{j=0}^{k-1} \frac{\frac{x+c}{a} + j}{\frac{x+b}{a} + j} = \delta \left[F\left(\frac{x+c}{a}, 1, \frac{x+b}{a}; 1 \right) - 1 \right] = \delta \, \frac{x+c}{b-c-a}.$$

Thus, in the special case $p = 1$, from Corollary 2.5 we deduce the following

Corollary 2.6. *Let a and δ be positive real numbers, and let $b \neq c$ be nonnegative real numbers such that $b - c - a > 0$. Further, let S be an unbounded subinterval of $\mathbb{R}_+^*$. If a function $g : S \to \mathbb{R}$ satisfies*

$$\left| g(x+a) - \frac{x+b}{x+c} g(x) \right| \leq \delta \quad \text{for all } x \in S,$$

then there exists a unique function $f : S \to \mathbb{R}$ such that

$$f(x+a) = \frac{x+b}{x+c} f(x) \quad \text{and} \quad |g(x) - f(x)| \leq \delta \frac{x+c}{b-c-a}$$

for all $x \in S$.

We conclude this section with an application of Theorem 2.1 to the Hyers–Ulam stability of the Schröder functional equation (1.4).

Corollary 2.7. *Let $s > 1$, let $\delta > 0$, let S be a nonempty set, and let $\varphi : S \to S$ be a given function. If $g : S \to \mathbb{K}$ satisfies*

$$|g(\varphi(x)) - s g(x)| \leq \delta \quad \text{for all } x \in S$$

then there exists a unique function $f : S \to \mathbb{K}$ such that for each point x in S, f satisfies (1.4) and

$$|g(x) - f(x)| \leq \frac{\delta}{s-1}.$$

PROOF. Follows from Theorem 2.1 for $X = \mathbb{K}$, $\psi(x) = 0$, $\phi(x) = s$, and $\varepsilon(x) = \delta$, taking into account that

$$\omega(x) = \delta \sum_{k=0}^{\infty} \frac{1}{s^{k+1}} = \frac{\delta}{s-1}$$

for all $x \in S$. □

3. Stability in the sense of R. Ger of the functional equation (1.5)

Following S.-M. JUNG [7], in this section a modified Hyers–Ulam stability of the functional equation (1.5) is investigated in the spirit of R. Ger.

Theorem 3.1. *Let S be a nonempty set and let $\varphi : S \to S$, $\phi : S \to \mathbb{R}_+^*$, and $\varepsilon : S \to]0,1[$ be given functions such that*

$$\alpha(x) := \prod_{j=0}^{\infty} [1 - \varepsilon(\varphi_j(x))] > 0 \quad \text{and} \quad \beta(x) := \prod_{j=0}^{\infty} [1 + \varepsilon(\varphi_j(x))] < \infty$$

for all $x \in S$. If a function $g : S \to \mathbb{R}_+^*$ satisfies

$$\left| \frac{g(\varphi(x))}{\phi(x)g(x)} - 1 \right| \le \varepsilon(x) \tag{3.1}$$

for all $x \in S$, then there exists a unique function $f : S \to \mathbb{R}_+^*$ such that for each point x in S, f satisfies (1.5) and

$$\alpha(x) \le \frac{f(x)}{g(x)} \le \beta(x). \tag{3.2}$$

PROOF. For each positive integer n let $f_n : S \to \mathbb{R}_+^*$ be the function defined by

$$f_n(x) := \frac{g(\varphi_n(x))}{\prod_{j=0}^{n-1} \phi(\varphi_j(x))}.$$

Let x be any point in S. For all positive integers m and n with $m < n$ we have

$$\frac{f_n(x)}{f_m(x)} = \frac{g(\varphi_n(x))}{g(\varphi_m(x))} \cdot \frac{1}{\prod_{j=m}^{n-1} \phi(\varphi_j(x))} = \prod_{j=m}^{n-1} \frac{g(\varphi_{j+1}(x))}{\phi(\varphi_j(x))g(\varphi_j(x))}.$$

Taking into account the inequality (3.1), we then obtain

$$\prod_{j=m}^{n-1}\left[1-\varepsilon(\varphi_j(x))\right] \le \frac{f_n(x)}{f_m(x)} \le \prod_{j=m}^{n-1}\left[1+\varepsilon(\varphi_j(x))\right],$$

or

$$\sum_{j=m}^{n-1}\ln\left(1-\varepsilon(\varphi_j(x))\right) \le \ln f_n(x)-\ln f_m(x) \le \sum_{j=m}^{n-1}\ln\left(1+\varepsilon(\varphi_j(x))\right).$$

Since

$$\sum_{j=0}^{\infty}\ln\left(1-\varepsilon(\varphi_j(x))\right)=\ln\alpha(x) \quad \text{and} \quad \sum_{j=0}^{\infty}\ln\left(1+\varepsilon(\varphi_j(x))\right)=\ln\beta(x),$$

it follows that

$$\lim_{m\to\infty}\sum_{j=m}^{\infty}\ln\left(1-\varepsilon(\varphi_j(x))\right)=\lim_{m\to\infty}\sum_{j=m}^{\infty}\ln\left(1+\varepsilon(\varphi_j(x))\right)=0.$$

Therefore, $(\ln f_n(x))$ is a Cauchy sequence for each $x \in S$. Hence we can define the function $f : S \to \mathbb{R}_+^*$ by

$$f(x) := e^{\lim_{n\to\infty}\ln f_n(x)}.$$

In fact, this definition is equivalent to $f(x)=\lim_{n\to\infty} f_n(x)$. Taking this into account, we have

$$f(\varphi(x))=\lim_{n\to\infty} f_n(\varphi(x))=\lim_{n\to\infty}\phi(x)f_{n+1}(x)=\phi(x)f(x)$$

for every $x \in S$. Consequently, f is a solution of the functional equation (1.5).

