Publ. Math. Debrecen
88/1-2 (2016), 139–161
DOI: 10.5486/PMD.2016.7305

Adjoint preenvelopes and precovers of modules

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Abstract. Let $\mathcal{C}$ be a class of left R-modules and $(-)^+ = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the ring of integers and $\mathbb{Q}$ is the ring of rational numbers. A left R-module homomorphism $M \to N$ is said to be a $\mathcal{C}$-adjoint preenvelope of M if $N \in \mathcal{C}$ and the sequence $0 \to C^+ \otimes M \to C^+ \otimes N$ is exact for any $C \in \mathcal{C}$. $M \to N$ is called a $\mathcal{C}$-adjoint precover of N if $M \in \mathcal{C}$ and the sequence $0 \to N^+ \otimes C \to M^+ \otimes C$ is exact for any $C \in \mathcal{C}$. We investigate the existence and properties of adjoint preenvelopes and adjoint precovers. The relationships among adjoint preenvelopes, adjoint precovers, preenvelopes and precovers are obtained. As a consequence, we characterize several important rings in terms of adjoint preenvelopes and adjoint precovers.

1. Introduction

Preenvelopes and precovers were introduced in the early eighties of the 20th century by Enochs and play crucial roles in relative homological algebra [9], [11], [13], [22]. Let $\mathcal{C}$ be a class of left R-modules. Recall that a left R-module homomorphism $M \to N$ is a $\mathcal{C}$-*preenvelope* of M if $N \in \mathcal{C}$ and the sequence $\mathrm{Hom}(N, C) \to \mathrm{Hom}(M, C) \to 0$ is exact for any $C \in \mathcal{C}$. Dually we have the definition of $\mathcal{C}$-precovers.

On the other hand, the notion of purity has also a substantial role in module and ring theory [2], [5], [12], [14], [20], [21]. There are two kinds of different notions of purity defined in terms of Hom and $\otimes$ respectively. According to Azumaya [2], an epimorphism $A \to B$ of left R-modules is called C-*pure* if $\mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B) \to 0$ is exact. A monomorphism $X \to Y$ of left R-modules is called *weakly* Z-*pure* if the sequence $0 \to Z \otimes X \to Z \otimes Y$ is exact. An exact sequence

Mathematics Subject Classification: 16D40, 16D50, 16D90, 16E30.
Key words and phrases: preenvelope, precover, adjoint preenvelope, adjoint precover.

$0 \to M \to N \to L \to 0$ of left R-modules is called *pure* if $N \to L$ is C-pure for any finitely presented left R-module C, or equivalently, if $M \to N$ is weakly Z-pure for any (finitely presented) right R-module Z by [20, Proposition 3]. In this case, $M \to N$ is called a *pure monomorphism* and $N \to L$ is called a *pure epimorphism*. A left R-module G is called *pure-injective* (resp. *pure-projective*) if the functor $\mathrm{Hom}(-, G)$ (resp. $\mathrm{Hom}(G, -)$) preserves the exactness of any pure exact sequence of left R-modules.

Our starting point is an easy observation. Let $\mathcal{C}$ be a class of left R-modules. Then an epimorphism $A \to B$ with $A \in \mathcal{C}$ is a $\mathcal{C}$-precover of B is just equivalent to the condition that $A \to B$ is C-pure for all $C \in \mathcal{C}$. So the concept of precovers may be viewed as a generalization of C-pure epimorphisms. It is natural to ask whether there exists a corresponding generalization of weakly Z-pure monomorphisms in terms of the functor $\otimes$, which leads to the arising of the concepts of adjoint preenvelopes and adjoint precovers in the present paper.

