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Optimal control for a general class of stochastic initial boundary value problems subject to distributed and boundary noise

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Abstract. In this paper we consider a class of stochastic evolution equations arising from parabolic initial boundary value problems subject to both boundary and distributed noise. We prove existence and regularity of mild solutions. Then we consider a controlled version of the model and prove the existence of optimal controls for partially observed problems using a class of relaxed controls containing both distributed controls and point controls.

1. Introduction

Over the last two decades great interest has been shown in the area of stochastic maximum principle for finite dimensional stochastic systems [2], [3]. See also the extensive references given therein. In recent years necessary conditions of optimality for infinite dimensional stochastic systems have also appeared in several papers [4]–[6], [8]. [11]–[16], [19], [22]. For a brief survey on recent developments in systems and control theory, the reader is referred to [22]. The subject continues to attract many researchers in the field and continues to expand. Most of these papers develop necessary conditions of optimality in the form of maximum principle (or minimum principle) without proving the existence of optimal controls with the exception of few papers [5], [6], [8], [19], [21]. In [6], the drift and the diffusion operators are assumed to map within the same Hilbert space. In [19], this is generalized admitting drift and the diffusion operators mapping a smaller space to a larger state space. In this paper, we generalize this further.

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We consider more general drift and diffusion operators mapping a subspace of the original Hilbert space, generated by the fractional powers of the semigroup generator, to an extended Hilbert space obtained by completion of the original space in the norm topology determined by the negative fractional powers of the semigroup generator. This certainly applies to a broader class of systems with unbounded drift and diffusion operators and described by stochastic partial differential equations subject to noise in the spatial domain as well as in its boundary. In [21], Guatteri and Masiero also consider stochastic control problems with distributed and boundary noise for second order spde. Our system model covers those of [21] and beyond. They use stochastic Hamiltonian system to prove the existence of optimal control for Bolza problem. Our technique is natural and direct, based on purely functional analysis. We consider also standard Bolza problem as in [21] and also several nonstandard control problems giving results on the the existence of optimal controls. It appears that the technique used by Guaterri and Masiero will not work for the nonstandard problems considered in this paper. The rest of the paper is organized as follows.

In Section 2, we introduce the system model subject to spatial as well as boundary noise leading to a class of nonlinear stochastic integral equations determined by the semigroup generated by the linear part of the partial differential operator. In Section 3, after presenting the basic assumptions, we prove the existence of mild solutions and present the regularity properties thereof. In Section 4, we introduce a metric topology on the space of relaxed controls and prove continuous dependence of solutions on controls with respect to this metric topology. In Section 5, we consider the question of existence of optimal controls proving the existence of partially observed controls which are signed Borel measure valued stochastic processes containing both point and distributed controls. Some interesting nonstandard control problems are also considered in this section. The paper is concluded with an example.

2. System model with distributed and boundary noise

A very Large class of dynamic systems arising in physical sciences and engineering can be described by the following class of partial differential equations:

$$\begin{aligned} \partial\varphi/\partial t + \mathcal{A}\varphi = f(t,\xi,\varphi) + \sigma(t,\xi,\varphi)V_d(t,\xi), \quad (t,\xi) \in I \times \Sigma, \\ (\mathcal{B}\varphi)(t,\xi) = V_b(t,\xi), \quad (t,\xi) \in I \times \partial\Sigma \\ \varphi(0,\xi) = \varphi_0(\xi), \quad \xi \in \Sigma \end{aligned} \tag{1}$$

subject to distributed and boundary noise $\{V_d, V_b\}$ defined on the domain $\Sigma \subset R^n$ and its boundary $\partial\Sigma$ respectively. The domain Σ is an open bounded set with smooth boundary and $I = (0, T]$ is an interval. The operator $\mathcal{A}$ is generally given by

$$(\mathcal{A}\varphi)(\xi) \equiv \sum_{|\alpha| \leq 2m} a_\alpha(\xi) D^\alpha \varphi, \text{ on } \Sigma, \tag{2}$$

with multi index $\alpha = \{\alpha_i\}_{i=1}^n,\ |\alpha| \equiv \sum_{i=1}^n \alpha_i,\ \alpha_i \in N_0 \equiv \{0, 1, 2, \dots\}$.

The boundary operator $\mathcal{B}$ is also a partial differential operator of order at most $2m - 1$, given by

$$\mathcal{B}\varphi = \{\mathcal{B}_j, j = 1, 2, \dots, m\} \quad (\mathcal{B}_j\varphi)(\xi) \equiv \sum_{|\beta| \leq m_j \leq 2m-1} b_\beta^j(\xi) D^\beta \varphi, \quad \xi \in \partial\Sigma, \tag{3}$$

where $\beta = \{\beta_i\}_{i=1}^n,\ |\beta| \equiv \sum \beta_i,\ \beta_i \in N_0,\ i = 1, 2, \dots, n$. The nonlinear operators $\{f, \sigma\}$ are defined shortly. Under fairly general assumptions on the coefficients $\{a_\alpha, b_\beta\}$ and smoothness of the boundary $\partial\Sigma$, the system (1) can be formulated as a first order evolution equation on the Hilbert space $E \equiv L_2(\Sigma)$.

For nonhomogeneous boundary conditions one needs the trace theorem which states that under sufficient smoothness conditions on the boundary $\partial\Sigma$ and the coefficients $\{b_\beta,\ |\beta| \leq 2m - 1\}$, the boundary operator $\mathcal{B}|_{\operatorname{Ker}\mathcal{A}}$ is an isomorphism of $W_2^{2m}(\Sigma)/\operatorname{Ker}\mathcal{B}$ on to $\Pi_{j=1}^m W_2^{2m-m_j-1/2}(\partial\Sigma)$ called the trace space. Thus it has a bounded inverse denoted by $\mathcal{R} \equiv (\mathcal{B}|_{\operatorname{Ker}\mathcal{A}})^{-1}$. Then the system (1) can formulated as an abstract differential equation given by

$$\begin{aligned} d/dt(z + \mathcal{R}V_b) + Az &= f(t, z + \mathcal{R}V_b) + \sigma(t, z + \mathcal{R}V_b)V_d, \\ z(0) + (\mathcal{R}V_b)(0) &= \varphi_0, \quad t \in I, \end{aligned} \tag{4}$$

where z is the solution of the homogeneous boundary value problem and $\varphi = z + \mathcal{R}V_b$ is the solution of the initial boundary value problem. Strictly speaking, φ is the solution of the following equivalent stochastic integral equation on E,

$$\begin{aligned} \varphi(t) = S(t)\varphi_0 &+ \int_0^t S(t-\tau) f(\tau, \varphi(\tau)) d\tau + \int_0^t AS(t-\tau)\mathcal{R}\ dW_b(\tau) \\ &+ \int_0^t S(t-\tau)\sigma(\tau, \varphi(\tau)) dW_d(\tau), \quad t \geq 0, \end{aligned} \tag{5}$$

where $\{V_b, V_d\}$ are the distributional derivatives of abstract Wiener processes $\{W_b, W_d\}$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$. For detailed proof leading to the above results and more on nonhomogeneous nonlinear boundary value problems see [9, Chapter 3, p. 59] and [10, Example 3.2.8, p. 85]. In case $\{f, \sigma\}$ are also dependent on control, the integral equation (5) turns into

$$\varphi(t) = S(t)\varphi_0 + \int_0^t S(t-\tau)f(\tau, \varphi(\tau), u_\tau)d\tau + \int_0^t AS(t-\tau)\mathcal{R}\; dW_b(\tau) + \int_0^t S(t-\tau)\sigma(\tau, \varphi(\tau), u_\tau)\; dW_d(\tau), \quad t \geq 0. \tag{6}$$

We consider this system on a finite time interval $I \equiv [0, T]$.

3. Basic assumptions, control and system analysis

In order to study control problems involving the system (1) we must now define the drift and the diffusion operators $\{f, \sigma\}$ with controls in their arguments. Let U be a compact metric space and $\mathcal{B}(U)$ the class of Borel subsets of U and $\mathcal{M}(U)$ the space of signed Borel measures on $\mathcal{B}(U)$. Let $\mathcal{G}_t$, $t \geq 0$, denote any family of complete non decreasing subsigma algebras of the sigma algebra $\mathcal{F}_t$, $t \geq 0$. For admissible controls we choose the space $L_\infty^a(I, \mathcal{M}(U))$ which consists of weak star measurable $\mathcal{G}_t$-adapted random processes defined on I with values in the space of signed Borel measures $\mathcal{M}(U)$. Let $L_1^a(I, C(U))$ denote the space of $\mathcal{G}_t$-adapted Bochner integrable random processes with values in the Banach space $C(U)$, the space of bounded continuous functions on U with the standard sup norm topology. It is easy to see that the topological dual of this space is given by $L_\infty^a(I, \mathcal{M}(U))$. In other words any continuous linear functional ℓ on $L_1^a(I, C(U))$ has the representation

$$\ell(\eta) \equiv \mathbf{E} \int_{I \times U} \eta(t, \zeta)\mu_t(d\zeta)dt \tag{7}$$

for some $\mu \in L_\infty^a(I, \mathcal{M}(U))$. For admissible control policies we choose a suitable subset $\mathcal{U}_{ad}$ of the space $L_\infty^a(I, \mathcal{M}(U))$. Let M_c be a norm (variation norm) bounded weak star closed convex subset of $\mathcal{M}(U)$ and choose

$$\mathcal{U}_{ad} \equiv \{u \in L_\infty^a(I, \mathcal{M}(U)) : u_t \in M_c \text{ for all } t \in I\}$$

as the set of admissible controls. There are several possible choices for U giving several possible physically important class of control policies as discussed below in (C1)–(C3).

