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A categorical treatment of numerical semigroups

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Abstract. A subsemigroup of the additive semigroup of positive integers $\mathbb{P}$ which generates the group of integers $\mathbb{Z}$ as a group bears the label of numerical. The category of numerical semigroups and homomorphisms is compared with several categories. The relationship turns out to be either isomorphism or equivalence of categories. These categories have objects of diverse nature: subsemigroups of $\mathbb{P}$, abstract semigroups satisfying strong conditions, partial orders on $\mathbb{Z}$, infinite cyclic groups partially ordered, pairs of the form (r, I) where $r \in \mathbb{P}$ and I is a $r \times r$-matrix over nonnegative integers $\mathbb{N}$, functions from $\mathbb{Z}/(r)$ into $\mathbb{N}$, all of these satisfying numerous conditions.

This is an "external" study of numerical semigroups: comparison with different objects which may shed some light on their own structure.

1. Introduction and summary

The semigroup $\mathbb{P}$ of positive integers under addition is probably almost as old as Mathematics. Very likely, the set $\mathbb{P}$ is the first infinite set encountered by Man. From the point of semigroups, or general mathematics, it seems surprising that so little attention has been paid to such natural objects as subsemigroups of $\mathbb{P}$.

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Numerical semigroups are those subsemigroups of $\mathbb{P}$ which generate the group $\mathbb{Z}$ of integers as a group. They admit other natural characterizations and have appeared in several branches of mathematics. Every subsemigroup of $\mathbb{P}$ is isomorphic to a unique numerical semigroup, so that, abstractly, it suffices to study numerical semigroups. They have attracted more attention than general subsemigroups of $\mathbb{P}$ so there is a greater body of information about them. The literature dealing with them is not only extensive but is also varied in the aspects studied.

The purpose of this work is the study of the category of numerical semigroups and their homomorphisms by constructing a number of categories related to it. These categories have objects: subsemigroups of $\mathbb{P}$, a class of abstract semigroups, certain partial orders on $\mathbb{Z}$, certain partially ordered infinite cyclic groups, and pairs that occur in Tamura's representation of a class of commutative cancellative semigroups. The relationships of these categories are either equivalences or isomorphisms.

Section 2 contains a minimal amount of needed notation and terminology. Various auxiliary statements are proved in Section 3. In Section 4 we study the mutual relationship of three categories of semigroups and their homomorphisms. Section 5 contains relationship of the category of numerical semigroups with some categories of orders and groups invoked above. Isomorphisms of the former category with some categories whose objects are related to certain constructions of T. Tamura are established in Sections 6 and 7. We wind up our discussion in Sections 8, 9 and 10 with some general statements and an example.

The diagram on the next page represents the categories and functors discussed in the paper. The functors are labelled with the number of the section in which they are treated. We will see that O_1, T_1 and Φ_1 are isomorphisms (with inverses O_2, T_2 and Φ_2, respectively) and the remaining ones are equivalences of categories except $I_{\mathcal{T},\mathcal{U}}$.

$$
\begin{tikzcd}
 & & \mathcal{S} \arrow[d, shift left, "\Sigma"] & \mathcal{U} & \\
\mathcal{G} \arrow[r, shift left, "\Gamma_2"] \arrow[r, phantom, "\scriptscriptstyle 5"] & \mathcal{O} \arrow[l, shift left, "\Gamma_1"] \arrow[r, shift left, "O_2"] \arrow[r, phantom, "\scriptscriptstyle 5"] & \mathcal{N} \arrow[l, shift left, "O_1"] \arrow[u, shift left, "I_{\mathcal{N},\mathcal{S}}"] \arrow[u, phantom, "\scriptscriptstyle 4"] \arrow[r, shift left, "T_1"] \arrow[r, phantom, "\scriptscriptstyle 6"] \arrow[d, shift left, "I_{\mathcal{N},\mathcal{P}}"] & \mathcal{T} \arrow[l, shift left, "T_2"] \arrow[u, "I_{\mathcal{T},\mathcal{U}}"'] \arrow[u, phantom, "\scriptscriptstyle 9"'] \arrow[r, shift left, "\Phi_1"] \arrow[r, phantom, "\scriptscriptstyle 7"] & \mathcal{F} \arrow[l, shift left, "\Phi_2"] \\
 & & \mathcal{P} \arrow[u, shift left, "\Pi"] \arrow[u, phantom, "\scriptscriptstyle 4"] & &
\end{tikzcd}
$$

2. Notation and terminology

We employ the standard symbolism and nomenclature which can be found in texts on semigroups, categories and ordered groups. For definiteness or emphasis, we state the following.

The symbols $\mathbb{P}$, $\mathbb{N}$ and $\mathbb{Z}$ stand for positive integers, nonegative integers and integers respectively, all under addition.

If A is a nonempty subset of a semigroup S, $\langle A\rangle$ denotes the subsemigroup of S generated by A. If A is a nonempty subset of a group G, $[A]$ denotes the subgroup of G generated by A. A semigroup S is *power joined* if for any $a, b \in S$, there exist $m, n \in \mathbb{P}$ such that $a^m = b^n$; it is *power cancellative* if for any $a, b \in S$ and $n \in \mathbb{P}$, $a^n = b^n$ implies $a = b$. For a nonempty subset A of $\mathbb{P}$, $\gcd(A)$ stands for the greatest common divisor of all elements of A. All our semigroups (and groups) are commutative. For any sets A and B, we write $A \setminus B = \{a \in A \mid a \notin B\}$.

We write functors, and most functions, on the right of the argument. We denote categories by capital script and functors by capital Greek letters. If $\mathcal{X}$ is a subcategory of a category $\mathcal{Y}$, $I_{\mathcal{X},\mathcal{Y}}$ denotes the inclusion functor $\mathcal{X} \to \mathcal{Y}$. In particular $I_{\mathcal{X}} = I_{\mathcal{X},\mathcal{X}}$ is the identity functor on $\mathcal{X}$. For an object X of a category, id_X denotes the identity morphism on X.

3. Lemmas

We prove here a number of auxiliary statements to be used later some of which are of independent interest.

Lemma 3.1. (i) *If X is a nonempty subset of $\mathbb{Z}$, then* $[X] = \gcd(X)\mathbb{Z}$.
(ii) *If S is a subsemigroup of $\mathbb{Z}$, then* $[S] = \{a - b \mid a, b \in S\}$.

We have defined a numerical semigroup in Section 1. Paraphrasing: a subsemigroup S of $\mathbb{P}$ is numerical if (and only if) $[S] = \mathbb{Z}$.

The next six lemmas are known. Since they will be used several times, we give them a short proof.

Lemma 3.2. *The following conditions on a subsemigroup S of $\mathbb{P}$ are equivalent.*

(i) *S is numerical.* (ii) $\gcd(S) = 1$. (iii) *$\mathbb{P} \setminus S$ is finite.*

Proof. *(i) and (ii) are equivalent.* This is a direct consequence of Lemma 3.1(i).

(ii) implies (iii). By hypothesis, $1 = \sum_{i=1}^{m} n_i s_i$ for some $n_i \in \mathbb{Z} \setminus \{0\}$ and $s_i \in S$. We may suppose that $n_i > 0$ if and only if $i \leq r$ for some $1 \leq r \leq m$. This way, we may write $1 = s - t$ where both $s = \sum_{i=1}^{r} n_i s_i$ and $t = \sum_{i=r+1}^{m} (-n_i) s_i$ are in S.

We show next that for all $n \in \mathbb{P}$ such that $n > t^2 + t$, we have $n \in S$. Hence let $n > t^2 + t$ and write $n = qt + p$ where $0 \leq p < t$. First note that $q > t$. It follows that

$$n = pt + p + qt - pt = p(t + 1) + (q - p)t = ps + (q - p)t,$$

whence we see that $n \in S$. Therefore $\mathbb{P} \setminus S \subseteq \{1, \dots, t^2 + t\}$ and it is finite.

(iii) implies (i). In view of the hypothesis, there exists $s \in S$ such that $s + 1 \in S$. By Lemma 3.1(ii), we get $1 \in [S]$ whence $[S] = \mathbb{Z}$ and S is numerical. □

Lemma 3.3. *Let S be a subsemigroup of $\mathbb{P}$. Then S is finitely generated.*

Proof. Let $d = \gcd(S)$. The mapping $s \mapsto \frac{s}{d}$ is an isomorphism of S onto a numerical semigroup, so we may assume that S is numerical.

By Lemma 3.2(iii), there exists $m \in S$ such that $m+k \in S$ for all $k \in \mathbb{N}$. For each $i = 0, \ldots, m-1$, let

$$p_i = \min\{s \in S \mid s \equiv i \pmod{m}\}.$$

Note that $p_0 = m$. We show next that the set $\{p_0, \ldots, p_{m-1}\}$ generates S. Let $s \in S$. Then $s \equiv p_i (\mathrm{mod}\, m)$ for some $0 \leq i < m$, so that $s = qm + p_i$ for some $q \in \mathbb{Z}$. Since $s \geq p_i$, we must have $q \in \mathbb{N}$ and thus $s = qm + p_i = qp_0 + p_i \in \langle p_0, \ldots, p_{m-1}\rangle$. □

Lemma 3.4. *Let S and T be subsemigroups of $\mathbb{P}$.*

(i) *Every homomorphism of S into T has the form*

$$\varphi_d : s \mapsto d\,\frac{\gcd(T)}{\gcd(S)}\,s \quad (s \in S)$$

for some (unique) $d \in \mathbb{P}$.