In order to prove (3.2), let us remark that

$$\frac{f_n(x)}{g(x)}=\prod_{j=0}^{n-1}\frac{g(\varphi_{j+1}(x))}{\phi(\varphi_j(x))g(\varphi_j(x))}$$

for each positive integer n and all $x \in S$. This equality together with (3.1) ensure that

$$\prod_{j=0}^{n-1}\big[1-\varepsilon(\varphi_j(x))\big] \le \frac{f_n(x)}{g(x)} \le \prod_{j=0}^{n-1}\big[1+\varepsilon(\varphi_j(x))\big],$$

for each positive integer n and all $x \in S$. By passing to the limit when $n \to \infty$, we see that (3.2) holds true, too.

Now, it remains only to prove the uniqueness of f. To this end, assume that $\tilde{f} : S \to \mathbb{R}_+^*$ is some function satisfying

$$\tilde{f}(\varphi(x)) = \phi(x)\tilde{f}(x) \tag{3.3}$$

and

$$\alpha(x) \le \frac{\tilde{f}(x)}{g(x)} \le \beta(x) \tag{3.4}$$

for all $x \in S$.

Let $x \in S$ be arbitrarily chosen. From (1.5) and (3.3) it follows that

$$\frac{f(x)}{\tilde{f}(x)} = \frac{f(\varphi_n(x))}{\tilde{f}(\varphi_n(x))} = \frac{f(\varphi_n(x))}{g(\varphi_n(x))} : \frac{\tilde{f}(\varphi_n(x))}{g(\varphi_n(x))}$$

for each positive integer n. By virtue of (3.2) and (3.4), we deduce that

$$\frac{\alpha(\varphi_n(x))}{\beta(\varphi_n(x))} \le \frac{f(x)}{\tilde{f}(x)} \le \frac{\beta(\varphi_n(x))}{\alpha(\varphi_n(x))} \tag{3.5}$$

for each positive integer n. Since

$$\alpha(\varphi_n(x)) = \frac{\alpha(x)}{\prod_{j=0}^{n-1}\left[1 - \varepsilon(\varphi_j(x))\right]} \quad \text{and} \quad \beta(\varphi_n(x)) = \frac{\beta(x)}{\prod_{j=0}^{n-1}\left[1 + \varepsilon(\varphi_j(x))\right]},$$

we conclude that $\alpha(\varphi_n(x)) \to 1$ and $\beta(\varphi_n(x)) \to 1$ as $n \to \infty$. By passing to the limit in (3.5) when $n \to \infty$, we see that $f(x) = \tilde{f}(x)$. Thus, the uniqueness of f is proved. □

Corollary 3.2 ([7]). *Let a, δ, and θ be positive real numbers, let S be an unbounded subinterval of $]\delta^{1/(1+\theta)}, \infty[$, and let $\phi : S \to \mathbb{R}_+^*$ be a given function. If $g : S \to \mathbb{R}_+^*$ satisfies*

$$\left| \frac{g(x+a)}{\phi(x)g(x)} - 1 \right| \le \frac{\delta}{x^{1+\theta}} \quad \text{for all } x \in S,$$

then there exists a unique function $f : S \to \mathbb{R}_+^*$ such that for each point $x \in S$, f satisfies (1.3) and

$$\alpha(x) \le \frac{f(x)}{g(x)} \le \beta(x),$$

where

$$\alpha(x) := \prod_{j=0}^{\infty}\left[1 - \frac{\delta}{(x+ja)^{1+\theta}}\right] \quad \text{and} \quad \beta(x) := \prod_{j=0}^{\infty}\left[1 + \frac{\delta}{(x+ja)^{1+\theta}}\right].$$

PROOF. Follows from Theorem 3.1 for $\varphi(x) = x + a$ and $\varepsilon(x) = \frac{\delta}{x^{1+\theta}}$. □

Corollary 3.3. *Let $s > 1$, let $\delta > 0$, let S be an unbounded subinterval of $]\delta, \infty[$, and let $\phi : S \to \mathbb{R}_+^*$ be a given function. If $g : S \to \mathbb{R}_+^*$ satisfies*

$$\left|\frac{g(sx)}{\phi(x)g(x)} - 1\right| \le \frac{\delta}{x} \quad \text{for all } x \in S,$$

then there exists a unique function $f : S \to \mathbb{R}_+^*$ such that for each point x in S, f satisfies

$$f(sx) = \phi(x)f(x) \quad \text{and} \quad \alpha(x) \le \frac{f(x)}{g(x)} \le \beta(x),$$

where

$$\alpha(x) := \prod_{j=0}^{\infty}\left(1 - \frac{\delta}{s^j x}\right) \quad \text{and} \quad \beta(x) := \prod_{j=0}^{\infty}\left(1 + \frac{\delta}{s^j x}\right).$$

PROOF. Follows from Theorem 3.1 for $\varphi(x) = sx$ and $\varepsilon(x) = \frac{\delta}{x}$. □

A similar corollary can be stated for $0 < s < 1$ if we take $S =]0, 1/\delta[$ and $\varepsilon(x) = \delta x$.

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