(C1): Let Σ_c be a closed bounded, and hence a compact, subset of $\Sigma \subset R^n$ equipped with any standard metric topology of R^n and take $U = \Sigma_c$ and $\mathcal{M}(U) =$

$\mathcal{M}(\Sigma_c)$. This gives a class of controls which are signed measures on a part of the spatial domain Σ.

(C2): Let $U \equiv \{\zeta_1, \zeta_2, \dots \zeta_m\} \subset \Sigma$, $m \in N$, and take $\mathcal{M}(U) \equiv \mathcal{M}(\{\zeta_1, \zeta_2, \dots \zeta_m\})$. This gives a class of point controls (Dirac measures) supported at any finite set of distinct points $\{\zeta_1, \zeta_2, \dots \zeta_m\}$ of Σ. A control u in this class has the representation

$$u_t(\cdot) \equiv \sum_{i=1}^{m} \pi_i(t)\delta_{\zeta_i}(\cdot)$$

with $\pi_i \in L^a_\infty(I)$, the class of essentially bounded $\mathcal{G}_t$-adapted random processes where $\delta_{\zeta_i}(\cdot)$ is the Dirac measure concentrated at the point $\zeta_i \in \Sigma$.

(C3): Let U be a closed bounded subset of the Banach space $L_\infty(\Sigma_c)$ which is the topological dual of $L_1(\Sigma_c)$. By Alaoglu's theorem, the set U is weak star compact and since $L_1(\Sigma_c)$ is separable, it follows from Theorem V.5.1 [[7], Dunford & Schwartz, p. 426] that it is metrizable. Thus with respect to this metric topology, U is a compact metric space. In this case, the admissible controls are signed measures on $\mathcal{B}(U)$, the class of Borel subsets of U. In particular, through Dirac measures, we obtain (regular) controls which are functions of time with values in $U \subset L_\infty(\Sigma_c)$. In other words, the controls are elements of $L^a_\infty(I, U) \subset L^a_\infty(I, L_\infty(\Sigma_c))$. For example, for any $\psi \in L^a_1(I, C(U))$, and any $\mu \in L^a_\infty(I, \mathcal{M}(U))$ we have the standard duality pairing

$$\mu(\psi) \equiv \int_{I\times U} \psi(t, v)\mu_t(dv)dt.$$

If μ is taken as the Dirac measure $\delta_{v(t)}(\cdot)$ concentrated along the path $v(t) \in U$, the above integral reduces to $\mu(\psi) \equiv \int_I \psi(t, v(t))dt$.

Remark 3.1. Regular controls denoted by $\mathcal{U}_r$ are bounded measurable functions from $I \times \Sigma_c$ to R. Note that this class of controls is a special case of the class of admissible controls described above in (C3). Clearly, the set of control policies $\mathcal{U}_{ad}$ contain such controls and therefore our control policies are far more general. Of crucial importance is that U is not required to be convex. In fact U can be a set of discrete points (so non convex) admitting point controls. This is not possible with regular controls and may even lead to chattering controls and nonexistence of optimal controls if they are used. Further, by Krien-Milman theorem $\mathcal{U}_{ad} = \text{clco}(\text{ext}(\mathcal{U}_{ad}))$ where the extreme points are the Dirac measures along paths in U and $\mathcal{U}_r$ is dense in $\mathcal{U}_{ad}$. Thus any relaxed control can be approximated (to any degree of accuracy) by regular controls.

Now we are prepared to introduce the basic assumptions.

Assumptions:

(A1): $-A$ is the infinitesimal generator of an analytic semigroup $S(t), t \geq 0$, on the Hilbert space E satisfying

$$\sup\{\|S(t)\|_{\mathcal{L}(E)},\ t \in I\} \leq M < \infty.$$

Without loss of generality we may assume that $0 \in \rho(A)$, the resolvent set of A. If not, one can choose a large enough positive number c such that $0 \in \rho(cI + A)$ and compensate it by adding $c\varphi$ to the drift $f(t, \varphi)$.

In view of the above assumption, we can construct a family of interpolation spaces by using the domain of the fractional powers of A giving $E_\alpha = [D(A^\alpha)]$ where $[D(A^\alpha)]$ is a Banach space with respect to the norm topology $|x|_\alpha \equiv |A^\alpha x|_E$. Clearly

$$[D(A)] \equiv E_1 \hookrightarrow E_\alpha \hookrightarrow E_\beta \hookrightarrow E_0 \equiv E$$

for $0 \leq \beta < \alpha \leq 1$. In fact these are Hilbert spaces endowed with the natural inner product: $(x, y)_{E_\alpha} \equiv (A^\alpha x, A^\alpha y)_E$ for $\alpha \in [0, 1]$. For $\alpha \geq 0$, let $E_{-\alpha}$ denote the completion of the space E with respect to the norm topology $|x|_{E_{-\alpha}} \equiv |A^{-\alpha} x|_E$. Clearly these are Banach spaces. Any continuous linear functional ℓ on E_α has the representation $\ell(v) = (\zeta, v)_{E_{-\alpha}, E_\alpha}$ for some $\zeta \in E_{-\alpha}$. Thus $E_{-\alpha} \cong (E_\alpha)^*$, the topological dual of E_α. This leads to the following chain of Banach spaces along with their topological duals

$$E_1 \hookrightarrow E_\alpha \hookrightarrow E_\beta \hookrightarrow E_0 \equiv E \hookrightarrow E_{-\beta} \hookrightarrow E_{-\alpha} \hookrightarrow E_{-1}.$$

(A2): There exists a $\gamma \in [0, 1)$, $\theta \in [0, 1)$ such that $f : I \times E_\gamma \times U \longrightarrow E_{-\theta}$ is measurable in the first argument and continuous with respect to the second and third. Further, there exists a constant $K_\theta \neq 0$ such that

$$|f(t, e, \xi)|^2_{E_{-\theta}} \leq K^2_\theta\{1 + |e|^2_{E_\gamma}\}, \quad |f(t, e_1, \xi) - f(t, e_2, \xi)|^2_{E_{-\theta}} \leq K^2_\theta\{|e_1 - e_2|^2_{E_\gamma}\}$$

for all $e, e_1, e_2 \in E_\gamma$ and $(t, \xi) \in I \times U$.

(A3): The incremental covariance of the Brownian motion W_d is denoted by $Q_d \in \mathcal{L}^+(H)$ (positive not necessarily nuclear). $\sigma : I \times E_\gamma \times U \longrightarrow \mathcal{L}(H, E_{-\theta})$ is Borel measurable in the first argument and continuous in the second and third and there exists a constant $K_{\theta,d} \neq 0$ such that for all $(t, e, \xi) \in I \times E_\gamma \times U$ and $e_1, e_2 \in E_\gamma$

$$|\sigma(t, e, \xi)|^2_{d,\theta} \equiv \mathrm{Tr}((A^{-\theta}\sigma)Q_d(A^{-\theta}\sigma)^*) \leq K^2_{d,\theta}\{1 + |e|^2_{E_\gamma}\},$$

$$\text{and} \quad |\sigma(t, e_1, \xi) - \sigma(t, e_2, \xi)|^2_{d,\theta} \leq K^2_{d,\theta}\{|e_1 - e_2|^2_{E_\gamma}\}$$

where $|\sigma|^2_{d,\theta} = \mathrm{Tr}(A^{-\theta}\sigma Q_d(A^{-\theta}\sigma)^*)$.

(A4): The numbers $\gamma \in [0,1)$, $\theta \in [0,1)$ are such that $0 \leq \gamma + \theta < 1/2$.