(ii) *If S and T are numerical and φ is an isomorphism of S onto T, then $\varphi = \mathrm{id}_S$.*

Proof. (i) Let $\varphi : S \to T$ be a homomorphism. Fix $a \in S$. For any $s \in S$, there exist $p, q \in \mathbb{P}$ such that $pa = qs$. Then $p(a\varphi) = q(s\varphi)$ whence

$$s\varphi = \frac{p}{q}(a\varphi) = \frac{s}{a}(a\varphi) = \frac{a\varphi}{a}s.$$

In particular, $S\varphi = \frac{a\varphi}{a}S \subseteq T$ which implies that $\frac{a\varphi}{a}[S] \subseteq [T]$. By Lemma 3.1(i), we get

$$\frac{a\varphi}{a}\gcd(S)\mathbb{Z} \subseteq \gcd(T)\mathbb{Z}$$

whence $d = \frac{a\varphi}{a}\frac{\gcd(S)}{\gcd(T)} \in \mathbb{P}$ so that $\varphi = \varphi_d$.

(ii) Let φ_d be an isomorphism of S onto T where S and T are numerical. Then $dS = T$ which implies that d is a common divisor of all elements of T forcing $d = 1$. But $\varphi_1 = \mathrm{id}_S$. □

The next lemma shows that the conjunction of power joinedness and power cancellation yields unexpected results.

Lemma 3.5. *Let S be a nontrivial commutative power joined power cancellative semigroup. Then S is idempotent-free and cancellative.*

Proof. Suppose that S has an idempotent e. For any $a \in S$, by power joinedness, there exist $m, n \in \mathbb{P}$ such that $a^m = e^n$. But then $a^m = e^m$ so by power cancellativity, $a = e$ contradicting the hypothesis that S in nontrivial. Therefore S is idempotent-free. In particular, every element of S is of infinite order.

Next let $ax = ay$ for some $a, x, y \in S$. By power joindness, there exist $m, n, p, q \in \mathbb{P}$ such that $a^m = x^n$ and $a^p = y^q$. The equality $(ax)^{nq} = (ay)^{nq}$ implies that $a^{nq+mq} = a^{nq+np}$ whence $nq + mq = nq + np$ so that $mq = np$. But then $x^{nq} = a^{mq} = a^{np} = y^{nq}$ and power cancellativity yields that $x = y$. Therefore S is cancellative. □

On any semigroup S we define relations $\leq$ and θ by

$$a \leq b \Longleftrightarrow a^m = b^n \text{ for some } m, n \in \mathbb{P},\ m \geq n,$$

$$a \ \theta \ b \Longleftrightarrow a^n = b^n \text{ for some } n \in \mathbb{P}. \tag{1}$$

Lemma 3.6. *Let S be a commutative idempotent-free semigroup.*

(i) *The relation θ is the symmetric part of $\leq$ and is the least power cancellative congruence on S.*

(ii) *The preorder $\leq$ is antisymmetric (and is thus a partial order) if and only if S is power cancellative.*

(iii) *The preorder $\leq$ is a total order if and only if S is power joined and power cancellative.*

Proof. Straightforward. □

Lemma 3.7. *Let S be a nontrivial commutative finitely generated power cancellative power joined semigroup.*

(i) *The preorder $\leq$ is a total order with a least element, say f_S.*

(ii) *There exists $m \in \mathbb{P}$ such that for all $a \in S$, we have $a^m \in \langle f_S \rangle$.*

Proof. (i) By Lemma 3.5, S is idempotent-free and, by Lemma 3.6 (iii), $\leq$ is a total order. By hypothesis, S has a finite set of generators, say $\{a_1, \dots, a_k\}$. We may suppose that $a_1 < a_2 < \cdots < a_k$, and will show that a_1 is the least element of S.

Let $a, b \in S$. Since S is power joined, we have $a^p = b^q$ for some $p, q \in \mathbb{P}$. Then $(ab)^q = a^q b^q = a^{p+q}$ which implies that $a \leq ab$. Now for any $a \in S$, we have $a = a_1^{m_1} \dots a_k^{m_k}$ where $m_j \in \mathbb{P}$ for some j. Hence $a_j \leq a$ and thus $a_1 \leq a$ for all $a \in S$.

(ii) Continuing with the same notation and $f_S = a_1$, since $f_S \leq a_i$ for $i = 1, \dots, k$, we get $a_i^{m_i} = f_S^{n_i}$ for some $m_i, n_i \in \mathbb{P}$ such that $m_i \leq n_i$. Letting $m = m_1 \cdots m_k$, we obtain $a_i^m \in \langle f_S \rangle$ for $i = 1, \dots, k$, and thus $a^m \in \langle f_S \rangle$ for all $a \in S$. $\square$

4. Categories of semigroups

We consider here the relationship of the following categories:

$\mathcal{N}$ – numerical semigroups,

$\mathcal{P}$ – subsemigroups of $\mathbb{P}$,

$\mathcal{S}$ – nontrivial commutative finitely generated power joined power cancellative semigroups (written multiplicatively),

all with homomorphisms.

Observe that, by Lemma 3.5, the hypotheses imposed upon objects S of $\mathcal{S}$ imply that S is idempotent-free and cancellative. We will make full use of Lemma 3.7 for objects of $\mathcal{S}$. For the relationship of the first two categories above, we observe first that $\mathcal{N}$ is a full subcategory of $\mathcal{P}$. Define

$$S\Pi = \left\{ \frac{m}{\gcd(S)} \mid m \in S \right\} \quad (S \in Ob\,\mathcal{P}),$$

and for $\varphi \in \mathrm{Hom}_{\mathcal{P}}(S, T)$,

$$\varphi\Pi : m \mapsto \frac{(\gcd(S)m)\varphi}{\gcd(T)} \quad (m \in S\Pi).$$

It is immediate that Π is a functor. For it and $I_{\mathcal{N},\mathcal{P}}$, we need a natural transformation: $\pi : I_{\mathcal{P}} \to \Pi I_{\mathcal{N},\mathcal{P}}$ defined for every $S \in Ob\,\mathcal{P}$ by

$$S\pi : m \mapsto \frac{m}{\gcd(S)} \quad (m \in S).$$

Theorem 4.1. *The quadruple* $(\Pi, I_{\mathcal{N},\mathcal{P}}, \pi, \mathrm{id})$ *is an equivalence of categories* $\mathcal{N}$ *and* $\mathcal{P}$.

Proof. It is immediate that π is a natural transformation. Since $I_{\mathcal{N},\mathcal{P}}\Pi = I_{\mathcal{N}}$, it only remains to observe that for any $S \in Ob\,\mathcal{P}$, we have that $S\pi : S \to S\Pi$ is an isomorphism. But the mapping

$$m \mapsto \gcd(S)\, m \quad (m \in S\Pi)$$

is clearly the inverse of $S\pi$. □

For the relationship of the first and the third categories above, we first note that by Lemma 3.3 every $S \in Ob\,\mathcal{P}$ is finitely generated. The other properties of objects of $\mathcal{S}$ are clearly satisfied by S so that $\mathcal{P}$ is a full subcategory of $\mathcal{S}$ (ignoring the additive notation in $Ob\,\mathcal{P}$ and multiplicative notation in $Ob\,\mathcal{S}$). Here we need only the existence of the inclusion functor $I_{\mathcal{N},\mathcal{P}} : \mathcal{N} \to \mathcal{S}$. For a functor in the opposite direction, we proceed as follows.

In view of Lemma 3.7(ii), for $S \in Ob\,\mathcal{S}$, we may define

$$m_S = \min\{m \in \mathbb{P} \mid a^m \in \langle f_S \rangle \text{ for all } a \in S\}.$$

For every $a \in S$, we have the equality $a^{m_S} = f_S^{m_a}$ for some $m_a \in \mathbb{P}$ which depends only upon a since a is of infinite order. Hence we may define

$$S\sigma : a \mapsto m_a \quad (a \in S), \qquad S\Sigma = \{m_a \mid a \in S\}.$$

Lemma 4.2. *For every $S \in Ob\,\mathcal{S}$, $S\Sigma$ is a numerical semigroup and the mapping $S\sigma$ is an isomorphism of S into $S\Sigma$.*

Proof. Let $a, b \in S$. Then

$$(ab)^{m_S} = a^{m_S} b^{m_S} = f_S^{m_a} f_S^{m_b} = f_S^{m_a+m_b}$$

and thus $m_{ab} = m_a + m_b$. This shows that $S\Sigma$ is a subsemigroup of $\mathbb{P}$ and that $S\sigma$ is a homomorphism. If $m_a = m_b$, then $a^{m_S} = b^{m_S}$ and power cancellativity yields $a = b$. Hence $S\sigma$ is injective and thus is an isomorphism of S onto $S\Sigma$.

Let $d = \gcd(S\Sigma)$. For every $a \in S$, we have $m_a = dn_a$ where $n_a \in \mathbb{P}$. Write $f = f_S$ and $m = m_S$, and observe that $m = m_f = dn_f$. But then for all $a \in S$, we have $a^{dn_f} = f^{dn_a}$ which implies that $a^{n_f} = f^{n_a}$ by power cancellativity. Hence $m \le n_f$, and since $m = dn_f$, we get $d = 1$. Therefore $S\Sigma$ is numerical. □

For $\varphi \in \mathrm{Hom}_{\mathcal{S}}(S, T)$, let

$$\varphi\Sigma = (S\sigma)^{-1}\,\varphi\,(T\sigma) : S\Sigma \to T\Sigma\,.$$

Theorem 4.3. *The quadruple $(I_{\mathcal{N},\mathcal{S}}, \Sigma, \mathrm{id}, \sigma)$ is an equivalence of categories $\mathcal{N}$ and $\mathcal{S}$.*

Proof. It is straightforward to verify that Σ is a functor. If $S \in Ob\mathcal{N}$, then by Lemma 4.2, $S\sigma : S \to \Sigma$ is an isomorphism of numerical semigroups which by Lemma 3.4(ii) implies that $S = S\Sigma$ and $S\sigma = \mathrm{id}_S$. Thus $I_{\mathcal{N},\mathcal{S}}\Sigma = I_{\mathcal{N}}$. That $\sigma : I_{\mathcal{S}} \to \Sigma I_{\mathcal{N},\mathcal{S}}$ is a natural transformation is evident from the form of homomorphisms, and that it is an isomorphism is a consequence of Lemma 4.2. $\square$

We have seen in Lemma 4.2 that $S\sigma$ is an isomorphism of $S \in Ob\mathcal{S}$ onto $S\Sigma$. Such an S has the properties announced in Lemma 3.7. This provides S with a total order $\leq$ in which S has a least element f_S. The next observation establishes the relationship between this order on S and the natural order on $S\Sigma$.

