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ε-convergent splines difference scheme

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Abstract. We derive the spline difference scheme which is first order ε-convergent in the uniform norm for the selfadjoint singular perturbation problem.

Introduction

Singular perturbation problem in one dimension has been used as a model equation for reaction-diffusion processes (see [8]). We are concentrated on the selfadjoint non-turning point case of the singularly perturbed equation

$$(1)\qquad Lu = -\varepsilon u'' + p(x)u = f(x),\ x \in [0,1],\ u(0) = \alpha_0,\ u(1) = \alpha_1,$$

where $p, f \in C^2[0,1],\ 0 < \varepsilon \ll 1,\ p(x) \geq \beta > 0$. Under these assumptions (1) has boundary layers at both endpoints.

Many methods have been proposed in the literature for the solution of this problem. A selection can be found in, say [4], [6], [5]. We require these numerical methods to be independent of the value $(h/\sqrt{\varepsilon})$ and to model the solution with given accuracy inside and outside the boundary layers. The difference schemes which are uniformly accurate, model well the behaviour of the solution inside the layers, but they are not strict enough outside of them. To impose stricter criteria than uniform convergence Doolan et. al. in [2] introduced criteria of optimal convergence for the initial value problems. The schemes optimal in ε have reduced schemes that model the reduced problem accurately. It turns out that these schemes perform better than the uniform ones. The investigation of optimal schemes for initial value problems was contiuned by Farrell in [3]. He found a class of

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initial differential schemes which are first order accurate. In [9] is given the first order optimally convergent difference scheme for the non-selfadjoint perturbation problem, and in [10] we find the difference scheme which is second order optimally accurate for the selfadjoint perturbation problem.

In [9] we propose ε-convergence as a criterion, interesting for a very small ε, to improve results in that case. We generate the difference schemes for the non-selfadjoint perturbation problem which are dependent only on ε in the whole region, i.e., the ε-convergent difference schemes.

We define ε-convergence in the following way:

A scheme is ε-convergent of order q if

$$|u(x_i) - u_i| \le M(\sqrt{\varepsilon})^q,$$

where M is a constant independent of ε, h, i and $q > 0$; $u(x_i)$ is the exact solution of the differential equation and u_i is the solution of the approximate equation at the point x_i.

In this paper we derive the difference scheme for the selfadjoint problem (1) which is first order ε-convergent in uniform norm. We generate the scheme in Section 2. In Section 3 we give the proof of ε-convergence. Section 4 contains a numerical experiment which confirms the theoretical predictions.

Notation. We assume a uniform partition of the interval $[0, 1]$ with step size $h = 1/N, N$ being an integer. Throughout the paper M denotes a generic constant independent of the step size h or the perturbation parameter ε. Let $\bar{\rho} = h/\sqrt{\varepsilon},\ \rho_i = p_i\bar{\rho}$ where $p_i = p(x_i)$. Then, $|u(x_i) - u_i|$ denotes nodal errors, where $u(x_i)$ and u_i are the exact solution and the approximate solution at the mesh point x_i, respectively. Truncation error of the difference scheme is defined as $\tau_i(u) = R(u(x_i) - u_i) = Ru_i - QLu_i$ where R and L are the corresponding, operators.

2. A scheme generation

Consider as in [7] the "comparison" problem

$$LS_\Delta(x) = -\varepsilon S''_\Delta(x) + \bar{p}_i S_\Delta(x) = \bar{f}_i,\ x \in (x_{i-1}, x_i), \tag{2}$$

which satisfies boundary conditions on $[x_{i-1}, x_i] : S_\Delta(x_{i-1}) = u_{i-1}, S_\Delta(x_i) = u_i$, $\bar{p}_i = p(x_i)$, and $\bar{f}_i = f_{i-1/2}[2/h^2(x - x_{i-1/2})(x - x_i)] + f_i[2/h^2(x - x_{i-1/2})\ (x - x_{i-1})] + f_{i-1/2}[-4/h^2(x - x_{i-1})(x - x_i)]$. By solving (2) and by using the continuity condition of the second derivative of the spline at the mid-points, we obtain the following discretization of (1):

$$Ru_i = QLu_i, \quad u(0) = \alpha_0, \quad u(1) = \alpha_1,$$

where

$$Ru_i = u_{i-1}r_i^- + u_i r_i^c + u_{i+1}r_i^+,$$
$$Qf_i = q_{i-1}^- f_{i-1} + q_i^c f_i + q_{i+1}^+ f_{i+1} + q_{i1/2}^- f_{i-1/2} + q_{i1/2}^+ f_{i+1/2}.$$