Before we can proceed with the analysis of the integral equation (8) we must find an appropriate function space $Y(\partial\Sigma)$ for the Brownian motion W_b representing the boundary noise. Towards this goal, we construct a family of interpolation spaces involving the Dirichlet map (boundary operator) $\mathcal{B}/\operatorname{Ker}(\mathcal{A})$ and the (quotient) Sobolev spaces $W_2^{2m}(\Sigma)/\operatorname{Ker}(\mathcal{B})$ and the corresponding trace spaces $Y(\partial\Sigma) \equiv \Pi_{j=1}^m W_2^{2m-m_j-1/2}(\partial\Sigma)$. We have already noted that the operator

$$\mathcal{B}/\operatorname{Ker}(\mathcal{A}) \in iso\big(W_2^{2m}(\Sigma)/\operatorname{Ker}(\mathcal{B}), Y(\partial\Sigma)\big).$$

For $\alpha \in (0,1]$, let us introduce the extrapolation (interpolation) spaces

$$X_\alpha \equiv W_2^{2\alpha m}(\Sigma)/\operatorname{Ker}(\mathcal{B}) \quad \text{and} \quad Y_\alpha \equiv \Pi_{j=1}^m W_2^{2\alpha m-m_j-1/2}(\partial\Sigma).$$

Clearly, for each $\alpha \in (0,1]$, the operator $\mathcal{B}/\operatorname{Ker}(\mathcal{A}) \in \mathrm{iso}(X_\alpha, Y_\alpha)$. Hence we have $(\mathcal{B}/\operatorname{Ker}(\mathcal{A}))^{-1} \equiv \mathcal{R} \in \mathcal{L}(Y_\alpha, X_\alpha)$. The state space for the Brownian motion W_b can then be chosen as any of the interpolation spaces Y_α with $\alpha \in (0,1]$ in the sense that, for any $t \geq 0$ and $y^* \in Y_\alpha^*$ (the dual of Y_α), we have

$$P\{|(W_b(t), y^*)_{Y_\alpha, Y_\alpha^*}| < \infty\} = 1$$

and $(W_b(t), y^*)_{Y_\alpha, Y_\alpha^*}$ is an $\mathcal{F}_t$-adapted real valued Gaussian random process with mean zero and variance $t(Q_b y^*, y^*)$ with Q_b being a positive nuclear operator from Y_α^* to Y_α.

To prove the existence, uniqueness and regularity properties of solutions of integral equations like (6) we must introduce the appropriate spaces where they may reside. Let $B_\infty^a(I, E_\gamma)$ denote the vector space of E_γ valued $\mathcal{F}_t$-adapted random processes having square integrable norms (with respect to the measure P) which are bounded on I. Furnished with the norm topology,

$$\|x\|_{B_\infty^a(I,E_\gamma)} \equiv (\sup\{\mathbf{E}|x(t)|^2_{E_\gamma},\ t \in I\})^{1/2},$$

$B_\infty^a(I, E_\gamma)$ is a closed subspace of the Banach space $L_\infty^a(I, L_2(\Omega, E_\gamma))$ and hence a Banach space.

For convenience of presentation, throughout the rest of the paper we use the notation

$$f(t,x,u) \equiv \int_U f(t,x,\xi)u(d\xi), \sigma(t,x,u) \equiv \int_U \sigma(t,x,\xi)u(d\xi) \tag{8}$$

for any $u \in \mathcal{M}(U)$. Now we can prove the following existence result.

Theorem 3.2. *Consider the evolution equation* (1) *modeled as the controlled integral equation* (6) *rewritten as*

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)f(\tau, x(\tau), u_\tau)d\tau + \int_0^t AS(t-\tau)\mathcal{R}\ dW_b(\tau)$$
$$+ \int_0^t S(t-\tau)\sigma(\tau, x(\tau), u_\tau)\ dW_d(\tau), \quad t \geq 0, \tag{9}$$

with $\{x_0, W_d, W_b\}$ *being mutually statistically independent. Suppose the assumptions* (A1)–(A4) *hold and that the state space for the Brownian motion* W_d *is* H *with incremental covariance operator* $Q_d \in \mathcal{L}_s^+(H)$, *and that for* W_b *is the space* Y_α *for any* $\alpha \in (\gamma+1/2, 1]$ *with incremental covariance operator* $Q_b \in \mathcal{L}_s^+(Y_\alpha^*, Y_\alpha)$. *Then, for every* $\mathcal{F}_0$*-measurable* E_γ *valued random variable* $x_0 \in L_2(\Omega, E_\gamma)$, *and control* $u \in \mathcal{U}_{ad}$, *the integral equation has a unique solution* $x \in B_\infty^a(I, E_\gamma)$. *Further the solution has a continuous modification.*

Proof. Consider the operator F,

$$(Fx)(t) \equiv S(t)x_0 + \int_0^t S(t-\tau)f(\tau, x(\tau), u_\tau)d\tau + \int_0^t AS(t-\tau)\mathcal{R}\ dW_b(\tau)$$
$$+ \int_0^t S(t-\tau)\sigma(\tau, x(\tau), u_\tau)\ dW_d(\tau), \tag{10}$$

on $B_\infty(I, E_\gamma)$ for any fixed $u \in \mathcal{U}_{ad}$ and any E_γ-valued $\mathcal{F}_0$-measurable initial state x_0 having finite second moment. Since both W_d and W_b are $\mathcal{F}_t$-adapted and $x(t), t \in I$, is $\mathcal{F}_t$ -adapted and u_t is $\mathcal{G}_t(\subset \mathcal{F}_t)$-adapted, we conclude that $(Fx)(t)$ is $\mathcal{F}_t$-adapted. We prove that $F : B_\infty^a(I, E_\gamma) \longrightarrow B_\infty^a(I, E_\gamma)$. Let $x \in B_\infty^a(I, E_\gamma)$ with $x(0) = x_0$. Since $-A$ is the generator of an analytic semigroup, the fractional powers $\{A^\alpha\}, \alpha \in [0, 1]$, are well defined and $[D(A^\alpha)] = E_\alpha$. Also recall [7, Ahmed, Theorem 3.3.16, p. 101] that for each $\alpha \in [0, 1]$ there exists a positive constant C_α such that

$$\|A^\alpha S(t)\|_{\mathcal{L}(E)} \leq C_\alpha / t^\alpha, \quad \forall\ t > 0.$$

We use the generic constant $C_\alpha, \alpha \in [0, 1]$. For simplicity of presentation, let $\{z_1, z_2, z_3, z_4\}$ denote the first, second, third and the fourth term on the right hand side of the expression (10). Considering first $\{z_1, z_2, z_4\}$, it follows from straightforward computation using assumptions (A1)–(A4) that

$$\mathbf{E}|z_1(t)|_{E_\gamma}^2 \equiv |A^\gamma z_1(t)| = |A^\gamma S(t)x_0|_E = |S(t)A^\gamma x_0|_E \leq M^2 \mathbf{E}|x_0|_{E_\gamma}^2 \quad \forall\ t \in I, \tag{11}$$

$$\mathbf{E}|z_2(t)|^2_{E_\gamma} \le \frac{t^{1-2(\gamma+\theta)}}{(1-2(\gamma+\theta))}(C_{\gamma+\theta}K)^2 \int_0^t \big(1+\mathbf{E}|x(s)|^2_{E_\gamma}\big)ds$$
$$\le \frac{t^{2(1-(\gamma+\theta))}}{(1-2(\gamma+\theta))}(C_{\gamma+\theta}K)^2\big(1+\sup_{0\le s\le t}\mathbf{E}|x(s)|^2_{E_\gamma}\big) \quad \forall\, t\in I, \tag{12}$$

and

$$\mathbf{E}|z_4(t)|^2_{E_\gamma}$$
$$= \mathbf{E}\int_0^t \mathrm{tr}\big(A^{\gamma+\theta}S(t-s)A^{-\theta}\sigma(s,x(s),u_s)\big)Q_d\big(A^{\gamma+\theta}S(t-s)A^{-\theta}\sigma(s,x(s),u_s)\big)^* ds$$
$$\le \int_0^t \|A^{\gamma+\theta}S(t-s)\|^2_{\mathcal{L}(E)}K^2_{d,\theta}(1+\mathbf{E}|x(s)|^2_\gamma)ds$$
$$\le \frac{t^{1-2(\gamma+\theta)}}{(1-2(\gamma+\theta))}(C_{\gamma+\theta}K_{d,\theta})^2\big(1+\sup_{0\le s\le t}\mathbf{E}|x(s)|^2_{E_\gamma}\big). \tag{13}$$

By assumption (A4) all the terms on righthand side of (12) and (13) are positive and finite for all $t\in I\equiv[0,T]$. For the third term z_3 given by the stochastic integral related to the boundary noise,

$$z_3(t)\equiv\int_0^t AS(t-\tau)\mathcal{R}\; dW_b(\tau), t\ge 0,$$

we use the interpolation space Y_α and the fractional powers of the operator A. Since the Dirichlet map $\mathcal{R}\in\mathcal{L}(Y_\alpha,X_\alpha)$ for any $\alpha\in(0,1]$ and $A^\alpha: X_\alpha\longrightarrow E$ we have $A^\alpha\mathcal{R}\in\mathcal{L}(Y_\alpha,E)$. It follows from the property of analytic semigroups as mentioned above that $A^{1-\alpha}S(t)\in\mathcal{L}(E)$ for $t>0$. Thus we can rewrite the above expression as

$$z_3(t)=\int_0^t A^{1-\alpha}S(t-\tau)A^\alpha\mathcal{R}\; dW_b(\tau), \quad t\ge 0.$$