Proposition 4.4. *Let $S \in Ob\,\mathcal{S}$ and $a, b \in S$. Then $a \leq b$ if and only if $a(S\sigma) \leq b(S\sigma)$.*

Proof. By power joinedness, we have $a^p = b^q$ for some $p, q \in \mathbb{P}$. Hence $a^{pm_S} = b^{qm_S}$ whence $f_S^{pm_a} = f_S^{qm_b}$ which yields $pm_a = qm_b$. Therefore $a \leq b \Longleftrightarrow p \geq q \Longleftrightarrow m_a \leq m_b$. $\square$

Let $S \in Ob\,\mathcal{S}$. We can arrive at the isomorphism $S\sigma$ in a different way as follows. Fix $a \in S$. According to ([3], Theorems 3 and 5), the function ζ defined on S by

$$\zeta : x \mapsto \frac{m}{n} \quad \text{if } a^m = x^n$$

is an isomorphism of S onto a subsemigroup of the additive semigroup of positive rationals. Since S is finitely generated, there exists $p \in \mathbb{P}$ such that $p(S\zeta) \subseteq \mathbb{P}$. Hence $p(S\zeta) \in Ob\,\mathcal{P}$. We can further divide all elements of $p(S\zeta)$ by $d = \gcd(S\zeta)$ thereby obtaining $\frac{p}{d}(S\zeta) \in Ob\mathcal{N}$. Hence the mapping

$$x \mapsto \frac{p}{d}\frac{m}{n} \quad \text{if } a^m = x^n \quad (a \in S)$$

is an isomorphism of S onto the numerical semigroup $\frac{p}{d}(S\zeta)$. But this semigroup is isomorphic to S and thus, by Lemma 4.2, to $S\Sigma$. Now Lemma 3.4(ii) implies that $\frac{p}{d}(S\zeta) = S\Sigma$.

It is remarkable that we made an arbitrary choice of the element $a \in S$ and that we did not insist that p be taken as small as possible. And it seems that d depends on both the choice of a and p. Nevertheless, we have arrived at the same semigroup $S\Sigma$.

5. Categories of orders and groups

We introduce here the categories $\mathcal{O}$ and $\mathcal{G}$ and prove that $\mathcal{O}$ is isomorphic to $\mathcal{N}$ and is equivalent to $\mathcal{G}$. A partial order ξ on an additive group G is said to be *dense* if for every $g \in G$, there exists $h \in G$ such that $0 \;\xi\; h$ and $0 \;\xi\; g + h$. Define

$$\begin{aligned} Ob\,\mathcal{O} = \{&\text{compatible dense partial orders on } \mathbb{Z} \\ &\text{contained in the natural order}\}, \end{aligned}$$

and for $\xi, \eta \in Ob\,\mathcal{O}$,

$$\mathrm{Hom}_{\mathcal{O}}(\xi, \eta) = \{d \in \mathbb{P} \mid m \;\xi\; n \Rightarrow dm \;\eta\; dn \text{ for all } m, n \in \mathbb{Z}\}.$$

The composition of morphisms is their product and the identity morphism is $d = 1$.

For the functors between $\mathcal{N}$ and $\mathcal{O}$, we define: for $S \in Ob\,\mathcal{N}$, let

$$m \; SO_1 \; n \Longleftrightarrow n = m + s \text{ for some } s \in S \qquad (m, n \in \mathbb{Z}),$$

and for $\varphi_d \in \mathrm{Hom}_{\mathcal{N}}(S, T)$, let $\varphi_d O_1 = d$; for $\xi \in Ob\,\mathcal{O}$, let

$$\xi O_2 = \{n \in \mathbb{P} \mid 0 \;\xi\; n\},$$

and for $d \in \mathrm{Hom}_{\mathcal{O}}(\xi, \eta)$, let $dO_2 = \varphi_d$.

Theorem 5.1. *The functor O_1 is an isomorphism of $\mathcal{N}$ onto $\mathcal{O}$ with inverse O_2.*

PROOF. *1. O_1 is a functor.* Let $S \in Ob\,\mathcal{N}$. It is immediate that SO_1 is a compatible partial order on $\mathbb{Z}$, and since $S \subseteq \mathbb{P}$, that it is contained in the natural order of $\mathbb{Z}$. Moreover, since S is numerical, it generates all of $\mathbb{Z}$ as a group. Hence for every $m \in \mathbb{Z}$, there exist $p, q \in S$ such that $m = p - q$ so that $p = m + q$. Since $0\ SO_1\ q$ and $0\ SO_1\ m+q$, the order SO_1 is dense. Therefore $SO_1 \in Ob\,\mathcal{O}$.

Next let $\varphi_d \in \mathrm{Hom}_{\mathcal{N}}(S, T)$. Then $dS \subseteq T$. Hence $n = m + q$ with $q \in S$ implies that $dn = dm + dq$ where $dq \in T$. Thus $m\ SO_1\ n$ implies $dm\ TO_1\ dn$ for all $m, n \in \mathbb{Z}$. Consequently $d = \varphi_d O_1 \in \mathrm{Hom}_{\mathcal{O}}(SO_1, TO_1)$. Clearly O_1 respects compositions and identities. Therefore O_1 is a functor.

2. O_2 is a functor. Let $\xi \in Ob\,\mathcal{O}$. It is immediate that ξO_2 is a subsemigroup of $\mathbb{Z}$, and since ξ is contained in the natural order of $\mathbb{Z}$, we obtain that $\xi O_2 \subseteq \mathbb{P}$. Moreover, since ξ is dense, for every $m \in \mathbb{Z}$, there exists $q \in \mathbb{Z}$ such that $0\ \xi\ q$ and $0\ \xi\ m+q$, which implies that $m = (m+q) - q$ belongs to the subgroup generated by ξO_2 since $q, m+q \in \xi O_2$. Consequently $\xi O_2 \in Ob\,\mathcal{N}$.

Now let $d \in \mathrm{Hom}_{\mathcal{O}}(\xi, \eta)$. Then $d \in \mathbb{P}$ and $0\ \xi\ n$ implies $0\ \eta\ dn$ for all $n \in \mathbb{Z}$. It follows that $d(\xi O_2) \subseteq \eta O_2$ and thus $\varphi_d = dO_2 \in \mathrm{Hom}_{\mathcal{N}}(\xi O_2, \eta O_2)$. Clearly O_2 respects compositions and identities. Therefore O_2 is a functor.

3. $O_1 O_2 = I_{\mathcal{N}}$. For every $S \in Ob\,\mathcal{N}$ and $n \in \mathbb{P}$, we have $0\ SO_1\ n$ if and only if $n \in S$. Hence $SO_1O_2 = S$ and O_1O_2 is the identity on objects. Since it is the identity on morphisms, we conclude that $O_1O_2 = I_{\mathcal{N}}$.

4. $O_2O_1 = I_{\mathcal{O}}$. Let $\xi \in Ob\,\mathcal{O}$. Let $m, n \in \mathbb{Z}$ be such that $m\ \xi O_2 O_1\ n$. Then $n = m+q$ for some $q \in \xi O_2$, that is $0\ \xi\ q$. But then $m+0\ \xi\ m+q$ and $m\ \xi\ n$. Therefore $\xi O_2 O_1 \subseteq \xi$. Conversely, let $m\ \xi\ n$. Taking $n = m + q$, it follows that $0\ \xi\ q$ and $q \in \xi O_2$ whence $m\ \xi O_2 O_1\ n$. Thus $\xi \subseteq \xi O_2 O_1$ and equality prevails. Therefore O_2O_1 is the identity on objects and it is plainly the identity on morphisms. Consequently $O_2O_1 = I_{\mathcal{O}}$. □

Our second category in this section follows. Define

$$\begin{aligned} Ob\,\mathcal{G} = \{(G, \xi) \mid\ & G \text{ is a infinite cyclic group written additively and} \\ & \xi \text{ is a compatible dense partial order on } G\}, \end{aligned}$$

and for $(G,\xi),(H,\eta)\in Ob\,\mathcal{G}$, let

$$\mathrm{Hom}_{\mathcal{G}}\big((G,\xi),(H,\eta)\big)=\{\theta:G\to H\mid \theta \text{ is a nontrivial homomorphism of partially ordered groups}\}.$$

For the functor from $\mathcal{O}$ to $\mathcal{G}$, we define: for $\xi\in Ob\,\mathcal{O}$, let $\xi\Gamma_1=(\mathbb{Z},\xi)$, and for $d\in\mathrm{Hom}_{\mathcal{O}}(\xi,\eta)$, let $d\Gamma_1=\varphi_d$.

In order to define Γ_2, we shall need the following result.

Proposition 5.2. *Let $(G,\xi)\in Ob\,\mathcal{G}$ and let g be a generator of G. Then exactly one of the following assertions holds:*

(a) *there exists $n\in\mathbb{P}$ such that $0\ \xi\ ng$,*

(b) *there exists $n\in\mathbb{P}$ such that $0\ \xi\ n(-g)$.*

Proof. We show first that at least one of these two alternatives takes place. Since ξ is dense, there exists $mg\in G$ such that $0\ \xi\ mg$ and $0\ \xi\ mg+g$. If $m\in\mathbb{N}$, then from the second expression, we get option (a) with $n=m+1$. If $-m\in\mathbb{P}$, then from the first expression we obtain $0\ \xi\ (-m)(-g)$ and we have option (b) with $n=-m$.