The layout of the paper is as follows:

In Section 2, we first give the notions of adjoint preenvelopes and adjoint precovers, which may be viewed the adjoint of preenvelopes and precovers by replacing the functor Hom with $\otimes$. Some basic properties of adjoint preenvelopes and adjoint precovers are obtained. Then we study the existence of $\mathcal{C}$-adjoint preenvelopes and $\mathcal{C}$-adjoint precovers for some special module classes $\mathcal{C}$. For example, it is shown that every left R-module has a pure-injective (cotorsion) adjoint preenvelope and a pure-projective (projective, flat) adjoint precover. Finally we characterize copure flat modules introduced by Enochs and Jenda [10] using adjoint preenvelopes and adjoint precovers. For a left coherent ring R, we prove that: (1) A right R-module M is a cokernel of a flat adjoint preenvelope if and only if M is copure flat. (2) A left R-module N is a kernel of an absolutely pure (injective) adjoint precover if and only if N^+ is copure flat.

In Section 3, we explore the connections between adjoint preenvelopes (adjoint precovers) and preenvelopes (precovers). Let $\mathcal{C}$ be a class of left R-modules. We prove that: (1) If $\mathcal{C}^{++} \subseteq \mathcal{C}$ and $\phi : M \to N$ is a (resp. special) $\mathcal{C}$-preenvelope of M, then $\phi : M \to N$ is a (resp. special) $\mathcal{C}$-adjoint preenvelope of M. (2) If every left R-module in $\mathcal{C}$ is pure-injective and $\phi : M \to N$ is a (resp. special) $\mathcal{C}$-adjoint preenvelope of M, then $\phi : M \to N$ is a (resp. special) $\mathcal{C}$-preenvelope of M. (3) If $\mathcal{C}^{++} \subseteq \mathcal{C}$, then $\phi : M \to N$ with $M \in \mathcal{C}$ is a (resp. special) $\mathcal{C}$-adjoint precover of N if and only if $\phi^{++} : M^{++} \to N^{++}$ is a (resp. special) $\mathcal{C}$-precover of N^{++}. It is also proven that: (1) A left R-module homomorphism $\phi : M \to N$ is a (resp. special) $\mathcal{C}$-adjoint preenvelope of M if and only if $\phi^+ : N^+ \to M^+$ is a (resp. special) $\mathcal{C}^+$-precover of M^+. (2) A left R-module homomorphism $\phi : M \to N$ is a (resp. special) $\mathcal{C}$-adjoint precover of N if and only if $\phi^+ : N^+ \to M^+$ is a (resp. special) $\mathcal{C}^+$-preenvelope of N^+.

Section 4 is devoted to some applications. Let $\mathcal{C}$ be a class of left R-modules and $\mathcal{D}$ be a class of right R-modules such that $\mathcal{C}^+ \subseteq \mathcal{D}$ and $\mathcal{D}^+ \subseteq \mathcal{C}$. We first prove that $\phi : M \to N$ with $M \in \mathcal{C}$ is a $\mathcal{C}$-adjoint precover of N if and only if $\phi^+ : N^+ \to M^+$ is a $\mathcal{D}$-adjoint preenvelope of N^+. Then we characterize some kinds of rings such as left coherent rings, left Noetherian rings, left Artinian rings, left coherent right perfect rings and left semihereditary rings in terms of adjoint preenvelopes and adjoint precovers. For example, we prove that the following are equivalent: (1) R is a left coherent ring. (2) Every left R-module has an absolutely pure (injective) adjoint preenvelope. (3) Every right R-module has a flat adjoint preenvelope. (4) A left R-module homomorphism $\phi : M \to N$ is an absolutely pure adjoint precover of N if and only if $\phi^+ : N^+ \to M^+$ is a flat adjoint preenvelope of N^+. (5) A left R-module homomorphism $\phi : M \to N$ is an absolutely pure adjoint preenvelope of M if and only if $\phi^+ : N^+ \to M^+$ is a flat adjoint precover of M^+.