The state space for W_b is Y_α for any $\alpha\in(\gamma+1/2,1]$, and by our assumption its incremental covariance operator $Q_b\in\mathcal{L}_1^+(Y_\alpha^*,Y_\alpha)$ (positive nuclear). Using the properties of the fractional powers of the operator A as indicated above, and computing the expected value of the square of the norm and recalling that $\alpha\in(\gamma+1/2,1]$, we have

$$\mathbf{E}|A^\gamma z_3(t)|^2_E = \int_0^t \mathrm{tr}\{(A^{1+\gamma-\alpha}S(t-s)A^\alpha\mathcal{R})Q_b(A^{1+\gamma-\alpha}S(t-s)A^\alpha\mathcal{R})^*\}\; ds$$
$$= \int_0^t |A^{1+\gamma-\alpha}S(t-s)A^\alpha\mathcal{R}|^2_{Q_b}ds$$
$$\le \mathrm{Tr}(A^\alpha\mathcal{R}Q_b(A^\alpha\mathcal{R})^*)\int_0^t \|A^{1+\gamma-\alpha}S(t-s)\|^2_{\mathcal{L}(E)}ds$$

$$
\begin{aligned}
&= |A^{\alpha}\mathcal{R}|^2_{Q_b} \int_0^t \|A^{1+\gamma-\alpha}S(t-s)\|^2_{\mathcal{L}(E)} ds \\
&\leq |A^{\alpha}\mathcal{R}|^2_{Q_b} \int_0^t C^2_{1+\gamma-\alpha}/(t-s)^{2(1+\gamma-\alpha)} ds \\
&\leq C^2_{1+\gamma-\alpha}|A^{\alpha}\mathcal{R}|^2_{Q_b} t^{2(\alpha-\gamma)-1}/(2(\alpha-\gamma)-1) < \infty \ \forall\ t \in I. \qquad (14)
\end{aligned}
$$

Thus it follows from the inequalities (11)–(14) that

$$
\begin{aligned}
\|Fx\|^2_{B^a_\infty(I,E_\gamma)} \leq \Big\{ & M^2\mathbf{E}|x_0|^2_E + (C_{\gamma+\theta}K)^2 \frac{T^{2(1-(\gamma+\theta))}}{1-2(\gamma+\theta)}(1+\sup_{t\in I}\{\mathbf{E}|x(t)|^2_{E_\gamma}\}) \\
&+ \frac{T^{1-2(\gamma+\theta)}}{(1-2(\gamma+\theta))}(C_{\gamma+\theta}K_{d,\theta})^2(1+\sup_{t\in I}\{\mathbf{E}|x(t)|^2_{E_\gamma}\}) \\
&+ (C^2_{1+\gamma-\alpha}/(2(\alpha-\gamma)-1))T^{(2(\alpha-\gamma)-1)}|A^{\alpha}\mathcal{R}|^2_{Q_b} \Big\} < \infty. \qquad (15)
\end{aligned}
$$

It follows from the above inequality that the operator F maps $B^a_\infty(I,E_\gamma)$ to $B^a_\infty(I,E_\gamma)$. We prove that it has a fixed point in $B^a_\infty(I,E_\gamma)$. Using the expression (10), with any pair $\{x,y\} \in B^a_\infty(I,E_\gamma)$ satisfying $x(0) = y(0) = x_0$, and the basic assumptions (A1)–(A4) it is easy to verify that

$$
\sup_{0\leq s\leq t} \mathbf{E}|F(x)(s) - F(y)(s)|^2_{E_\gamma} \leq \eta(t) \sup_{0\leq s\leq t} \mathbf{E}|x(s)-y(s)|^2_{E_\gamma} \qquad (16)
$$

where the function η is given by

$$
\eta(t) \equiv 2\frac{t^{1-2(\gamma+\theta)}}{1-2(\gamma+\theta)}\big\{t(C_{\gamma+\theta}K)^2 + (C_{\gamma+\theta}K_{d,\theta})^2\big\}, \quad t \in I. \qquad (17)
$$

Clearly, it follows from the assumptions (A2)–(A4) that η is a continuous monotone increasing function of its argument and bounded on bounded intervals such as $I \equiv [0,T]$ with $\eta(0) = 0$. If $\eta(t) < 1$ for all $t \in I$, take $T_1 = T$. If not, then there exists a $T_1 > 0$ such that $\eta(T_1) < 1$. Using the expression (16) for the interval $I_{T_1} \equiv [0,T_1]$ one can easily deduce that

$$
\|Fx - Fy\|_{B^a_\infty(I_{T_1},E_\gamma)} \leq \sqrt{\eta(T_1)}\|x-y\|_{B^a_\infty(I_{T_1},E_\gamma)}. \qquad (18)
$$

Thus the operator F is a contraction in the Banach space $B^a_\infty(I_{T_1},E_\gamma)$ and hence it has a unique fixed point in $B^a_\infty(I_{T_1},E_\gamma)$. Since $I \equiv [0,T]$ is a compact interval it can be covered by a finite number of intervals of length T_1. Thus the solution can be extended to cover the entire interval I in a finite number of steps. Therefore, we conclude that the operator F has a unique fixed point in $B^a_\infty(I,E_\gamma)$. Hence the integral equation (9) has a unique solution in $x \in B^a_\infty(I,E_\gamma)$. Further, it follows from the well known factorization technique due to Da Prato and Zabczyk [1] that x has continuous modification. This completes the proof. □

Corollary 3.3. *Consider the system* (9) *and suppose the assumptions of Theorem 3.2 hold and the set of admissible controls $\mathcal{U}_{ad}$ is given by the topological space $L^a_\infty(I, M_c) \subset L^a_\infty(I, \mathcal{M}(U))$. Then the solution set $\Xi \equiv \{x(u), u \in \mathcal{U}_{ad}\}$ is a bounded subset of $B^a_\infty(I, E_\gamma)$.*

Proof. For any $u \in \mathcal{U}_{ad}$, let $x(u) \in B^a_\infty(I, E_\gamma)$ denote the unique solution of equation (9). Then using the integral equation (9) and following similar procedure as in the proof of theorem 3.2, one obtains the following inequality

$$\|x(u)\|^2_{B^a_\infty(I_T,E_\gamma)} \leq c_1(x_0, T) + c_2(T)\|x(u)\|^2_{B^a_\infty(I_T,E_\gamma)}. \tag{19}$$

where

$$c_1(x_0,T) \equiv 2^3\left\{M^2\mathbf{E}|x_0|^2_{E_\gamma} + \frac{T^{2(1-(\gamma+\theta))}}{1-2(\gamma+\theta)}(C_{\gamma+\theta}K)^2 + \frac{T^{(1-2(\gamma+\theta))}}{1-2(\gamma+\theta)}(C_{\gamma+\theta}K_{d,\theta})^2 \right.$$
$$\left. + \frac{T^{2(\alpha-\gamma)-1}}{2(\alpha-\gamma)-1}C^2_{1+\gamma-\alpha}|A^\alpha\mathcal{R}|^2_{Q_b}\right\},$$

and

$$c_2(T) \equiv 2^3\left\{\frac{T^{2(1-(\gamma+\theta))}}{1-2(\gamma+\theta)}(C_{\gamma+\theta}K)^2 + \frac{T^{(1-2(\gamma+\theta))}}{1-2(\gamma+\theta)}(C_{\gamma+\theta}K_{d,\theta})^2\right\}.$$

Since $\alpha \in (\gamma + 1/2, 1]$, and by assumption (A4), $\gamma + \theta < 1/2$, both $c_1(x_0, T)$ and $c_2(T)$ are finite for every $T < \infty$. Note that $c_2(T)$ is also a continuous monotone increasing function of $T \in [0, \infty)$ and $c_2(0) = 0$. Hence there exists a finite $\tau_1 \in [0, \infty)$ such that $c_2(\tau_1) < 1$. Thus for $T \equiv \tau_1$, it follows from (19) that

$$\|x(u)\|^2_{B^a_\infty(I_{\tau_1},E_\gamma)} \leq \frac{c_1(\tau_1)}{(1-c_2(\tau_1))} < \infty. \tag{20}$$