Now suppose that both alternatives take place. Let $n\in\mathbb{P}$ be the least verifying (a) and $m\in\mathbb{P}$ the least verifying (b). Since $0\ \xi\ ng$ and $0\ \xi\ m(-g)$ imply $0\ \xi\ (n-m)g$ and $n-m<n$, we cannot have $n-m\in\mathbb{P}$ and thus $m-n\in\mathbb{N}$. But since $0\ \xi\ (m-n)(-g)$ and $m-n<m$, we cannot have $m-n\in\mathbb{P}$. Therefore $m=n$. But then $0\ \xi\ ng$ and $0\ \xi\ n(-g)$ whence $0\ \xi\ ng$ and $ng\ \xi\ 0$ which would imply $ng=0$, contradicting $n\in\mathbb{P}$. $\square$

For $([g],\xi)\in Ob\,\mathcal{G}$, let

$$\hat{g}=\begin{cases} g & \text{if } 0\ \xi\ ng \text{ for some } n\in\mathbb{P}\\ -g & \text{otherwise,}\end{cases}$$

and define

$$m\ ([g],\xi)\Gamma_2\ n \Longleftrightarrow m\hat{g}\ \xi\ n\hat{g}\quad (m,n\in\mathbb{Z}),$$

and for $\theta\in\mathrm{Hom}_{\mathcal{G}}\big(([g],\xi),([h],\eta)\big)$, let $\theta\Gamma_2=d$ where $\hat{g}\theta=d\hat{h}$.

We will also need: for $([g],\xi)\in Ob\,\mathcal{G}$, let

$$([g],\xi)\gamma: m\hat{g}\mapsto m\quad (m\in\mathbb{Z}).$$

Theorem 5.3. *The quadruple* $(\Gamma_1, \Gamma_2, \mathrm{id}, \gamma)$ *is an equivalence of categories* $\mathcal{O}$ *and* $\mathcal{G}$.

Proof. *1.* Γ_1 *is a functor.* This requires straightforward verification.

2. Γ_2 *is a functor.* Let $(G, \xi) \in Ob\,\mathcal{G}$ where $G = [g]$. Since G is an infinite cyclic group and $\hat{g}$ is one of its two generators, the function $(G, \xi)\gamma$ is an isomorphism of G onto $\mathbb{Z}$. Given that $(G, \xi)\Gamma_2$ is exactly the translate of ξ via the isomorphism $(G, \xi)\gamma$, we deduce that $(G, \xi)\Gamma_2$ is a compatible dense partial order on $\mathbb{Z}$, which by Proposition 5.2 satisfies the implication: $0\ (G, \xi)\Gamma_2\ n$ implies $n \in \mathbb{P}$. Hence $(G, \xi)\Gamma_2$ is contained in the natural order of $\mathbb{Z}$. Therefore $(G, \xi)\Gamma_2 \in Ob\,\mathcal{O}$.

Next let $\theta \in \mathrm{Hom}_{\mathcal{G}}\big(([g], \xi), ([h], \eta)\big)$. Suppose that $\hat{g}\theta = d\hat{h}$ for some $d \in \mathbb{Z}$. Since θ is nontrivial, we must have $d \neq 0$. Moreover, $0\ \xi\ n\hat{g}$ for some $n \in \mathbb{P}$ by Proposition 5.2. It follows that $0\ \eta\ n(\hat{g}\theta)$ and $0\ \eta\ nd\hat{h}$. By the same reference, we get that $nd \in \mathbb{P}$ which implies that $d = \theta\Gamma_2 \in \mathbb{P}$.

Now let $m\ ([g], \xi)\Gamma_2\ n$ for $m, n \in \mathbb{Z}$. Then $m\hat{g}\ \xi\ n\hat{g}$ which implies that $m(\hat{g}\theta)\ \eta\ n(\hat{g}\theta)$, that is $(md)\hat{h}\ \eta\ (nd)\hat{h}$. But then $dm\ ([h], \eta)\Gamma_2\ dn$. Therefore $d = \theta\Gamma_2 \in \mathrm{Hom}_{\mathcal{O}}\big(([g], \xi)\Gamma_2, ([h], \eta)\Gamma_2\big)$.

Let $([g], \xi) \xrightarrow{\theta} ([g'], \xi') \xrightarrow{\theta'} ([g''], \xi'')$ be two composable $\mathcal{G}$-morphisms, and $d = \theta\Gamma_2$, $d' = \theta'\Gamma_2$. Then

$$\hat{g}(\theta\theta') = (\hat{g}\theta)\theta' = (d\widehat{g'})\theta' = d(\widehat{g'}\theta') = d(d'\widehat{g''}) = (dd')\widehat{g''}$$

and thus $(\theta\theta')\Gamma_2 = dd' = (\theta\Gamma_2)(\theta'\Gamma_2)$. Hence Γ_2 preserves composition. Clearly $\mathrm{id}_{(G,\xi)}\,\Gamma_2 = 1$. Therefore Γ_2 is a functor.

3. $\Gamma_1\Gamma_2 = I_{\mathcal{O}}$. Let $\xi \in Ob\,\mathcal{O}$. Then $\xi\Gamma_1 = (\mathbb{Z}, \xi)$ and since ξ is contained in the natural order of $\mathbb{Z}$, there exists $n \in \mathbb{P}$ such that $0\ \xi\ n1$. Hence in this case $\hat{1} = 1$. Thus for any $m, n \in \mathbb{Z}$, we have $m\ \xi\Gamma_1\Gamma_2\ n$ if and only if $m\ \xi\ n$. Therefore $\xi\Gamma_1\Gamma_2 = \xi$. Let $d \in \mathrm{Hom}_{\mathcal{O}}(\xi, \eta)$. Then $d\Gamma_1 = \varphi_d$ and in particular, $\hat{1} = 1 \mapsto d = d\hat{1}$. Thus $d\Gamma_1\Gamma_2 = d$. This proves that $\Gamma_1\Gamma_2 = I_{\mathcal{O}}$.

4. $\gamma : I_{\mathcal{G}} \to \Gamma_2\Gamma_1$ *is a natural isomorphism.* Let $(G, \xi) \in Ob\,\mathcal{G}$ where $G = [g]$. Then $(G, \xi)\Gamma_2\Gamma_1 = (\mathbb{Z}, (G, \xi)\Gamma_2)$ where

$$m\ (G, \xi)\Gamma_2\ n \Longleftrightarrow m\hat{g}\ \xi\ n\hat{g} \quad (m, n \in \mathbb{Z}).$$

Since $\hat{g}$ is a generator of G, which is infinite cyclic, it is clear that

$$(G,\xi)\gamma : m\hat{g} \mapsto m \qquad (m,n \in \mathbb{Z})$$

is a $\mathcal{G}$-isomorphism between (G,ξ) and $(G,\xi)\Gamma_2\Gamma_1$.

To prove the naturalness of γ, let $\theta \in \mathrm{Hom}_{\mathcal{O}}\big((G,\xi),(H,\eta)\big)$. We must show commutativity of the diagram

$$\begin{tikzcd}
(G,\xi) \arrow[r, "(G,\xi)\gamma"] \arrow[d, "\theta"] & (G,\xi)\Gamma_2\Gamma_1 \arrow[d, "\theta\Gamma_2\Gamma_1"] \\
(H,\eta) \arrow[r, "(H,\eta)\gamma"] & (H,\eta)\Gamma_2\Gamma_1
\end{tikzcd}$$

Let $G = [g]$, $H = [h]$, and assume that $\hat{g}\theta = d\hat{h}$. With $d = \theta\Gamma_2$, we get

$$\begin{aligned}
\hat{g}[(G,\xi)\gamma \cdot (\theta\Gamma_2\Gamma_1)] &= (\hat{g}(G,\xi)\gamma)(d\Gamma_1) = 1(d\Gamma_1) = d1 = d \\
&= (d\hat{h})[(H,\eta)\gamma] = (\hat{g}\theta)[(H,\eta)\gamma] = \hat{g}[\theta \cdot (H,\eta)\gamma].
\end{aligned}$$

Since $\hat{g}$ is a generator of G, we deduce that $(G,\xi)\gamma \cdot (\theta\Gamma_2\Gamma_1) = \theta \cdot [H,\eta)\gamma]$. □

6. A category of pairs

Here we establish an isomorphism between $\mathcal{N}$ and a category whose objects are pairs of the form (r, I) where I is a $r \times r$-matrix over $\mathbb{N}$. We start with the relevant construction.

Let G be a (abelian) group and $I : G \times G \to \mathbb{N}$ be a function. We will need the following conditions on the function I: for any $a, b, c \in G$,

(A) $I(a,b) + I(ab,c) = I(a,bc) + I(b,c)$,

(C) $I(a,b) = I(b,a)$,

(N) $I(e,e) = 1$,

where e is the identity of G.

When G is finite, the following notation will prove very handy:

$$I_a = \sum_{x \in G} I(a,x) \quad (a \in G). \tag{2}$$

We may think of I as a matrix so that I_a is the sum of the matrix entries in the row (or column in view of condition (C)) of a.

Now let $r \in \mathbb{P}$ and $G = \mathbb{Z}_r$ $(= \mathbb{Z}/(r))$. For each $m \in \mathbb{P}$, let $\overline{m}$ be the residue class modulo r. In this case, we will need two more conditions:

(P) $\overline{I_{\bar{1}}} = \bar{1}$, $\quad (Q)$ $I_a \geq r \qquad (a \in \mathbb{Z}_r)$.