Here the coefficients of the scheme are given by:

1. In the case $S''_{\Delta_i}(x_{i-1}+h/2) = S''_{\Delta_{i+1}}(x_{i-1}+h/2)$ we have

(3)
$$\begin{aligned}
r_i^- &= \rho_i^2 \sinh(\rho_i/2)/\sinh(\rho_i), r_i^+ = \rho_{i+1}^2 \sinh(\rho_{i+1}/2)/\sinh(\rho_{i+1})\\
r_i^c &= \rho_i^2/\sinh(\rho_i)\sinh(\rho_i/2) - \rho_{i+1}^2/\sinh(\rho_{i+1})\sinh(3\rho_{i+1}/2),\\
q_i^- &= 1/p_{i-1}[-4 + \sinh(\rho_i/2)/\sinh(\rho_i)(8+\rho_i^2)],\\
q_i^+ &= 1/p_{i+1}[\sinh(\rho_{i+1}/2)/\sinh(\rho_{i+1})(\rho_{i+1}^2+4)\\
&\quad - 4\sinh(3\rho_{i+1}/2)/\sinh(\rho_{i+1}) + 4],\\
q_i^c &= 1/p_i[\sinh(\rho_i/2)/\sinh(\rho_i)(\rho_i^2+8)\\
&\quad + 4\sinh(\rho_{i+1}/2)/\sinh(\rho_{i+1})\\
&\quad - \sinh(3\rho_{i+1}/2)/\sinh(\rho_{i+1})(4+\rho_{i+1}^2)],\\
q_{i-1/2}^- &= 1/p_{i-1/2}[-16\sinh(\rho_i/2)/\sinh(\rho_i) + 8],\\
q_{i+1/2}^+ &= 1/p_{i+1/2}[-8\sinh(\rho_{i+1}/2)/\sinh(\rho_{i+1})\\
&\quad + 8\sinh(3\rho_{i+1}/2)/\sinh(\rho_{i+1}) - 8].
\end{aligned}$$

2. In the case $S''_{\Delta_i}(x_i+h/2) = S''_{\Delta_{i+1}}(x_i+h/2)$ we have:

(4)
$$\begin{aligned}
r_i^- &= -\rho_i^2 \sinh(\rho_i/2)/\sinh(\rho_i),\\
r_i^+ &= -\rho_{i+1}^2 \sinh(\rho_{i+1}/2)/\sinh(\rho_{i+1}),\\
r_i^c &= \rho_i^2/\sinh(\rho_i)\sinh(\rho_i/2) - \rho_{i+1}^2/\sinh(\rho_{i+1})\sinh(3\rho_{i+1}/2),\\
q_i^- &= 1/p_{i-1}[4 + \sinh(\rho_i/2)/\sinh(\rho_i)(4+\rho_i^2)\\
&\quad - 4/\sinh(\rho_i)\sinh(3\rho_i/2)],\\
q_i^+ &= 1/p_{i+1}[\sinh(\rho_{i+1}/2)/\sinh(\rho_{i+1})(-\rho_{i+1}^2 - 8) + 4],\\
q_i^c &= 1/p_i[\sinh(3\rho_i/2)/\sinh(\rho_i)(\rho_i^2+4)\\
&\quad - \sinh(\rho_{i+1}/2)/\sinh(\rho_{i+1})(\rho_{i+1}^2+8)\\
&\quad - 4\sinh(\rho_i/2)/\sinh(\rho_i)],\\
q_{i1/2}^- &= 1/p_{i-1/2}[-8\sinh(3\rho_i/2)/\sinh(\rho_i)\\
&\quad + 8/\sinh(\rho_{i+1})\sinh(\rho_{i+1}/2) - 8],\\
q_{i1/2}^+ &= 1/p_{i+1/2}[16\sinh(\rho_{i+1}/2)/\sinh(\rho_{i+1}) - 8].
\end{aligned}$$