Hence the set $\{x(u), u \in \mathcal{U}_{ad}\}$ is a bounded subset of $B^a_\infty(I_{\tau_1}, E_\gamma)$. Clearly this implies that the attainable set $\mathcal{A}(\tau_1, x_0) \equiv \{x(\tau_1, u), u \in \mathcal{U}_{ad}\}$ is a bounded subset of $L_2(\Omega, E_\gamma)$ and that they are $\mathcal{F}_{\tau_1}$ adapted. Now considering the interval $[\tau_1, T]$ with the initial state $x(\tau_1) \in \mathcal{A}(\tau_1, x_0)$ the reader can easily verify that the inequality (19) takes the form

$$\|x(u)\|^2_{B^a_\infty(I_{[\tau_1,T]},E_\gamma)} \leq c_1(x(\tau_1), T-\tau_1) + c_2(T-\tau_1)\|x(u)\|^2_{B^a_\infty(I_{[\tau_1,T]},E_\gamma)}. \tag{21}$$

Choosing $T \equiv \tau_2$ so that $c_2(\tau_2 - \tau_1) < 1$, it follows from (21) that

$$\|x(u)\|^2_{B^a_\infty(I_{[\tau_1,\tau_2]},E_\gamma)} \leq \frac{c_1(x(\tau_1), \tau_2-\tau_1)}{(1-c_2(\tau_2-\tau_1))}. \tag{22}$$

Since $x(\tau_1) \in \mathcal{A}(\tau_1, x_0)$ and it is a bounded subset of $L_2(\Omega, E_\gamma)$ it follows from (22) that $\|x(u)\|^2_{B^a_\infty(I_{[\tau_1,\tau_2]}, E_\gamma)} < \infty$. Continuing this process we conclude that

$$\sup\{\|x(u)\|_{B^a_\infty(I,E_\gamma)},\ u \in \mathcal{U}_{ad}\} < \infty.$$

This completes the proof. □

Remark 3.4. Note that we have assumed that $\gamma, \theta \in [0,1)$ and that $\gamma + \theta < (1/2)$. We do not know if the above results can be proved for the critical value $1/2$.

4. Continuous dependence of solutions on control

In the study of optimal control we need the continuity of the map $u \longrightarrow x$, that is, control to solution map. This is crucial in the proof of existence of optimal controls. Since continuity is critically dependent on the topology, we must mention the topologies used for the control space and the solution space. For the solution space we have already the norm topology on $B^a_\infty(I, E_\gamma)$ as seen in the previous section. So we must consider the control space.

The natural topology for admissible controls is the weak star topology in $L^a_\infty(I, M_c) \subset L^a_\infty(I, \mathcal{M}(U))$. This topology is too weak. We need to consider a slightly stronger topology, for example, a metric topology as follows. Consider the measure space $(I \times \Omega, \mathcal{B}(I) \times \mathcal{F}, \lambda(dt) \times P(d\omega))$ where λ denotes the Lebesgue measure. Let $\mathcal{P}$ denote the sigma algebra of predictable subsets of the set $I \times \Omega$ with respect to the filtration $\mathcal{G}_{t\geq 0} \subseteq \mathcal{F}_{t\geq 0}$. Let $\mu = \mu(dt \times d\omega)$ denote the restriction of the product measure $\lambda(dt) \times P(d\omega)$ to the predictable sigma field $\mathcal{P}$. Recall that $L^a_\infty(I, \mathcal{M}(U))$ denotes the space of random processes adapted to the filtration $\mathcal{G}_{t\geq 0}$ in the weak star sense and taking values in the space of signed Borel measures. Consider now the predictable measure space $(I \times \Omega, \mathcal{P}, \mu)$ and introduce the following topological space $\Lambda \equiv \Lambda((I\times\Omega, \mathcal{P}, \mu), M_c)$ of $\mathcal{P}$ measurable $M_c(\subset \mathcal{M}(U))$ valued random processes. We can introduce a suitable metric topology on this space and turn this into a complete metric space as follows. First note that U is a compact metric space and therefore the Banach space $C(U)$, furnished with usual norm topology, is separable. Let $\{\varphi_n\}$ denote any countable set dense in the closed unit ball of the B-space $C(U)$ and μ the measure introduced above which is defined on the predictable sigma algebra $\mathcal{P}$. Define the function $\rho : \Lambda \times \Lambda \longrightarrow [0, \infty]$ by

$$\rho(u,v) \equiv \sum_{n=1}^{\infty} (1/2^n) \frac{\left(\int_{I\times\Omega} |u_{t,\omega}(\varphi_n) - v_{t,\omega}(\varphi_n)|^2 d\mu\right)^{1/2}}{1 + \left(\int_{I\times\Omega} |u_{t,\omega}(\varphi_n) - v_{t,\omega}(\varphi_n)|^2 d\mu\right)^{1/2}},$$

where

$$u_{t,\omega}(\varphi) \equiv \int_U \varphi(\eta) u_{t,\omega}(d\eta)$$

for $\varphi \in C(U)$. This defines a metric on the topological space Λ and that completion of this with respect to this metric topology turns this into a complete metric space. Convergence in this metric is equivalent to w^*-convergence in M_c for μ-a.e $(t,\omega) \in I \times \Omega$. We denote this metric topology by $\tau_{\mu w*}$ and choose any compact subset $\mathcal{U}_{ad} \subset (\Lambda, \tau_{\mu w*})$ as the class of admissible controls. Now we present a result on continuity of the control to solution map.

Theorem 4.1. *Consider the system* (9) *driven by the control* $u \in \mathcal{U}_{ad}$ *and suppose the assumptions of Theorem 3.2 hold. Then the control to solution map* $u \longrightarrow x$ *is continuous with respect to the relative* $\tau_{\mu w^*}$ *topology on* $\mathcal{U}_{ad}$ *and the strong (norm) topology on* $B^a_\infty(I, E_\gamma)$.

Proof. Consider the net $u^\alpha \in \mathcal{U}_{ad}, \alpha \in D$ (a countably accessible directed set), and suppose $u^\alpha \xrightarrow{\tau_{\mu w*}} u^o$. Let $\{x^\alpha, x^o\} \in B^a_\infty(I, E_\gamma)$ denote the solutions of the integral equation (9) corresponding to the controls $\{u^\alpha, u^o\}$ respectively. We show that $x^\alpha \xrightarrow{s} x^o$ in $B^a_\infty(I, E_\gamma)$. Clearly it follows from equation (9) corresponding to the controls $\{u^o, u^\alpha\}$ that

$$x^o(t) - x^\alpha(t) = \int_0^t S(t-s)\big(f(s, x^o(s), u^o_s) - f(s, x^\alpha, u^\alpha_s)\big)\, ds$$
$$+ \int_0^t S(t-s)\big(\sigma(s, x^o(s), u^o_s) - \sigma(s, x^\alpha, u^\alpha_s)\big)\, dW_d(s), \quad t \in I. \tag{23}$$

The reader can easily verify that this identity is equivalent to the following one

$$\begin{aligned} x^o(t) - x^\alpha(t) = &\int_0^t S(t-s)\big(f(s, x^o(s), u^\alpha_s) - f(s, x^\alpha(s), u^\alpha_s)\big)\, ds \\ &+ \int_0^t S(t-s)\big(\sigma(s, x^o(s), u^\alpha_s) - \sigma(s, x^\alpha(s), u^\alpha_s)\big)\, dW_d(s) \\ &+ e^\alpha_1(t) + e^\alpha_2(t), \quad t \in I, \end{aligned} \tag{24}$$

where $\{e^\alpha_1, e^\alpha_2\}$ are given by

$$e^\alpha_1(t) \equiv \int_0^t S(t-s)\big(f(s, x^o(s), u^o_s) - f(s, x^o(s), u^\alpha_s)\big)\, ds, \quad t \in I, \tag{25}$$

$$e^\alpha_2(t) \equiv \int_0^t S(t-s)\big(\sigma(s, x^o(s), u^o_s) - \sigma(s, x^o(s), u^\alpha_s)\big)\, dW_d(s), \quad t \in I. \tag{26}$$

Taking expected value of the norm square and using the assumptions (A1)–(A4) and carrying out some computations, it follows from (24) that for all $t \in I$, we have

$$\sup_{0\le s\le t} \mathbf{E}|x^o(s)-x^\alpha(s)|^2_{E_\gamma} \le 2^3\Big\{\eta(t)\sup_{0\le s\le t}\mathbf{E}|x^o(s)-x^\alpha(s)|^2_{E_\gamma} + \sup_{0\le s\le t}\big(\mathbf{E}|e_1^\alpha(s)|^2_{E_\gamma}+\mathbf{E}|e_2^\alpha(s)|^2_{E_\gamma}\big)\Big\}. \quad (27)$$

Since we can choose $t_1 > 0$ such that $2^3\eta(t_1) < 1$, it follows from the above inequality that

$$\sup_{0\le s\le t_1} \mathbf{E}|x^o(s)-x^\alpha(s)|^2_{E_\gamma} \le \frac{1}{(1-2^3\eta(t_1))}\sup_{0\le s\le t_1}\big(\mathbf{E}|e_1^\alpha(s)|^2_{E_\gamma}+\mathbf{E}|e_2^\alpha(s)|^2_{E_\gamma}\big). \quad (28)$$