Throughout this paper, R is an associative ring with identity and all modules are unitary. We denote Mod-R (resp. R-Mod) for the category of right (resp. left) R-modules. M_R (resp. ${}_RM$) denotes a right (resp. left) R-module. For an R-module M, $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ will stand for the character module and $\delta_M : M \to M^{++}$ will mean the canonical pure embedding. Let M and N be R-modules. $\mathrm{Hom}(M, N)$ and $M \otimes N$ will mean $\mathrm{Hom}_R(M, N)$ and $M \otimes_R N$ respectively, and similarly for derived functors $\mathrm{Ext}^1(M, N)$ and $\mathrm{Tor}_1(M, N)$. All classes of modules are assumed to be closed under isomorphisms. For a class $\mathcal{C}$ of R-modules, we write $\mathcal{C}^+ = \{M^+ : M \in \mathcal{C}\}$. For unexplained concepts and notations, we refer the reader to [1], [3], [11], [13], [14], [18], [21], [22].

2. Adjoint preenvelopes, adjoint precovers and copure flat modules

Let us start with the following definition.

Definition 2.1. Let $\mathcal{C}$ be a class of left R-modules. A left R-module homomorphism $\phi : M \to N$ is called a $\mathcal{C}$*-adjoint preenvelope* of M if $N \in \mathcal{C}$ and the sequence $0 \to C^+ \otimes M \xrightarrow{1\otimes\phi} C^+ \otimes N$ is exact for any $C \in \mathcal{C}$. $\phi : M \to N$ is called a $\mathcal{C}$*-adjoint precover* of N if $M \in \mathcal{C}$ and the sequence $0 \to N^+ \otimes C \xrightarrow{\phi^+\otimes 1} M^+ \otimes C$ is exact for any $C \in \mathcal{C}$.

Remark 2.2. Let $\mathcal{C}$ be a class of left R-modules.

(1) A left R-module monomorphism $\phi : M \to N$ is a $\mathcal{C}$-adjoint preenvelope of M if and only if $N \in \mathcal{C}$ and $\phi : M \to N$ is weakly C^+-pure for any $C \in \mathcal{C}$. A left R-module epimorphism $\phi : M \to N$ is a $\mathcal{C}$-adjoint precover of N if and only if $M \in \mathcal{C}$ and $\phi^+ : N^+ \to M^+$ is weakly C-pure for any $C \in \mathcal{C}$.

(2) Any pure monomorphism $M \to N$ with $N \in \mathcal{C}$ is clearly a $\mathcal{C}$-adjoint preenvelope of M. Any pure epimorphism $M \to N$ with $M \in \mathcal{C}$ is obviously a $\mathcal{C}$-adjoint precover of N.

We note that $\mathcal{C}$-adjoint preenvelopes need not be monomorphisms and $\mathcal{C}$-adjoint precovers need not be epimorphisms in general. But we have

Proposition 2.3. *Let $\mathcal{C}$ be a class of left R-modules.*

(1) *If $(R_R)^+ \in \mathcal{C}$, then any $\mathcal{C}$-adjoint preenvelope is a monomorphism.*

(2) *If ${}_RR \in \mathcal{C}$, then any $\mathcal{C}$-adjoint precover is an epimorphism.*

Proof. (1) Let $M \xrightarrow{\phi} N$ be any $\mathcal{C}$-adjoint preenvelope of M. Consider the following commutative diagram:

$$\begin{array}{ccccc} M & \xrightarrow{\cong} & R\otimes M & \xrightarrow{\delta_R\otimes 1_M} & R^{++}\otimes M \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle 1_R\otimes\phi} & & \downarrow{\scriptstyle 1_{R^{++}}\otimes\phi} \\ N & \xrightarrow{\cong} & R\otimes N & \xrightarrow{\delta_R\otimes 1_N} & R^{++}\otimes N. \end{array}$$

Since $(R_R)^+ \in \mathcal{C}$, $R^{++} \otimes M \xrightarrow{1_{R^{++}}\otimes\phi} R^{++} \otimes N$ is monic. Also $R \otimes M \xrightarrow{\delta_R\otimes 1_M} R^{++} \otimes M$ is monic, so $R \otimes M \xrightarrow{1_R\otimes\phi} R \otimes N$ is monic. Thus $M \xrightarrow{\phi} N$ is a monomorphism.