We are now ready to introduce the category invoked above. Let

$$\begin{aligned} Ob\,\mathcal{T} = \{(r, I) \mid\ & r \in \mathbb{P} \text{ and } I : \mathbb{Z}_r \times \mathbb{Z}_r \to \mathbb{N} \text{ is} \\ & \text{a function satisfying } (A), (C), (N), (P), (Q)\}. \end{aligned}$$

For $(r, I), (s, J) \in Ob\,\mathcal{T}$, let

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}}\big((r, I), (s, J)\big) = \{d \in \mathbb{P} \mid\ & \text{for any } a \in \mathbb{Z}_r, \text{ there exist} \\ & n \in \mathbb{N} \text{ and } b \in \mathbb{Z}_r \text{ such that } dI_a = ns + J_b\}. \end{aligned}$$

The composition of morphisms is their product and the identity morphism is $d = 1$.

The next two lemmas will prove very useful in our development.

Lemma 6.1. *Let $r \in \mathbb{P}$ and $I : \mathbb{Z}_r \times \mathbb{Z}_r \to \mathbb{N}$ be a function satisfying (A), (C) and (N).*

$$\text{(i)} \qquad I_a + I_b = rI(a, b) + I_{a+b} \quad (a, b \in \mathbb{Z}_r). \tag{3}$$

$$\text{(ii)} \qquad kI_{\bar{1}} - I_{\bar{k}} \in r\mathbb{N} \quad (1 \leq k \leq r).$$

$$\text{(iii)} \qquad I(a, \bar{0}) = 1 \quad (a \in \mathbb{Z}_r). \tag{4}$$

Proof. (i) Indeed,

$$\begin{aligned} I_a + I_b &= \sum_{x \in \mathbb{Z}_r} I(a, x) + \sum_{x \in \mathbb{Z}_r} I(b, x) = \sum_{x \in \mathbb{Z}_r} I(a, b + x) + \sum_{x \in \mathbb{Z}_r} I(b, x) \\ &= \sum_{x \in \mathbb{Z}_r} \Big(I(a, b + x) + I(b, x)\Big) \overset{(A)}{=} \sum_{x \in \mathbb{Z}_r} \Big(I(a, b) + I(a + b, x)\Big) \\ &= \sum_{x \in \mathbb{Z}_r} I(a, b) + \sum_{x \in \mathbb{Z}_r} I(a + b, x) = rI(a, b) + I_{a+b}. \end{aligned}$$

(ii) The proof is by induction on $1 \leq k \leq r$. The case $k = 1$ is trivial. Further,

$$(k+1)I_{\bar{1}} - I_{\overline{k+1}} \overset{(3)}{=} kI_{\bar{1}} + I_{\bar{1}} - \big(I_{\bar{1}} + I_{\bar{k}} - rI(\bar{1}, \bar{k})\big) = kI_{\bar{1}} - I_{\bar{k}} + rI(\bar{1}, \bar{k}).$$

The induction hypothesis is $kI_{\bar{1}} - I_{\bar{k}} \in r\mathbb{N}$ whence it follows that

$$(k+1)I_{\bar{1}} - I_{\overline{k+1}} \in r\mathbb{N}.$$

(iii) For any $a \in \mathbb{Z}_r$, by condition (A), we have

$$I(\bar{0}, \bar{0}) + I(\bar{0}, a) = I(\bar{0}, a) + I(\bar{0}, a)$$

so that $I(\bar{0}, a) = I(\bar{0}, \bar{0}) = 1$. $\square$

Note that property Lemma 6.1(iii) is valid in any group; we will use it without express mention.

Lemma 6.2. *Let* $(r, I) \in Ob\,\mathcal{T}$.

$$\text{(i)}\quad \overline{I_a} = a \qquad (a \in \mathbb{Z}_r). \qquad\qquad \text{(ii)}\quad I_{\bar{0}} = r. \tag{5}$$

Proof. (i) By Lemma 6.1(ii), for any $1 \leq k \leq r$, we obtain

$$\overline{I_{\bar{k}}} = \overline{kI_{\bar{1}}} = k\overline{I_{\bar{1}}} \overset{(P)}{=} \bar{k}.$$

(ii) $I_{\bar{0}} = \sum_{x \in \mathbb{Z}_r} I(\bar{0}, x) = \sum_{x \in \mathbb{Z}_r} 1 = r$. $\square$

Lemma 6.3. $\mathcal{T}$ *is a category.*

Proof. The only nontrivial part here is that the composition of morphisms in $\mathcal{T}$ is again a morphism in $\mathcal{T}$. Let

$$(r, I) \xrightarrow{d} (s, J) \xrightarrow{e} (t, K)$$

be composable morphisms in $\mathcal{T}$ and let $a \in \mathbb{Z}_r$. Then $dI_a = ns + J_b$ for some $n \in \mathbb{N}$ and $b \in \mathbb{Z}_s$, and

$$es \overset{(5)}{=} eI_{\bar{0}} = mt + K_c, \quad eJ_b = pt + K_q$$

for some $m, p \in \mathbb{N}$ and $c, q \in \mathbb{Z}_r$. It follows that

$$\begin{aligned} edI_a &= e(ns + J_b) = n(mt + K_c) + pt + K_q \\ &= (nm + p)t + K_c + K_q \overset{(5)}{=} (nm + p)t + tK(c, e) + K_{c+e} \\ &= (nm + p + K(c, e))t + K_{c+e}, \end{aligned}$$

whence $ed \in \mathrm{Hom}_{\mathcal{T}}\big((r, I), (t, K)\big)$. $\square$

In order to define a functor from $\mathcal{N}$ to $\mathcal{T}$, we need some preparation.

Let $S \in Ob\mathcal{N}$. We denote by r_S the least element of S in the natural order of $\mathbb{P}$, and for each $m \in S$, by $\overline{m}$ its class in $\mathbb{Z}_{r_S}$.

Since S is numerical, by Lemma 3.2(iii), there exists $c \in S$ such that $c + k \in S$ for all $k \in \mathbb{N}$. Hence the homomorphism $S \to \mathbb{Z}_{r_S}$, mapping $m \mapsto \overline{m}$, is surjective, so for each $a \in \mathbb{Z}_{r_S}$, there is an element

$$\iota_a = \min\{m \in S \mid \overline{m} = a\}. \tag{6}$$

Note that

$$\iota_{\bar{0}} = r_S. \tag{7}$$

For any $m \in S$, if $\overline{m} = a$ then $\overline{m} = \overline{\iota_a}$ and hence

$$m = rr_S + \iota_a \tag{8}$$

for some integer r. Since $m \geq \iota_a$, we must have $r \in \mathbb{N}$. The representation of m in (8) is unique, that is, if $rr_S + \iota_a = nr_S + \iota_b$ with $r, n \in \mathbb{N}$ and $a, b \in \mathbb{Z}_{r_S}$, then forming the classes modulo r_S, we see that $a = b$ and thus also $r = n$.

If $a, b \in \mathbb{Z}_{r_S}$, since $\overline{\iota_a + \iota_b} = \overline{\iota_a} + \overline{\iota_b} = a + b$, we get

$$\iota_a + \iota_b = I_S(a, b)r_S + \iota_{a+b} \tag{9}$$

for a (unique) $I_S(a, b) \in \mathbb{N}$. In this way, formula (9) defines a function

$$I_S : \mathbb{Z}_{r_S} \times \mathbb{Z}_{r_S} \to \mathbb{N}.$$

For every $S \in Ob\mathcal{N}$, define $ST_1 = (r_S, I_S)$, and for $\varphi_d \in \mathrm{Hom}_{\mathcal{N}}(S, T)$ (see Lemma 3.4(i)), let $\varphi_d T_1 = d$.

Lemma 6.4. *$T_1 : \mathcal{N} \to \mathcal{T}$ is a functor.*

Proof. Let $S \in Ob\,\mathcal{N}$. We adopt the notation introduced above and write $r = r_S$, $I = I_S$. As is well known, condition (A) for I follows from the identity in S: $(\iota_a + \iota_b) + \iota_c = \iota_a + (\iota_b + \iota_c)$ and condition (C) from the equality $\iota_a + \iota_b = \iota_b + \iota_a$. By (7), the equality $\iota_{\bar 0} + \iota_{\bar 0} = r + \iota_{\bar 0 + \bar 0}$ implies tat $I(\bar 0, \bar 0) = 1$ which verifies condition (N).

In order to prove conditions (P) and (Q) for I, we first observe that from (9) follows the equality

$$\sum_{x \in \mathbb{Z}_r} (\iota_a + \iota_x) = \sum_{x \in \mathbb{Z}_r} (I(a,x)r + \iota_{a+x})$$

whence

$$r\iota_a + \sum_{x \in \mathbb{Z}_r} \iota_x = r \sum_{x \in \mathbb{Z}_r} I(a,x) + \sum_{x \in \mathbb{Z}_r} \iota_{a+x}.$$

Since $\sum_{x \in \mathbb{Z}_r} \iota_x = \sum_{x \in \mathbb{Z}_r} \iota_{a+x}$, we deduce that

$$\iota_a = \sum_{x \in \mathbb{Z}_r} I(a,x) = I_a. \tag{10}$$

It follows that conditions (P) and (Q) for I are equivalent to $\overline{\iota_{\bar 1}} = \bar 1$ and $\iota_a \geq r$ for all $a \in \mathbb{Z}_r$, respectively. The latter relations are obvious in view of (6) and the minimality of r in S.

We have proved that $ST_1 \in Ob\,\mathcal{T}$.