For any compact interval like I, this process can be continued till the entire interval is covered. So it suffices to verify that the righthand member of (28) converges to zero. Starting with (25) we note that

$$\begin{aligned}\mathbf{E}|e_1^\alpha(t)|^2_{E_\gamma} &\le \mathbf{E}\bigg(\int_0^t \|A^{\gamma+\theta}S(t-s)\|_{\mathcal{L}(E)}|A^{-\theta}[f(s,x^o(s),u_s^o)-f(s,x^o(s),u_s^\alpha)]|_E ds\bigg)^2\\ &\le \int_0^t \|A^{\gamma+\theta}S(t-s)\|^2_{\mathcal{L}(E)}ds\ \mathbf{E}\int_0^t |f(s,x^o(s),u_s^o)-f(s,x^o(s),u_s^\alpha)|^2_{E_{-\theta}}ds. \quad (29)\end{aligned}$$

By assumption (A4) it follows from (29) that

$$\mathbf{E}|e_1^\alpha(t)|^2_{E_\gamma} \le C^2_{\gamma+\theta}\frac{t^{1-2(\gamma+\theta)}}{1-2(\gamma+\theta)}\mathbf{E}\int_0^t |f(s,x^o(s),u_s^o)-f(s,x^o(s),u_s^\alpha)|^2_{E_{-\theta}}ds. \quad (30)$$

By assumption (A2) we have $|f(s,x^o(s),u_s^o)-f(s,x^o(s),u_s^\alpha)|^2_{E_{-\theta}} \le 4K_\theta^2(1+|x^o(s)|^2_{E_\gamma})$ and therefore by Theorem 3.2 the integrand in (30) is dominated by the integrable function as displayed above. Further the integrand converges to zero μ-a.e as $u^\alpha \xrightarrow{\tau_{\mu w^*}} u^o$. Since D is a countably accessible directed set, Lebesgue dominated convergence theorem applies [20]. Thus it follows from Lebesgue dominated convergence theorem applied to (30) that $\lim_\alpha \mathbf{E}|e_1^\alpha(t)|^2_{E_\gamma} = 0$ as $u^\alpha \xrightarrow{\tau_{\mu w^*}} u^o$ for each $t \in I$ (and even uniformly on I). Considering (26) we note that

$$\begin{aligned}\mathbf{E}|e_2^\alpha(t)|^2_{E_\gamma} = \mathbf{E}\int_0^t \mathrm{tr}\Big(&A^{\gamma+\theta}S(t-s)A^{-\theta}\big[\sigma(s,x^o(s),u_s^o)-\sigma(s,x^o(s),u_s^\alpha)\big]Q_d\\ &\times\big(A^{\gamma+\theta}S(t-s)A^{-\theta}\big[\sigma(s,x^o(s),u_s^o)-\sigma(s,x^o(s),u_s^\alpha)\big]\big)^*\Big)ds.\end{aligned}$$

Using the basic properties of trace operation in Hilbert spaces it follows from the above expression that

$$\mathbf{E}|e_2^\alpha(t)|^2_{E_\gamma} \le \mathbf{E}\int_0^t \|A^{\gamma+\theta}S(t-s)\|^2_{\mathcal{L}(E)}\|A^{-\theta}(\sigma(s,x^o(s),u_s^o) \\ - \sigma(s,x^o(s),u_s^\alpha))\|^2_{Q_d}ds. \quad (31)$$

Since $u^\alpha \xrightarrow{\tau_{\mu w *}} u^o$, the integrand in the expression (31) converges to zero μ-a.e. On the other hand, by assumption (A3) we have

$$\|A^{-\theta}\big(\sigma(s,x^o(s),u_s^o) - \sigma(s,x^o(s),u_s^\alpha)\big)\|^2_{Q_d} \le 4K^2_{d,\theta}(1+|x^o(s)|^2_{E_\gamma})$$

and by Theorem 3.2, $x^o \in B^a_\infty(I,E_\gamma)$. Thus it is clear that the integrand on the right hand side of the inequality (31) is dominated by an integrable random process. Hence, again by Lebesgue dominated convergence theorem, we conclude that for each $t \in I$, $\lim_\alpha \mathbf{E}|e_2^\alpha(t)|^2_{E_\gamma} = 0$ as $u^\alpha \xrightarrow{\tau_{\mu w *}} u^o$. Further, it follows from (31) and the above inequality that

$$\mathbf{E}|e_2^\alpha(t)|^2_{E_\gamma} \le 4\big(K_{d,\theta}C_{\gamma+\theta}\big)^2(1+b^2)\frac{t^{1-2(\gamma+\theta)}}{1-2(\gamma+\theta)} \quad (32)$$

where $b^2 \equiv \|x^o\|^2_{B^a_\infty(I,E_\gamma)}$. Thus the convergence $\lim_\alpha \mathbf{E}|e_2^\alpha(t)|^2_{E_\gamma} = 0$ is also uniform in $t \in I$. This shows that the expression on the righthand side of the inequality (28) converges to zero. As stated above this process can be continued exhausting the interval I. This proves the continuity as stated. □

Remark 4.2. As pointed out by a reviewer of this paper, in general Lebesgue dominated convergence (LDC) theorem does not apply to net convergence. However, if the directed set is countably accessible [20], LDC does hold.

Remark 4.3. Given the topology on $B^a_\infty(I,E_\gamma)$, it does not seem possible to relax the topology $\tau_{\mu w^*}$ further.

5. Existence of optimal controls

5.1. Standard control problem. We consider the following control problem. The cost functional is given by

$$J(u) = \mathbf{E}\Big\{\int_I \ell(t,x(t),u_t)dt + \Phi(x(T))\Big\} \quad (33)$$

where $x \in B^a_\infty(I, E_\gamma)$ is the solution of the integral equation (9) (mild solution of the controlled version of system (1)) corresponding to the control $u \in \mathcal{U}_{ad}$. This is a Bolza problem. The objective is to find a control $u^o \in \mathcal{U}_{ad}$ that minimizes the functional J. The first problem we consider is the question of existence of such controls.

Theorem 5.1. *Consider the system* (9) *with the cost functional given by* (33)*. Suppose the assumptions of Theorem 4.1 hold, the function $\ell : I \times E_\gamma \times M_c \longrightarrow \overline{R}$ is measurable in $t \in I$, lower semicontinuous in $(x, u) \in E_\gamma \times M_c$; Φ is lower semi continuous on E_γ, and further there exists a finite positive number κ such that*

$$|\ell(t, x, u)| \leq \kappa\{1 + |x|^2_{E_\gamma}\}, \quad |\Phi(x)| \leq \kappa(1 + |x|^2_{E_\gamma}) \ \forall \ (t, x, u) \in I \times E_\gamma \times M_c.$$

Then there exists an optimal control for the problem (33)*.*

Proof. Since the set of admissible controls $\mathcal{U}_{ad}$ is compact in the $\tau_{\mu w^*}$ topology, it suffices to prove that $u \longrightarrow J(u)$ is lower semicontinuous in this topology. Let $u^\alpha, \alpha \in D$ (a countably accessible directed set), be a net that converges in the $\tau_{\mu w^*}$ to $u^o \in \mathcal{U}_{ad}$. Let $\{x^\alpha, x^o\}$ denote the solutions of the integral equation (9) corresponding to the controls $\{u^\alpha, u^o\}$ respectively. The by Theorem 4.1, $x^\alpha \xrightarrow{s} x^o$ in $B^a_\infty(I, E_\gamma)$. Hence, along a subnet if necessary, $x^\alpha(t) \xrightarrow{s} x^o(t)$ in E_γ almost surely for all $t \in I$. Thus for almost all $t \in I$, it follows from our assumption on lower semicontinuity of ℓ that

$$\ell(t, x^o(t), u^o_t) \leq \underline{\lim}\, \ell(t, x^\alpha(t), u^\alpha_t), \quad \mu \text{ a.e.} \tag{34}$$

By our assumption on ℓ we have $|\ell(t, x^\alpha(t), u^\alpha_t)| \leq \kappa\{1 + |x^\alpha(t)|^2_E\}$ μ a.e, and by Corollary 3.2, the solution set is bounded and therefore there exists an $\mathcal{F}_t$-adapted nonnegative integrable process $Z(t), t \in I$, so that

$$\sup_{\alpha \in D}\{|\ell(t, x^\alpha(t), u^\alpha_t)|, \ |\ell(t, x^o(t), u^o_t)|\} \leq Z(t).$$

Hence Fatou's Lemma applies and from this lemma we may conclude that

$$\mathbf{E}\int_I \ell(t, x^o(t), u^o_t)\ dt \leq \underline{\lim}\, \mathbf{E}\int_I \ell(t, x^\alpha(t), u^\alpha_t)\ dt. \tag{35}$$

Since Φ is also lower semicontinuous on E_γ, and by Theorem 4.1, $x^\alpha(T) \xrightarrow{s} x^o(T)$ in E_γ P-a.s, it follows from the growth property of Φ that Fatou's lemma holds and we have

$$\mathbf{E}\Phi(x^o(T)) \leq \underline{\lim}\, \mathbf{E}\Phi(x^\alpha(T).$$

Thus we have demonstrated that each component of the functional $J(u)$ given by (33) is lower semicontinuous. It is well known that the sum of liminf is equal or less than the liminf of the sum. Hence we conclude that $u \longrightarrow J(u)$ is lower semicontinuous in the $\tau_{\mu w^*}$ topology. Since $\mathcal{U}_{ad}$ is compact in this topology, J attains its minimum on $\mathcal{U}_{ad}$. Hence an optimal control does exist. This completes the proof. □

Remark 5.2. Note that the lower semicontinuity assumption of ℓ on $E \times M_c$ is trivially satisfied if $x \longrightarrow \ell(t,x,\xi)$ is continuous on E for almost all $t \in I$ uniformly with respect to $\xi \in U$.