3. Proof of the ε-convergence

The proof of the ε-convergence will be given for the scheme (3). The same holds for (4). Since

$$\max_i |u(x_i) - u_i| \le M\|A^{-1}\| \max_i |\tau_i(u)|, \tag{5}$$

(see [11]) we must estimate the matrix A^{-1} in the maximum norm and the truncation error of the discretization (3) to obtain the nodal errors.

Matrix estimate. Since $\|A^{-1}\| \le \max_i \Delta_i$ where $\Delta_i = |r_i^- + r_i^c + r_i^+|$ and

$$\Delta_i = \rho_i^2/\sinh(\rho_i)(3\sinh(\rho_i/2) - \sinh 3(\rho_i/2)) \ge M \begin{cases} \bar\rho^4, & \bar\rho \le 1, \\ \bar\rho^2\, e^{M\bar\rho/2}, & \bar\rho \ge 1, \end{cases}$$

we obtain the matrix estimate

$$\|A^{-1}\| \le M \begin{cases} \bar\rho^4, & \bar\rho \le 1, \\ \bar\rho^2\, e^{M\bar\rho/2}, & \bar\rho \ge 1. \end{cases} \tag{6}$$

The estimate of the truncation error.

In order to estimate the truncation error we shall use the following

Lemma 1. [2]. *Let $f, p \in C^2([0,1])$ and $p'(0) = p'(1) = 0$. Then the solution of (1) can be expressed as*

$$u(x) = u_0(x) + \omega_0(x) + g(x), \tag{7}$$

$$u_0(x) = p_0\, e^{(-x\sqrt{p(0)/\varepsilon})},\ \omega_0(x) = p_1\, e^{(-(1-x)\sqrt{p(1)/\varepsilon})}, \tag{8}$$

p_0, p_1 are bounded functions of ε independent of x and

$$|g^{(i)}(x)| \le M(1 + \varepsilon^{1-i/2}), \qquad i = 0(1)n. \tag{9}$$

Hence, $\tau_i(u) = \tau_i(u_0) + \tau_i(\omega_0) + \tau_i(g)$.

Truncation error for g. The truncation error appropriate for g is

$$\begin{aligned} \tau_i(g) = {} & \tau_i^{(0)} g^{(0)} + \tau_i^{(1)} g^{(1)} + \tau_i^{(2)} g^{(2)} + \tau_i^{(3)} g^{(3)} + r_i^- h^4/4!\, g^{(4)}(\xi_1) \\ & + r_i^+ h^4/4!\, g^{(4)}(\xi_2) + \varepsilon q_i^- h^2/2 g^{(4)}(\xi_1) + \varepsilon h^2 q_i^+ g^{(4)}(\xi_4) \\ & + \varepsilon h^2/8 q_{i1/2}^- g^{(4)}(\xi_5) + \varepsilon h^2/8 q_{i1/2}^+ g^{(4)}(\xi_6) \\ & - p_{i-1} h^4/4!\, g^{(4)}(\xi_2) q_i^- - p_{i+1} h^4/4!\, g^{(4)}(\xi_2) q_i^+ \\ & - p_{i-1/2} h^4/(4!\,16) g^{(4)}(\xi_5) q_{i1/2}^- \\ & + q_{i1/2}^+ p_{i+1/2} h^4/(4!\,16) g^{(4)}(\xi_6), \end{aligned} \tag{10}$$

where

$$x_{i-1} \le \xi_5 \le x_{i-1/2} \le \xi_1 \le \xi_2 \le x_{i+1/2} \le \xi_6 \le x_{i+1},$$