Let $\varphi_d \in \mathrm{Hom}_{\mathcal{N}}(S,T)$. For any $a \in \mathbb{Z}_r$, we have $\iota_a \in S$ and $\iota_a\varphi = d\iota_a \in T$. According to equality (8) for elements of T, there exist $n \in \mathbb{N}$ and $b \in \mathbb{Z}_{r_T}$ such that $d\iota_a = nr_T + \iota_b$. We conclude from (10) for S and T, respectively, that $dI_a = nr_T + (I_T)_b$. Therefore $d = \varphi_d T_1 \in \mathrm{Hom}_{\mathcal{T}}(ST_1, TT_1)$, which proves that T_1 maps morphisms onto morphisms. That T_1 respects compositions and identities follows immediately from the definitions since $\varphi_d\varphi_e = \varphi_{de}$ and $\mathrm{id}_S = \varphi_1$. □

For each $(r, I) \in Ob\,\mathcal{T}$, let

$$(r,I)T_2 = \{mr + I_a \mid m \in \mathbb{N},\ a \in \mathbb{Z}_r\}, \tag{11}$$

and for $d \in \mathrm{Hom}_{\mathcal{T}}\big((r,I),(s,J)\big)$, let $dT_2 = \varphi_d$.

Lemma 6.5. *$T_2 : \mathcal{T} \to \mathcal{N}$ is a functor.*

Proof. Let $(r, I) \in Ob\,\mathcal{T}$. Since $I_a \geq r$ for every $a \in \mathbb{Z}_r$, it is clear that $(r, I)T_2 \subseteq \mathbb{P}$. That $(r, I)T_2$ is a subsemigroup of $\mathbb{P}$ follows from the equality

$$\begin{aligned}(mr + I_a) + (nr + I_b) &\overset{(3)}{=} (m + n)r + rI(a, b) + I_{a+b}\\ &= (m + n + I(a, b))r + I_{a+b}.\end{aligned}$$

By (5), we have $I_{\bar{0}} = r$ and by condition (P), $I_{\bar{1}} \equiv 1 \,(\mathrm{mod}\ r)$ so that $\gcd(I_{\bar{0}}, I_{\bar{1}}) = 1$. Since $I_{\bar{0}}, I_{\bar{1}} \in (r, I)T_2$, we conclude that $\gcd\big((r, I)T_2\big) = 1$ so that $(r, I)T_2 \in Ob\,\mathcal{N}$.

Let $d \in \mathrm{Hom}_{\mathcal{T}}\big((r, I), (s, J)\big)$. The elements of $(r, I)T_2$ are of the form $mr + I_a$ with $m \in \mathbb{N}$ and $a \in \mathbb{Z}_r$. By hypothesis $dI_a \in (s, J)T_2$. In particular, by Lemma 6.2(ii), we have $dI_{\bar{0}} = dr \in (s, J)T_2$ and thus

$$d(mr + I_a) \in (s, J)T_2$$

which implies that $dT_2 \in \mathrm{Hom}_{\mathcal{N}}\big((r, I)T_2, (s, J)T_2\big)$.

The remaining parts of the proof are routine. $\square$

Theorem 6.6. *The functor T_1 is an isomorphism of $\mathcal{N}$ onto $\mathcal{T}$ with inverse T_2.*

Proof. In view of Lemmas 6.4 and 6.5, it remains to prove that (a) $T_1T_2 = I_{\mathcal{N}}$ and (b) $T_2T_1 = I_{\mathcal{T}}$.

(a) Let $S \in Ob\,\mathcal{N}$ and $ST_1 = (r_S, I_S)$, see (6) and (7). Then

$$ST_1T_2 \overset{(11)}{=} \{mr_S + \iota_a \mid m \in \mathbb{N},\ a \in \mathbb{Z}_{r_S}\} \overset{(8)}{=} S.$$

Hence T_1T_2 is the identity on objects. The equality $\varphi_d T_1T_2 = dT_2 = \varphi_d$ shows the same for morphisms.

(b) Let $(r, I) \in Ob\,\mathcal{T}$. By Lemma 6.5, $(r, I)T_2$ is a numerical semigroup, which by condition (Q) has $r = I_{\bar{0}}$ as its least element, that is $r_{(r,I)T_2} = r$. Moreover, for every $a \in \mathbb{Z}_r$, we have $\overline{mr + I_a} = \overline{I_a} = a$ and thus

$$I_a = \min\{x \in (r, I)T_2 \mid \bar{x} = a\}.$$

Comparing (3) and (9), we see that $I = I_{(r,I)T_2}$. Consequently

$$(r, I)T_2T_1 = \big(r_{(r,I)T_2}, I_{(r,I)T_2}\big) = (r, I).$$

In addition, $dT_2T_1 = \varphi_d T_1 = d$ and thus T_2T_1 is the identity both on objects and on morphisms. □

Corollary 6.7. *Let $(r, I) \in Ob\,\mathcal{T}$. Then $(r, I)T_2$ is generated by the set $\{I_{\bar{0}}, \dots, I_{\overline{r-1}}\}$. Consequently, if $S \in Ob\,\mathcal{N}$, then S admits a set of generators having r_S elements.*

Proof. By Lemma 6.2(ii), we have $I_{\bar{0}} = r$. Now recalling formula (11) and the definition of the functor T_2, the first assertion follows. The second follows from Theorem 6.6 since for $S \in Ob\,\mathcal{N}$ we have $S \cong ST_1T_2$ and $r = r_S$. □

Conditions (A), (C) and (N) occur in various contexts, but mainly in Schreier-type extensions of groups and semigroups. We will encounter them also in the next section. As contrasted to this, conditions (P) and (Q) are new. The first one is a normalization type condition which can also be written more explicitly as $I_{\bar{1}} \equiv 1 \pmod{r}$, and bears only upon the row sum of $\bar{1}$. The second condition imposes a restriction upon all row sums, recall that $I_{\bar{0}} = r$. It should be noticed that on account of their strength, it was possible to achieve the isomorphism of categories $\mathcal{N}$ and $\mathcal{T}$.

7. A category of functions

There is a category whose objects are similar to those of the category $\mathcal{T}$ with the same morphisms. This is a consequence of the observation: the expression for the effect of the functor T_2 on objects of $\mathcal{T}$ uses only I_a's and no $I(a, b)$'s. Hence we can omit all reference to $I(a, b)$ and consider only I_a's. The latter form a r-tuple $(I_{\bar{0}}, \dots, I_{\overline{r-1}})$ of positive integers. The conditions these integers must satisfy can be derived from the conditions $I(a, b)$ must fulfill, that is (A), (C), (N), (P) and (Q). All this will be expressed categorically. Let

$$Ob\,\mathcal{F} = \Big\{(r, \alpha) \mid r \in \mathbb{P},\ \alpha : \mathbb{Z}_r \to \mathbb{P} \text{ is a function satisfying}$$

$$\left.\begin{array}{ll} (N') \ \ \bar{0}\alpha = r, \qquad (P') \ \bar{1}\alpha \equiv 1 \pmod r, \\ (Q') \ \ a\alpha \geq r \quad (a \in \mathbb{Z}_r), \\ (R) \ \ a\alpha + b\alpha - (a+b)\alpha \in r\mathbb{N} \quad (a, b \in \mathbb{Z}_r) \end{array}\right\},$$

and for $(r, \alpha), (s, \beta) \in Ob\,\mathcal{F}$, let

$$\begin{aligned} \mathrm{Hom}_{\mathcal{F}}\big((r,\alpha),(s,\beta)\big) = \{d \in \mathbb{P} \mid \ & \text{for every } a \in \mathbb{Z}_r, \text{ there exist} \\ & n \in \mathbb{P} \text{ and } b \in \mathbb{Z}_s \text{ such that } d(a\alpha) = ns + b\beta\}. \end{aligned}$$

The composition of morphisms is the product and the identity morphism is $d = 1$.

For $(r, I) \in Ob\,\mathcal{T}$, define $(r, I)\Phi_1 = (r, \alpha)$ where

$$\alpha : a \mapsto I_a \quad (a \in \mathbb{Z}_r)\,.$$

For $(r, \alpha) \in Ob\,\mathcal{F}$, let $(r, I)\Phi_2 = (r, I)$ where

$$I(a, b) = \frac{1}{r}(a\alpha + b\alpha - (a+b)\alpha) \quad (a, b \in \mathbb{Z}_r).$$

For $i = 1, 2$, let $d\Phi_i = d$.

Theorem 7.1. *The functor Φ_1 is an isomorphism of $\mathcal{T}$ onto $\mathcal{F}$ with inverse Φ_2.*

Proof. Let $(r, I) \in Ob\,\mathcal{T}$. Clearly, we have the implications

$$(N) \Longrightarrow (N'),\ (P) \Longrightarrow (P'),\ (Q) \Longrightarrow (Q'). \tag{12}$$

Moreover, by Lemma 6.1(i), (R) holds as well. Thus $(r, I)\Phi_1 \in Ob\,\mathcal{F}$. The remaining requirements for a functor follow easily. Hence $\Phi_1 : \mathcal{T} \to \mathcal{F}$ is a functor.

Next let $(r, \alpha) \in Ob\,\mathcal{F}$ and $(r, I) = (r, \alpha)\Phi_2$. Condition (R) assures that $I(a, b) \in \mathbb{N}$. For any $a, b, c \in \mathbb{Z}_r$, we get

$$\begin{aligned} I(a,b) + I(a+b,c) &= \frac{1}{r}\big(a\alpha + b\alpha - (a+b)\alpha\big) \\ &\quad + \frac{1}{r}\big((a+b)\alpha + c\alpha - (a+b+c)\alpha\big) \end{aligned}$$

$$\begin{aligned} I(a, b+c) + I(b, c) &= \frac{1}{r}\big(a\alpha + (b+c)\alpha - (a+b+c)\alpha\big) \\ &\quad + \frac{1}{r}\big(b\alpha + c\alpha - (b+c)\alpha\big) \end{aligned}$$

which verifies condition (A);

$$I(a, b) = a\alpha + b\alpha - (a+b)\alpha = b\alpha + a\alpha - (b+a)\alpha = I(b, a)$$

giving (C). Clearly, the reverse implications in (12) are valid. Therefore $(r, \alpha)\Phi_2 \in Ob\,\mathcal{T}$. The conditions for a functor are straightforward to verify. Therefore $\Phi_2 : \mathcal{F} \to \mathcal{T}$ is a functor.