Remark 5.3. If, for almost all $t \in I$, the optimal relaxed control $u_t^o(\cdot)$ is absolutely continuous with respect to Lebesgue measure then the optimal control is an element of the class of regular controls (measurable functions) $\mathcal{U}_r$.

For existence of optimal regular controls it is necessary to impose stronger assumptions on the drift and the diffusion $\{f, \sigma\}$ and the cost integrands including the control space U.

Theorem 5.4. *Suppose A generates a compact analytic semigroup $S(t)$, $t > 0$, and both $\{f, \sigma\}$ are linear in control, and the control domain $U \subset L_p(\Sigma_c, R^d)$, $p \in [2, \infty)$, is weakly compact and convex with admissible controls $\mathcal{U}_r \equiv L^a_\infty(I, U)$. Suppose the cost integrands ℓ and Φ satisfy the assumptions of Theorem 5.1 and further, ℓ is convex in the control variable and once continuously Gateaux differentiable. Then there exists an optimal control in the class $\mathcal{U}_r$.*

Proof. Under the given assumptions, $u \longrightarrow J(u)$ is weak star lower semicontinuous on $\mathcal{U}_r$ and $\mathcal{U}_r$ is weak star compact. Thus J attains its minimum on $\mathcal{U}_r$. □

5.2. Some nonstandard control problems. Consider the control system (9) and let μ_0 denote the measure induced by the initial state x_0 and $\mathcal{S}(\mu_0)$ its support. Let $K \subset E_\gamma$ be a closed set(target set) and suppose $\mathcal{S}(\mu_0) \cap K = \emptyset$. The objective is to maximize the probability of hitting K at a given time say T. Clearly, the objective functional to be maximized is given by

$$J_1(u) = P\{x^u(T) \in K\} = \mu^u_T(K), \tag{36}$$

where $x^u \in B^a_\infty(I, E_\gamma)$ is the solution of the integral equation (9) corresponding to the control $u \in \mathcal{U}_{ad}$ and $\mu^u \equiv \{\mu^u_t, t \in I\}$, is the associated probability measure valued function. Similarly, one may like to maximize the functional

$$J_2(u) \equiv \int_I P\{x^u(t) \in K\}\ dt = \int_I \mu^u_t(K) dt \tag{37}$$

which is an indicator of the time spent by the trajectory x^u in the set K. If $J_2(u) = 0$, it indicates that $x^u(t)$ never hits the set K. On the other hand if it equals $\ell(I)$, the length of the interval I, $x^u(t) \in K$ for all $t \in I$ and consequently $\mu_t^u(K) = 1$ for all $t \in I$. Many such problems can be formulated in terms of the measure valued functions $\{\{\mu_t^u,\ t \in I\},\ u \in \mathcal{U}_{ad}\}$ induced by the solutions of the control system (9). For this we must study the properties of these measures. For any given initial measure μ_0, define the reachable (or attainable) set of measures at any time $t \in I$ by

$$\mathcal{R}(t) \equiv \{\mu_t^u, u \in \mathcal{U}_{ad}\}.$$

In the following theorem we prove the weak compactness of the attainable set.

Theorem 5.5. *Let $\mathcal{M}_o(E_\gamma)$ denote the space of regular probability measures on $\mathcal{B}(E_\gamma)$, the Borel subsets of E_γ. Suppose the assumptions of Theorem 4.1 hold. Then, for each $t \in I$, the attainable set of measures $\mathcal{R}(t)$ induced by the control system* (9) *is a weakly compact subset of the space $\mathcal{M}_o(E_\gamma)$.*

Proof. The proof is similar to that of [6, Theorem 5.2]. □

In fact the probability measures induced by the solutions of the system (9) are much more regular than those of $\mathcal{M}_o(E_\gamma)$. Let

$$\mathcal{M}_2(E_\gamma) \equiv \left\{\mu \in \mathcal{M}_o(E_\gamma) : \int_{E_\gamma} |x|^2_{E_\gamma} \mu(dx) < \infty\right\}$$

denote the space of regular probability measures having finite second moments. It is interesting to mention that the reachable set is not only a weakly compact subset of $\mathcal{M}_o(E_\gamma)$, it is also a weakly compact subset of the space $\mathcal{M}_2(E_\gamma) \subset \mathcal{M}_0(E_\gamma)$. This is stated in the following theorem.

Theorem 5.6. *Under the assumptions of Theorem 5.5, for each $t \in I$, the reachable set of measures $\mathcal{R}(t)$ induced by the control system* (9) *is a weakly compact subset of $\mathcal{M}_2(E_\gamma)$.*

Proof. The proof is identical to that of [6, Theorem 5.4]. □

Using the above results we can prove the existence of optimal controls for the above mentioned problems.

Corollary 5.7. *Consider the control system* (9) *with the admissible controls $\mathcal{U}_{ad}$ endowed with the $\tau_{\mu w^*}$ topology and let K be a closed subset of E_γ giving the objective functionals $J_1(u)$ of* (36) *and $J_2(u)$ of* (37) *respectively. Suppose the assumptions of Theorem 4.1 hold. Then, there exist optimal controls maximizing the functionals $J_1(u)$ and $J_2(u)$ respectively.*

Proof. The proof follows from [6, Corollary 5.5, 5.6]. $\square$

Another problem of significant interest is: given a probability measure valued function ν over the interval I, can we find one induced by the control system (9) that is closest to it?. This can be formulated by use of the Prokhorov metric d_P on $\mathcal{M}_0(E_\gamma)$. The objective functional is taken as

$$J(u) = \int_I d_p(\mu_t^u, \nu_t)\ dt. \tag{38}$$

The problem is to find a control that minimizes this functional.

Corollary 5.8. *Consider the control system* (9) *with the admissible controls* $\mathcal{U}_{ad}$ *endowed with the* $\tau_{\mu w^*}$ *topology and the cost functional* (38). *Suppose the assumptions of Theorem 4.1 hold. Then, there exists an optimal control minimizing the functional* (38).

Proof. Since $\mathcal{U}_{ad}$ is compact in the $\tau_{\mu w^*}$ topology, it suffices to prove that $u \longrightarrow J(u)$ is lower semi-continuous with respect to this topology. First, we note that since E is separable, $E_\gamma, \gamma \in [0,1)$ is separable. Thus the topology induced by the Prokhorov metric on $\mathcal{M}_o(E_\gamma)$ is equivalent to the topology induced by weak convergence in $\mathcal{M}_o(E_\gamma)$. Let $\{u^\alpha\} \in \mathcal{U}_{ad}$ be any net converging to $u^o \in \mathcal{U}_{ad}$ in the $\tau_{\mu w^*}$ topology and let $\{x^\alpha, x^o\}$ denote the corresponding solutions of equation (9) and $\{\mu^\alpha, \mu^o\}$ denote the associated measure valued functions. Then it follows from Theorem 4.1 that $x^\alpha \longrightarrow x^o$ in $B_\infty^a(I, E_\gamma)$ as $u^\alpha \xrightarrow{\tau_{\mu w^*}} u^0$. Thus there exists a subnet, relabeled as the original net, such that for each $t \in I$, $x^\alpha(t) \xrightarrow{s} x^o(t)$ in E_γ P-a.s. Hence for any $\varphi \in C_b(E_\gamma)$, the space of bounded continuous functions on E_γ, $\varphi(x^\alpha(t)) \longrightarrow \varphi(x^o(t))$ for each $t \in I$ P-a.s. This means that $\int_{E_\gamma} \varphi(\xi)\mu_t^\alpha(d\xi) \longrightarrow \int_{E_\gamma} \varphi(\xi)\mu_t^o(d\xi)$ for each $\varphi \in C_b(E_\gamma)$. Clearly, then $\mu_t^\alpha \xrightarrow{w} \mu_t^o$ for each $t \in I$ and therefore by virtue of the equivalence mentioned above we conclude that $d_P(\mu_t^\alpha, \nu_t) \longrightarrow d_P(\mu_t^o, \nu_t)$ for each $t \in I$. Since $d_P(\mu_1, \mu_2) \leq 2$ for all $\mu_1, \mu_2 \in \mathcal{M}_0(E_\gamma)$ it follows from Lebesgue bounded convergence theorem that

$$\lim_\alpha \int_I d_P(\mu_t^\alpha, \nu_t) dt = \int_I d_P(\mu_t^o, \nu_t) dt.$$

This shows that the functional (38) is actually continuous with respect to $\tau_{\mu w^*}$ topology on $\mathcal{U}_{ad}$. Thus there exists a control $u^* \in \mathcal{U}_{ad}$ at which this functional attains its minimum. This proves the existence of an optimal control. $\square$

Remark 5.9. In fact the previous Corollary can be easily generalized giving

$$J(u) \equiv \int_0^T d_P(\mu_t^u, \nu_t)\lambda(dt), \tag{39}$$

where λ is any positive measure on the sigma field of Borel subsets of the set $I \equiv [0, T]$ having bounded total variation.