and

$$\begin{aligned}\tau_i^{(0)} &= r_i^- + r_i^c + r_i^+ - p_{i-1}q_i^- - p_iq_i^c - p_{i+1}q_i^+ - p_{i-1/2}q_{i1/2}^- \\ &\quad - p_{i+1/2}q_{i1/2}^+ = 0,\\ \tau_i^{(1)} &= h(r_i^+ - r_i^-) - h(-p_{i-1}q_i^- + p_{i+1}q_i^+ - p_{i-1/2}q_{i1/2}^- \\ &\quad + p_{i+1/2}q_{i1/2}^+) = 0.\end{aligned}$$

Since

$$\begin{aligned}\tau_i^{(2)} = h^2/2(r_i^+ + r_i^-) + \varepsilon(q_i^- + q_i^+ + q_i^c + q_{i1/2}^- + q_{i1/2}^+)\\ - h^2/2(p_{i-1}q_i^- + p_{i+1}q_i^+ + 1/4p_{i-1/2}q_{i1/2}^-);\end{aligned}$$

and $\tau_i^{(2)}(\rho_i) = 0$, where $\rho_i = \bar{\rho}p_i, p_i = const$ we have

$$\tau_i^{(2)} = \tau_i^{(2)}(\rho_i) + h^2/2(\rho_{i+1} - \rho_i)\frac{\partial\tau_i^{(2)}}{\partial\rho_{i+1}}(\rho_i) + Q,$$

where Q is the part in error estimate which is of the lower order than the previous ones. Because of $|\rho_{i+1} - \rho_i| \le Mh^2/\sqrt{\varepsilon}$, $\left|\frac{\partial\tau_i^{(2)}}{\partial\rho_{i+1}}(\rho_i)\right| \le Mh/\sqrt{\varepsilon}$, we obtain $|\tau_i^{(2)}g''| \le Mh^5/\varepsilon$ for $\bar{\rho} \le 1$. In the opposite case for $\bar{\rho} \ge 1$ we obtain $|\tau_i^{(2)}| \le Mh^2e^{M\bar{\rho}}$. Further,

$$\begin{aligned}\tau_i^{(3)} = h^3/6(r_i^+ - r_i^-) + \varepsilon h(-q_i^- + q_i^+ + 1/2(q_{i1/2}^+ - q_{i1/2}^-))\\ - h^3/6(-p_{i-1}q_i^- + p_{i+1}q_i^+ + 1/8(p_{i+1/2}q_{i1/2}^+ - p_{i-1/2}q_{i1/2}^-)),\end{aligned}$$

and $\tau_i^{(3)} = h^3/6\{10 - 5\sinh(3/2\rho_i)/\sinh(\rho_i) - 5\sinh(\rho_i/2)/\sinh(\rho_i)\}$. Clearly, $|\tau_i^{(3)}| \le Mh^3\bar{\rho}^2$, $\bar{\rho} \le 1, |\tau_i^{(3)}| \le Mh^3e^{M\bar{\rho}}$, $\bar{\rho} \ge 1$. With (9) it gives $|\tau_i^{(3)}g'''| \le Mh^3/\sqrt{\varepsilon}\bar{\rho}^2, \bar{\rho} \le 1$, and $|\tau_i^{(3)}g'''| \le Mh^3/\sqrt{\varepsilon}e^{M\bar{\rho}}$, $\bar{\rho} \ge 1$. Since the remainders are of the same or lower order we obtain

$$|\tau_i(g)| \le M\begin{cases} h^5/(\varepsilon\sqrt{\varepsilon}), & \bar{\rho} \le 1,\\ h^3/(\sqrt{\varepsilon})e^{M\bar{\rho}}, & \bar{\rho} \ge 1.\end{cases} \tag{11}$$

Truncation error for boundary layer terms.