(2) Let $M \xrightarrow{\phi} N$ be any $\mathcal{C}$-adjoint precover of N. Since ${}_RR \in \mathcal{C}$, $0 \to N^+ \otimes R \xrightarrow{\phi^+\otimes 1} M^+ \otimes R$ is exact. Because $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in Mod-$\mathbb{Z}$, $M \xrightarrow{\phi} N$ is an epimorphism. □

The following result gives the closure properties of adjoint preenvelopes and adjoint precovers.

Proposition 2.4. *Let $\mathcal{C}$ be a class of left R-modules.*

(1) *If $\mathcal{C}$ is closed under direct limits and $M_i \to N_i$ is a $\mathcal{C}$-adjoint preenvelope of M_i, $i \in I$, where I is a direct set, then $\varinjlim M_i \to \varinjlim N_i$ is a $\mathcal{C}$-adjoint preenvelope of $\varinjlim M_i$.*

(2) *If $\mathcal{C}$ is closed under direct sums and $M_i \to N_i$ is a $\mathcal{C}$-adjoint preenvelope of M_i, $i \in I$, then $\coprod_{i\in I} M_i \to \coprod_{i\in I} N_i$ is a $\mathcal{C}$-adjoint preenvelope of $\coprod_{i\in I} M_i$.*

(3) *If $\mathcal{C}$ is closed under finite direct sums and $M_i \to N_i$ is a $\mathcal{C}$-adjoint precover of N_i, $i = 1, 2, \ldots, n$, then $\coprod_{i=1}^{n} M_i \to \coprod_{i=1}^{n} N_i$ is a $\mathcal{C}$-adjoint precover of $\coprod_{i=1}^{n} N_i$.*

PROOF. (1) For any $C \in \mathcal{C}$ and $i \in I$, the sequence $0 \to C^+ \otimes M_i \to C^+ \otimes N_i$ is exact. So the sequence $0 \to \varinjlim(C^+ \otimes M_i) \to \varinjlim(C^+ \otimes N_i)$ is exact.

Consider the following commutative diagram:

$$\begin{array}{ccccc} 0 \longrightarrow & \varinjlim(C^+ \otimes M_i) & \longrightarrow & \varinjlim(C^+ \otimes N_i) \\ & \cong \downarrow & & \cong \downarrow \\ & C^+ \otimes \varinjlim M_i & \longrightarrow & C^+ \otimes \varinjlim N_i . \end{array}$$

Thus we get the exact sequence $0 \to C^+ \otimes \varinjlim M_i \to C^+ \otimes \varinjlim N_i$. Hence $\varinjlim M_i \to \varinjlim N_i$ is a $\mathcal{C}$-adjoint preenvelope of $\varinjlim M_i$.

The proof of (2) and (3) are straightforward. □

Next we discuss the existence of $\mathcal{C}$-adjoint preenvelopes and $\mathcal{C}$-adjoint precovers for some special classes of modules $\mathcal{C}$.

Recall that a left R-module M is *cotorsion* [11] if $\mathrm{Ext}^1(F, M) = 0$ for any flat left R-module F. By [11, Proposition 5.3.7], for any R-module M, M^+ is pure-injective and so is cotorsion by [11, Lemma 5.3.23].

Proposition 2.5. *Let R be a ring.*

(1) *Every left R-module has a pure-injective (cotorsion) adjoint preenvelope.*

(2) *Every left R-module has a pure-projective (projective, flat) adjoint precover.*

PROOF. (1) For any left R-module M, M^{++} is pure-injective (cotorsion). So the pure monomorphism $\delta_M : M \to M^{++}$ is a pure-injective (cotorsion) adjoint preenvelope of M by Remark 2.2(2).

(2) For any left R-module M, there is a pure epimorphism $N \to M$ with N pure-projective by [11, Example 8.3.2]. So $N \to M$ is a pure-projective adjoint precover of M by Remark 2.2(2).