Now let $(r, I) \in Ob\,\mathcal{T}$. Then $(r, I)\Phi_1\Phi_2 = (r, \alpha)\Phi_2 = (r, J)$ where for any $a, b \in \mathbb{Z}_r$, we have

$$J(a, b) = \frac{1}{r}(I_a + I_b - I_{a+b}) \overset{(3)}{=} I(a, b).$$

For $(r, \alpha) \in Ob\,\mathcal{F}$, we get $(r, \alpha)\Phi_2\Phi_1 = (r, I)\Phi_2 = (r, \beta)$, where for any $a \in \mathbb{Z}_r$,

$$\begin{aligned} a\beta = I_a &= \sum_{x \in \mathbb{Z}_r} I(a, x) = \sum_{x \in \mathbb{Z}_r} \frac{1}{r}\big(a\alpha + x\alpha - (a+x)\alpha\big) \\ &= \frac{1}{r}\Big(\sum_{x \in \mathbb{Z}_r} a\alpha + \sum_{x \in \mathbb{Z}_r} x\alpha - \sum_{x \in \mathbb{Z}_r} (a+x)\alpha\Big) = \frac{1}{r} r(a\alpha) = a\alpha. \end{aligned}$$

Since for $i = 1, 2$, $d\Phi_i = d$ we conclude that $\Phi_1\Phi_2 = I_{\mathcal{T}}$ and $\Phi_2\Phi_1 = I_{\mathcal{F}}$. □

8. The semigroup $\mathbb{N}(G, I)$

Here we impose some of the semigroup conditions we have studied so far on the semigroup $\mathbb{N}(G, I)$ defined below. In the next section we will relate these semigroups to numerical semigroups.

Let G be a (abelian) group and $I : G \times G \to \mathbb{N}$ be a function satisfying conditions (A), (C) and (N). On $\mathbb{N} \times G$ define a multiplication by

$$(m, a)(n, b) = (m + n + I(a, b), ab).$$

Denote the resulting groupoid by $\mathbb{N}(G, I)$.

By ([1], Theorem 4.3), $S = \mathbb{N}(G, I)$ is a commutative cancellative idempotent-free and subarchimedean (i.e., there exists $z \in S$ such that for every $a \in S$, there exist $n \in \mathbb{P}$ and $x \in S$ such that $z^n = ax$) semigroup, and conversely, every semigroup having these properties is isomorphic to some $\mathbb{N}(G, I)$.

Recall that a semigroup S is called a $\mathfrak{N}$-*semigroup* if S is a commutative cancellative idempotent-free and archimedean (i.e., every $z \in S$ has the property in the definition of subarchimedean above).

Proposition 8.1. *Let $S = \mathbb{N}(G, I)$.*

(i) *S is power joined if and only if G is periodic.*

(ii) *The following conditions are equivalent.*

(a) *S is archimedean and finitely generated.*

(b) *S is power joined and finitely generated.*

(c) *G is finite.*

Proof. (i) Let S be power joined. Then for any $a \in G$ there exist $m, n \in \mathbb{P}$ such that $(0, a)^m = (0, e)^n$. Hence $a^m = e^n = e$ and G is periodic. Conversely, assume that G is a periodic group and let $a \in G$ be of order k. Then for any $m \in \mathbb{N}$, we have

$$(m, a)^k \left(km + \sum_{i=1}^{k-1} I(a, a^i), a^k \right) = (l - 1, e),$$

where $l = km + \sum_{i=1}^{k-1} I(a, a^i) + 1$. Since $l > 0$ and

$$(0, e)^l = \left(\sum_{i=1}^{l-1} I(e, e), e \right) = (l - 1, e),$$

we get $(m, a)^k = (0, e)^l$. From this, we conclude that S is power joined.

(ii) (a) implies (b) implies (c). Then S is a $\mathfrak{N}$-semigroup and the assertions follow from ([2], Theorems II.7.3 and II.7.4).

(c) implies (a). By part (i), S is power joined and thus is a $\mathfrak{N}$-semigroup. The equalities

$$(m, a) = (0, e)^m (0, a) \quad (m \in \mathbb{P},\ a \in G)$$

show that the set $\{(0,a) \mid a \in G\}$ generates S. $\square$

Proposition 8.2. *Let $S \in Ob\,\mathcal{S}$. Every group homomorphic image of S is cyclic and finite.*

Proof. By Lemma 4.2, S is isomorphic to a numerical semigroup so that we may assume that $S \in Ob\,\mathcal{N}$. Let φ be a homomorphism of S onto a group G. Since S is numerical, we have $\mathbb{Z} = \{s - t \mid s, t \in S\}$. Then φ extends to an epimorphism $\bar{\varphi} : \mathbb{Z} \to G$ such that $(s-t)\bar{\varphi} = s\varphi\,(t\varphi)^{-1}$ for all $s, t \in S$. Hence G is a homomorphic image of $\mathbb{Z}$ and it is cyclic. If G were infinite, $\bar{\varphi}$ would be an isomorphism and thus $\varphi : S \to G$ would be also. But this would mean that S is a group which is manifestly impossible. $\square$

Theorem 8.3. *Let $S = \mathbb{N}(G, I)$. Then S is power joined finitely generated and power cancellative if and only if G is finite, say of order r and*

$$I_a \equiv I_b \pmod{r} \Longrightarrow a = b. \tag{13}$$

In such a case, G is also cyclic.

Proof. *Necessity.* Since S is power joined and finitely generated, by Proposition 8.1(ii), G must be finite. Power cancellativity in view of ([4], Proposition 3.7) implies condition (13).

Sufficiency. By Proposition 8.1, S is power joined and finitely generated. Hence S is a $\mathfrak{N}$-semigroup and the second reference above yields that S is also power cancellative.

Now assume all these conditions. Then S is nontrivial commutative power joined finitely generated and power cancellative so $S \in Ob\,\mathcal{S}$. The mapping

$$(m,a) \mapsto a \quad \big((m,a) \in S\big)$$

is a homomorphism of S onto G and Proposition 8.2 gives that G is cyclic. $\square$

Through the sequence of statements in Proposition 8.1 and Theorem 8.3, we started with power joined, strengthened by finitely generated, and finished with adding power cancellative. This of course can be ramified by other combinations of these properties.

9. Semigroups $\mathbb{N}(G, I)$ and a related category

We introduce here a category related to $\mathcal{T}$ and study the interplay of this category, semigroups $\mathbb{N}(G, I)$ satisfying the conditions in Theorem 8.3 and the category $\mathcal{N}$. First we generalize the category $\mathcal{T}$. Let

$$Ob\mathcal{U} = \{(r, I) \mid r \in \mathbb{P} \text{ and } I : \mathbb{Z}_r \times \mathbb{Z}_r \to \mathbb{N} \text{ is a function satisfying } (A), (C), (N)\}.$$

and for $\mathrm{Hom}_{\mathcal{U}}$ we adopt the same definition as for $\mathrm{Hom}_{\mathcal{T}}$ but now for objects of $\mathcal{U}$. Hence $\mathcal{U}$ is a category having $\mathcal{T}$ as a full subcategory. We also define T by the same formulas as for T_2 thereby obtaining a functor $T : \mathcal{U} \to \mathcal{P}$.

By $m \mid n$ we mean that m divides n.

Lemma 9.1. *Let $(r, I) \in Ob\mathcal{U}$ and $d \in \mathbb{P}$. Then the following statements are equivalent.*

(i) $d \,|\, I_{\bar{0}}$ *and* $d \,|\, I_{\bar{1}}$. (ii) $d \,|\, I_a$ *for all* $a \in \mathbb{Z}_r$. (iii) $d \,|\, s$ *for all* $s \in (r, I)T$.

Proof. *(i) implies (ii).* By Lemma 6.2(ii), $d \,|\, I_{\bar{0}} = r$ and by Lemma 6.1(i), we have $I_{\bar{1}} + I_{\bar{1}} = rI(\bar{1}, \bar{1}) + I_{\bar{2}}$ so that $d \,|\, I_{\bar{2}}$. Continuing this procedure, we get the assertion.

(ii) implies (iii). By Lemma 6.2(ii) and we have $I_{\bar{0}} = r$ and hence by (11), the definition of T_2 and thus also of T, we have that $I_{\bar{0}}, \dots, I_{\overline{r-1}}$ generate $(r, I)T$, whence follows the claim.

(iii) implies (i). This is trivial. $\square$

Theorem 9.2. *Let $(r, I) \in Ob\mathcal{U}$ and on $S = \mathbb{N}(\mathbb{Z}_r, I)$ define a function τ by*

$$\tau : (m, a) \mapsto mr + I_a. \tag{14}$$

Then τ is a homomorphism of S onto $(r, I)T$ which induces the least power cancellative congruence θ on S. Moreover, $\gcd\big((r, I)T\big) = \gcd(r, I_{\bar{1}})$.

Proof. For any $(m, a), (n, b) \in S$, we obtain

$$\begin{aligned}\big((m, a)(n, b)\big)\tau &= (m + n + I(a, b), a + b)\tau = r\big(m + n + I(a, b)\big) + I_{a+b} \\ &\overset{(3)}{=} rm + rn + I_a + I_b = (m, a)\tau + (n, a)\tau\end{aligned}$$

and τ is a homomorphism; it is surjective by the construction of $(r, I)T$.

Since $S\tau$ is a subsemigroup of $\mathbb{P}$, which is power cancellative, we deduce that $\bar{\tau}$, the congruence induced by τ, is power cancellative. By Lemma 3.6(i), it follows that $\theta \subseteq \bar{\tau}$ by the minimality of θ.