We shall give the estimate for the boundary layer $u_0(x)$. Since $\tau_i(u_0) = Ru_i - QLu_i$ where $Ru_i = u_{0i}(r_i^-e^{\rho_0} + r_i^c + r_i^+e^{-\rho_0})$ and $-QLu_i =$

$u_{0i}p_0/\varepsilon\ \{(p_0-p_{i-1})q_i^- e^{\rho_0} + (p_0-p_i)q_i^c + (p_0-p_{i+1})q_i^+ e^{-\rho_0} + (p_0-p_{i-1/2})$ $q_{i1/2}^- e^{\rho_0/2} + (p_0-p_{i+1/2})q_{i1/2}^+ e^{-\rho_0/2}\}$, $u_{0i} = e^{-x_i\sqrt{p_0/\varepsilon}}$, we have $Ru_i = u_{0i}(-1/16)(p_0-p_i)\rho_i^4 + O(h^5/(\varepsilon\sqrt{\varepsilon}))$ and $-QLu_i = u_{0i}(-1/16)(p_0-p_i)\rho_i^4 + O(h^5/(\varepsilon\sqrt{\varepsilon}))$. After cancelling the hardest parts in the error estimate we obtain that

$$|\tau_i(u_0)| \leq Mh^5/(\varepsilon\sqrt{\varepsilon}) \quad \text{for} \quad \bar{\rho} \leq 1. \tag{12}$$

Similarly,

$$|\tau_i(\omega_0)| \leq Mh^5/(\varepsilon\sqrt{\varepsilon}) \quad \text{for} \quad \bar{\rho} \leq 1. \tag{13}$$

In the opposite case, if $\bar{\rho} \geq 1$, since $Ru_i(\rho_0) = 0$, $\rho_0 = \sqrt{p_0/\varepsilon}h$, we have $Ru_i - Ru_i(\rho_0) = u_{0i}\{(r_i^- - r_i^-(\rho_0))\, e^{\rho_0} + (r_i^c - r_i^c(\rho_0)) + (r_i^+ - r_i^+(\rho_0))e^{-\rho_0}\}$. By using $|\overset{+}{r_i^-} - \overset{+}{r_i^-}(\rho_0)| \leq Mh/\sqrt{\varepsilon}\bar{\rho}^2\ x_{i+1}^2$, $|r_i^c - r_i^c(\rho_0)| \leq Mh/\sqrt{\varepsilon}\bar{\rho}^2 x_i^2 e^{\rho_i/2}$ because of $|x_i^2/\varepsilon e^{-\sqrt{p_0/\varepsilon}x_i}| \leq Me^{-\delta x_i/\varepsilon}$, where $0 < \delta < 1$, we obtain

$$|Ru_i| \leq Mh\varepsilon\rho_i^2 e^{\rho_i/2}, \quad \text{if } \bar{\rho} \geq 1. \tag{14}$$

Similarly, since $|p_0 - p_{(\cdot)}| \leq Mx_{(\cdot)}^2$ where $(\cdot) = x_{i\pm 1}$ or $x_{i\pm 1/2}$ we obtain

$$|QLu_i| \leq Mh\varepsilon\rho_i^2 e^{M\rho_i/2} \quad \text{if } \bar{\rho} \geq 1. \tag{15}$$

From (14), (15), follows that

$$|\tau_i(u_0)| \leq Mh\sqrt{\varepsilon}\bar{\rho}^2 e^{M\bar{\rho}}, \quad \bar{\rho} \geq 1. \tag{16}$$

By similar calculation we obtain

$$|\tau_i(\omega_0)| \leq Mh\sqrt{\varepsilon}\bar{\rho}^2 e^{M\bar{\rho}}, \quad \bar{\rho} \geq 1. \tag{17}$$

Hence, Lemma 1, (11), (12), (13), (16) and (17) give the truncation error for the discretization (3):

$$|\tau_i(u)| \leq M\begin{cases} h^5/(\varepsilon\sqrt{\varepsilon}), & \text{for } \bar{\rho} \leq 1, \\ h^3/\sqrt{\varepsilon}\, e^{M\bar{\rho}}, & \text{for } \bar{\rho} \geq 1. \end{cases} \tag{18}$$

From (5), (6), and (18) we obtain the nodal errors for the discretization (3)

$$|u_i - u(x_i)| \leq Mh\sqrt{\varepsilon},$$

in both cases.