Remark 5.10. In case of evasion problems, one must maximize the Prokhorov distance or the functional (38). Since $u \longrightarrow J(u)$ is continuous, as seen above, it attains both its minimum and maximum. Thus there exists an optimal control that maximizes the functional (38).

Many such interesting control problems such as target following problem, first hitting time problem, etc are considered in [6, Corollary 5.8, Corollary 5.9, Corollary 5.10]. Using the same technique one can consider similar problems for the system (9).

6. An example

Kuramoto-Sivashinsky Like Equation: Here we wish to consider control problems for the well known Kuramoto–Sivashinsky like equation in one dimension subject to both distributed and boundary noise. A generalized version of the KSE system is given

$$\partial_t v + a\Delta^2 v + b\Delta v = Df_1(v, Dv) + F_2 u + g_0(\xi)n_0(t), \quad \xi \in \Sigma \equiv (0, 1), \tag{40}$$

$$\mathcal{B}v = g_1(\xi)n_1(t) \quad \text{for } \xi \in \partial\Sigma = \{0, 1\} \tag{41}$$

where the boundary operator $\mathcal{B}$ is given by

$$(\mathcal{B}_1 v)(\xi) \equiv \beta_0 v(\xi), \quad (\mathcal{B}_2 v)(\xi) \equiv \beta_1 v(\xi) + \beta_2 D_\nu v(\xi) \quad \text{for } \xi \in \partial\Sigma \tag{42}$$

with $\beta_0, \beta_1, \beta_3 \neq 0$. Here we have used D^k to denote the spatial derivative of order k and D_ν the directional derivative at any point $\xi \in \partial\Sigma$ along the normal pointing outward of the boundary. The coefficients $\{a, b\}$ are real positive and those of the boundary operator $\mathcal{B}$ are assumed to be nonzero. The function $f_1 : R^2 \longrightarrow R$ is continuous with respect to its arguments, $g_0 \in L_2(\Sigma)$ and n_0 is the standard white noise. The function $g_1 \in L_2(\partial\Sigma, R^{2\times 2})$ where $R^{2\times 2}$ denotes the class of 2×2 square matrices (with entries real) and n_1 is an R^2 valued standard white noise.

Note that for $a > 0$ the operator $a\Delta^2$ is dissipative under homogeneous Dirichlet boundary condition while for $b > 0$ the operator $b\Delta$ is accretive or anti-dissipative. Define the differential operator $\mathcal{A}$ by $\mathcal{A}\varphi = a\Delta^2\varphi + b\Delta\varphi$. Then define the operator A by setting $D(A) = \{\varphi \in E : \mathcal{A}\varphi \in E \ \&\ \varphi|_{\partial\Sigma} = D\varphi|_{\partial\Sigma} = 0\} =$

$H^4 \cap H_0^2$. We show that under these assumptions the operator $-A$ generates an analytic semigroup on the Hilbert space $E \equiv L_2(\Sigma)$. Indeed, by simple integration by parts one can easily verify that

$$(A\varphi,\varphi) + (b/2\varepsilon)|\varphi|_E^2 \geq (a - b\varepsilon/2)|\Delta\varphi|_E^2 \tag{43}$$

for all $\varepsilon > 0$ and all $\varphi \in D(A)$. Choosing $\varepsilon = a/b$ we obtain

$$(A\varphi,\varphi) + (b^2/2a)|\varphi|_E^2 \geq (a/2)|\Delta\varphi|_E^2. \tag{44}$$

For every $\varphi \in D(A)$ it follows form elementary computation (or Poincaré inequality) that there exists a positive constant c such that $|\varphi|_E \leq c|\Delta\varphi|_E$. From the above inequalities we obtain the following resolvent inequality

$$|(\lambda I + A)^{-1}|_{\mathcal{L}(E)} \leq \frac{1}{\lambda + r_0} \ \forall\, \lambda > -r_0 \tag{45}$$

where $r_0 = (a^2 - b^2c^2)/2ac^2$. Note that the destabilizing influence of the anti-dissipative term is very well reflected in the resolvent inequality. From now on we use the same symbol A to denote its closed extension in E as an unbounded operator. One can easily verify that the operator A is self adjoint on the Hilbert space E but not positive. It is clear from the inequality (44) or (45) that for $\beta > ((b^2c^2 - a^2)/2ac^2)$, the operator $A_\beta \equiv (\beta I + A)$ is an unbounded positive self adjoint operator in E. Then it follows from (45) that the resolvent of the operator A_β satisfies the inequality $|(\lambda I + A_\beta)^{-1}|_{\mathcal{L}(E)} \leq 1/\lambda$, for $\lambda > 0$. Since A_β is closed and densely defined it follows from Hille–Yosida theorem that $-A_\beta$ generates a C_0-semigroup $S_\beta(t), t \geq 0$, of contractions on E. Using the operator A_β, we can rewrite the system (40), with homogeneous boundary condition, as an ordinary differential equation on the Hilbert space E in the abstract form

$$(d/dt)v + A_\beta v = F_1(v) + F_2 u + \dot{W}_0 \tag{46}$$

where $F_1(v) = \beta v + Df_1(v, Dv)$ and $\dot{W}_0 \equiv g_0 n_0$ is the space time white noise. Let C denote the field of complex numbers. Then, for $\lambda \in C$ given by $\lambda = \nu + i\tau$ with $\nu > 0$, one can easily verify that

$$|((\lambda I + A_\beta)\varphi, \varphi)| \geq |\tau|\, |\varphi|_E^2$$

and hence $|(\lambda I + A_\beta)\varphi| \geq |\tau|\, |\varphi|_E$ for all $\varphi \in D(A) = D(A_\beta)$. From this we obtain

$$|(\lambda I + A_\beta)^{-1}|_{\mathcal{L}(E)} \leq 1/|\tau|$$

for all $\nu = \mathrm{Re}\lambda > 0$ and $\tau \neq 0$. Thus it follows from Hille's characterization of analytic semigroups [10, Theorem 3.2.7, p. 82] see also [Pazy, 17, Theorem 5.2, p. 61] that $-A_\beta$ generates an analytic semigroup $S_\beta(t), t \geq 0$ in E. As a result, $-A$ generates an analytic semigroup $S(t) = S_\beta(t)e^{\beta t}$. Then the mild solution of equation (46) with homogeneous boundary condition $\mathcal{B}v = 0$ is given by the solution of the integral equation

$$\begin{aligned} v(t) = S_\beta(t)v_0 &+ \int_0^t S_\beta(t-s)F_1(v(s))ds \\ &+ \int_0^t S_\beta(t-s)F_2 u_s ds + \int_0^t S_\beta(t-s)dW_0 \end{aligned} \tag{47}$$

in the Hilbert space $E_\gamma \equiv D(A^\gamma)$ for a suitable $\gamma \in [0,1]$ that admits F_1 of the form given by the expression following equation (46). For the nonhomogeneous boundary data this equation takes the form

$$\begin{aligned} v(t) = S_\beta(t)v_0 &+ \int_0^t S_\beta(t-s)F_1(v(s))ds + \int_0^t S_\beta(t-s)F_2 u_s ds \\ &+ \int_0^t S_\beta(t-s)dW_0 + \int_0^t A_\beta S_\beta(t-s)\mathcal{R}\, dW_1(s) \end{aligned} \tag{48}$$

where $\mathcal{R}$ is the Dirichlet map given by $\mathcal{R} = (\mathcal{B}|_{\ker(\mathcal{A})})^{-1}$, and W_1 is the $L_2(\partial\Sigma, R^2)$ -valued Brownian motion with distributional derivative $\dot{W}_1(t) = g_1(\cdot)n_1(t)$, $t \geq 0$. This represents the boundary noise. System (48) is a special case of our system given by the stochastic integral equation (9) and therefore all the results on control of Section 5 hold for this case.

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