For the opposite inclusion, we proceed as follows. Let $a \in \mathbb{Z}_r$. Then

$$(0, a)^r = \left(\sum_{i=1}^{r-1} I(a, a^i), \bar{0}\right)$$

and applying τ, we get

$$rI_a = r\sum_{i=1}^{r-1} I(a, a^i) + I_{\bar{0}} \stackrel{(4)}{=} r\sum_{i=1}^{r-1} I(a, a^i) + r = r\left(\sum_{i=1}^{r-1} I(a, a^i) + 1\right)$$

whence

$$I_a = \sum_{i=1}^{r-1} I(a, a^i) + 1. \tag{15}$$

Now let $(m, a)\ \bar{\tau}\ (n, b)$. Then $(m, a)\tau = (n, b)\tau$ whence $mr + I_a = nr + I_b$ which by (15) yields

$$\begin{aligned}(m, a)^r &= \left(mr + \sum_{i=1}^{r-1} I(a, a^i), \bar{0}\right) = (mr + I_a - 1, \bar{0}) \\ &= (nr + I_b - 1, \bar{0}) = \left(nr + \sum_{i=1}^{r-1} I(b, b^i), \bar{0}\right) = (n, b)^r.\end{aligned}$$

Hence by (1), we have $(m, a)\ \theta\ (n, b)$. Therefore $\bar{\tau} \subseteq \theta$ and equality prevails.

The final assertion of the theorem follows directly from Lemma 9.1. $\square$

Theorem 9.3. *Let $(r, I) \in Ob\mathcal{U}$ and $S = \mathbb{N}(\mathbb{Z}_r, I)$. Then the following statements are equivalent*

(i) *S is power cancellative.*

(ii) *τ is injective (see (14)).*

(iii) *$(r, I)T$ is numerical.*

(iv) $\gcd(r, I_{\bar{1}}) = 1$.

(v) $I_a \equiv 0 \pmod{r} \Longrightarrow a = \bar{0}$ $(a \in \mathbb{Z}_r)$.

(vi) $I_a \equiv I_b \pmod{r} \Longrightarrow a = b$ $(a, b \in \mathbb{Z}_r)$.

Proof. By Theorem 9.2, (i) and (ii) are equivalent, and also (iii) and (iv) are equivalent.

(ii) implies (iii). Suppose that $\gcd(r, I_{\bar{1}}) = d > 1$. Then $r = dk$ and $I_{\bar{1}} = dl$ for some $k, l \in \mathbb{P}$. Hence $kI_{\bar{1}} = kdl = rl$. By Lemma 6.1(ii), $kI_{\bar{1}} - I_{\bar{k}} \in r\mathbb{N}$ so that $rl = I_{\bar{k}} + rt$ for some $t \in \mathbb{N}$. Thus $(l-1, \bar{0})\tau = (t, \bar{k})\tau$ which contradicts the hypothesis that τ is injective since $1 < k < r$ so that $\bar{k} \neq \bar{0}$.

(iv) implies (v). Let $I_{\bar{k}} \equiv 0 \pmod{r}$. By the cited reference, we deduce that $kI_{\bar{1}} = \bar{0}$ in $\mathbb{Z}_r$. By virtue of the hypothesis, $\overline{I_{\bar{1}}}$ is a unit in the ring $\mathbb{Z}_r$ and hence $\bar{k} = \bar{0}$.

(v) implies (iv). Let $I_a \equiv I_b \pmod{r}$. We may assume that $I_a \geq I_b$ so that $I_a = I_b + rm$ for some $m \in \mathbb{N}$. By Lemma 6.1(i), we have

$$I_{a-b} + I_b = rI(a-b, b) + I_a$$

whence

$$I_{a-b} + I_b = r\bigl(I(a-b, b) + m\bigr) + I_b$$

so that $I_{a-b} = r\bigl(I(a-b, b) + m\bigr) \in r\mathbb{N}$. By hypothesis, $a - b = \bar{0}$ and thus $a = b$.

(vi) implies (ii). Assume that $(m, a)\tau = (n, b)\tau$, that is $rm + I_a = rn + I_b$. Then $I_a \equiv I_b \pmod{r}$ and the hypothesis implies that $a = b$. But then $rm = rn$ so $m = n$. This proves that τ is injective. $\square$

With the conditions (A), (C), (N), (P) and (Q), we restricted the function I so strongly that we arrived at the following situation. Let $\mathbb{N}(\mathbb{Z}_r, I) \cong \mathbb{N}(\mathbb{Z}_s, J)$. Applying τ, we get $(r, I)T \cong (s, J)T$, see Theorem 9.3. By Lemma 3.4(ii), we get $(r, I)T_2 = (s, J)T_2$ since these semigroups are numerical. Now Theorem 6.6 yields that $(r, I) = (s, J)$ so that $r = s$ and $I = J$.

In constructing the Tamura representation of a $\mathfrak{N}$-semigroup, we generally arrive at many $\mathbb{N}(G, I)$ where two such $\mathbb{N}(G, I)$ and $\mathbb{N}(H, J)$ have very little in common. The Tamura representation is based on the choice of a "standard element". Here we may choose the least element of a numerical semigroup to be such an element. This is not sufficient to produce the uniqueness of the representation discussed in the preceding paragraph. The function I must be further restricted by condition (Q). This is, to our best knowledge, the only instance of a unique Tamura representation.

10. An example

There remains the herculean task of transferring various parameters associated to a numerical semigroup to the objects we have constructed in these categories. A yet larger labor would be required to establish the counterparts of statements valid for these parameters in the context of new objects, some of which are quite different from numerical semigroups. All this is relegated to the interested and assiduous reader. We limit ourselves here to small examples showing the various associations for a given numerical semigroup.

In the diagram of categories and functors at the end of Section 1, in the position of $\mathcal{N}$ we put the numerical semigroup $S = \{2, 3, \dots\}$. Inclusion functors $I_{\mathcal{N},\mathcal{S}}$, $I_{\mathcal{N},\mathcal{P}}$, $I_{\mathcal{T},\mathcal{U}}$ and the functor Γ_1 do not change objects and thus we may restrict our attention to the functors O_1, T_1 and Φ_1.

Recall that

$$m\ SO_1\ n \Longleftrightarrow n = m + s \text{ for some } s \in S \quad (m, n \in \mathbb{Z})$$

so that in our case

$$m\ SO_1\ n \Longleftrightarrow n - m \geq 2 \quad (m, n \in \mathbb{Z}).$$

Next $ST_1 = (r_S, I_S)$ where

r_S is the least element of S,

I_S satisfies equation (9) with definition (6).

Hence in our case: $r_S = 2$ and (9) becomes

$$\iota_a + \iota_b = 2\, I_S(a, b) + \iota_{a+b} \qquad (a, b \in \mathbb{Z}_2).$$

By (6), we have

$$\iota_{\bar{0}} = 2, \quad \iota_{\bar{1}} = 3 \tag{16}$$

and thus

$$\iota_a + \iota_b \qquad\qquad \iota_{a+b} \qquad\qquad \iota_a + \iota_b - \iota_{a+b}$$

$$\begin{array}{c|c|c|}
 & \bar{0} & \bar{1} \\ \hline
\bar{0} & 4 & 5 \\ \hline
\bar{1} & 5 & 6
\end{array}
\qquad
\begin{array}{c|c|c|}
 & \bar{0} & \bar{1} \\ \hline
\bar{0} & 2 & 3 \\ \hline
\bar{1} & 3 & 2
\end{array}
\qquad
\begin{array}{c|c|c|}
 & \bar{0} & \bar{1} \\ \hline
\bar{0} & 2 & 2 \\ \hline
\bar{1} & 2 & 4
\end{array}$$

whence

$$ST_1 = \left(2, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right).$$

For $(r, I) \in Ob\,\mathcal{T}$, by definition $(r, I)\Phi_1 = (r, \alpha)$ where

$$\alpha : a \mapsto I_a \quad (a \in \mathbb{Z}_r).$$

By (10), $I_a = \iota_a$ so by (16), we obtain $I_{\bar{0}} = 2$ and $I_{\bar{1}} = 3$ and hence

$$ST_1\Phi_1 = \left(2, \begin{pmatrix} \bar{0} & \bar{1} \\ 2 & 3 \end{pmatrix}\right).$$

Summarizing

$$\prec \;\xleftarrow{O_1}\; S \;\xrightarrow{T_1}\; \left(2, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right) \xrightarrow{\Phi_1} \left(2, \begin{pmatrix} \bar{0} & \bar{1} \\ 2 & 3 \end{pmatrix}\right).$$

where $m \prec n \Leftrightarrow n - m \geq 2$. □

The above details may serve as a prototype for handling more complex examples. For instance, the treatment of numerical semigroups such as $S = \{k, k+1, \dots\}$, or with holes, say $S = \{2, 4, 5, 6, \dots\}$, follows the same lines as above.

Hence for $S = \{k, k+1, \dots\}$, we have

$S \xrightarrow{O_1} \prec$ where $m \prec n \Leftrightarrow n - m \geq k \quad (m, n \in \mathbb{Z})$,

$$S \xrightarrow{T_1} (k, I) \xrightarrow{\Phi_1} \left(k, \begin{pmatrix} \bar{0} & \bar{1} & \cdots & \overline{k-1} \\ k & k+1 & \cdots & 2k-1 \end{pmatrix}\right),$$

where, for $0 \leq r, s < k$, it holds

$$I(\bar{r}, \bar{s}) = \begin{cases} 1 & r + s < k \\ 2 & r + s \geq k. \end{cases}$$

Similarly, for $S = \{2, 4, 5, 6, \dots\}$, we obtain

$S \xrightarrow{O_1} \prec$ where $m \prec n \Leftrightarrow n - m \geq 4$ or $n = m + 2 \quad (m, n \in \mathbb{Z})$,

$$S \xrightarrow{T_1} \left(2, \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}\right) \xrightarrow{\Phi_1} \left(2, \begin{pmatrix} \bar{0} & \bar{1} \\ 2 & 5 \end{pmatrix}\right).$$

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