We have just proved the following theorem:

Theorem 1. *Let $p, f \in C^2[0,1]$, and $p'(0) = p'(1) = 0$, in* (1). *Let $\{u_i\}$, $i = 0(1)n$, be the solution of the linear system* (3). *Then the nodal errors are*

$$|u(x_i) - u_i| \le Mh\sqrt{\varepsilon}.$$

The same result holds for the scheme (4).

Theorem 2. *Let $p, f \in C^2[0,1]$, and $p'(0) = p'(1) = 0$ in* (1). *Then the discretization* (4) *has the first order of the ε-convergence.*

4. Computational results

A technique to test the rate of uniform convergence is the well-known double mesh technique ([2]). The errors $e_{i,\varepsilon,h,k}$ determine

$$\mathrm{RATE}_{h,\varepsilon,k} = \frac{\ln(e_{\varepsilon,h,k}/e_{\varepsilon,h,k+1})}{\ln 2},$$

where

$$e_{\varepsilon,h,k} = \max_i |u_{i,\varepsilon,h/2^k} - u_{i,\varepsilon,h/2^{k+1}}|,$$

and $u_{i,\varepsilon,h/2^k}$ is the approximate values at the points $x_i = ih/2^k$. The next meshes, of lenght $h = 1/N$, were obtained by halving the previous ones. The first one has $N = 16$ and the last one contains $N = 1024$ points.

In Table 1 is given the test of ε-convergence for the scheme (3) as applied to the example

$$-\varepsilon u'' + u = -\cos^2(\pi x) - 2\varepsilon\pi^2\cos(2\pi x),\ u(0) = u(1) = 0, \tag{19}$$

whose known exact solution is

$$u(x) = (\exp(-(1-x)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon}))/(1 + \exp(-1/\sqrt{\varepsilon})) - \cos^2(\pi x)$$

taken from [2].

The rate of convergence in h is also one: $O(h)$. It can be seen from Table 1.

In Table 2 is given the difference between the exact and the approximate solution in the max norm

$$\mathrm{MAX}_{\varepsilon,N_k} = \max_{0\le i\le N} |u(x_i) - u_i|,$$

attained at mesh points for $N_k = 2^k,\ k = 4,5,6,7,8$.

From this Table 2 we can see that the convergence is on ε.

Table 1:
The $\mathrm{RATE}_{1/16,\varepsilon,k}$ to show the convergence of the scheme (3) for problem (19)

ε/k	1	2	3	4	5
1	1.19	1.09	1.04	1.02	1.01
2^{-1}	1.18	1.08	1.04	1.02	1.00
2^{-2}	1.16	1.07	1.03	1.01	1.00
2^{-3}	1.14	1.06	1.03	1.01	1.00
2^{-4}	1.11	1.05	1.02	1.01	1.00
2^{-5}	1.12	1.04	1.01	1.00	1.00
2^{-6}	1.14	1.04	1.01	1.00	1.00
2^{-7}	1.21	1.06	1.01	1.01	1.00
2^{-8}	1.31	1.11	1.03	1.01	1.00
2^{-9}	1.42	1.19	1.05	1.01	1.00
10^{-5}	0.91	0.84	1.06	1.01	1.00
10^{-7}	0.92	0.98	0.99	0.99	0.99
10^{-8}	0.92	0.98	0.99	0.99	0.99

Table 2:
The maximum error $\mathrm{MAX}_{\varepsilon,N_k}$ to show the ε-convergence.

ε/N_k	4	5	6	7	8
10^{-5}	.1930E-04	.108E-04	.510E-05	.183E-05	.654E-06
10^{-7}	.1915E-06	.965E-07	.484E-07	.243E-07	.122E-07
10^{-8}	.1917E-07	.965E-08	.484E-08	.242E-08	.121E-08

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