On the other hand, there is an epimorphism $P \to M$ with P projective (flat). It is easy to see that $P \to M$ is a projective (flat) adjoint precover of M. □

We will discuss further the existence of adjoint preenvelopes and adjoint precovers for other classes of modules in the latter sections.

Recall that a left R-module M is *absolutely pure* [16] if it is a pure submodule in every module that contains it. Absolutely pure modules are also known as FP-injective modules [19]. Obviously, any injective left R-module is absolutely pure.

By [16, Theorem 3], the left Noetherian rings are characterized by the condition that every absolutely pure left R-module is injective.

Following [10], a right R-module N is *copure flat* if $\mathrm{Tor}_1(N, E) = 0$ for any injective left R-module E, or equivalently, $\mathrm{Tor}_1(N, A) = 0$ for any absolutely pure left R-module A by [15, Remark 3.2].

We also recall that a ring R is *left coherent* if every finitely generated left ideal is finitely presented. By [4, Theorem 1], R is a left coherent ring if and only if a left R-module M being absolutely pure is equivalent to M^+ being flat.

Ding and Chen proved that, if R is left coherent, then a "finitely presented" right R-module M is copure flat if and only if M is a cokernel of a flat preenvelope (see [8, Theorem 3.4]). It is interesting to compare the result and the following theorem.

Theorem 2.6. *Let R be a left coherent ring. The following are equivalent for a right R-module M:*

(1) *M is copure flat.*

(2) *For every exact sequence $0 \to A \to B \to M \to 0$ with B flat, $A \to B$ is a flat adjoint preenvelope of A.*

(3) *M is a cokernel of a flat adjoint preenvelope $f : A \to B$.*

Proof. (1) $\Longrightarrow$ (2) For any flat right R-module F, F^+ is injective by [12, Theorem 2.1]. Then the exact sequence $0 \to A \to B \to M \to 0$ induces the exact sequence

$$0 = \mathrm{Tor}_1(M, F^+) \to A \otimes F^+ \to B \otimes F^+.$$

So $A \to B$ is a flat adjoint preenvelope of A.

(2) $\Longrightarrow$ (3) is clear since there always exists an exact sequence $0 \to A \to B \to M \to 0$ with B flat.

(3) $\Longrightarrow$ (1) There are an epimorphism $\pi : A \to \mathrm{im}(f)$ and a monomorphism $\lambda : \mathrm{im}(f) \to B$ such that $f = \lambda\pi$. For any injective left R-module E, E^+ is flat by [4, Theorem 1]. We get the following commutative diagram:

$$\begin{array}{ccc} A \otimes E^{++} & \xrightarrow{f \otimes 1_{E^{++}}} & B \otimes E^{++} \\ {\scriptstyle \pi \otimes 1_{E^{++}}} \downarrow & \nearrow {\scriptstyle \lambda \otimes 1_{E^{++}}} & \\ \mathrm{im}(f) \otimes E^{++} & & \end{array}$$

Since $A \otimes E^{++} \xrightarrow{f \otimes 1_{E^{++}}} B \otimes E^{++}$ is monic and $A \otimes E^{++} \xrightarrow{\pi \otimes 1_{E^{++}}} \mathrm{im}(f) \otimes E^{++}$ is epic, we have that $\mathrm{im}(f) \otimes E^{++} \xrightarrow{\lambda \otimes 1_{E^{++}}} B \otimes E^{++}$ is monic. The exact sequence

$0 \to \mathrm{im}(f) \xrightarrow{\lambda} B \to M \to 0$ yields the exactness of the sequence

$$0 = \mathrm{Tor}_1(B, E^{++}) \to \mathrm{Tor}_1(M, E^{++}) \to \mathrm{im}(f) \otimes E^{++} \xrightarrow{\lambda \otimes 1_{E^{++}}} B \otimes E^{++},$$

which forces $\mathrm{Tor}_1(M, E^{++}) = 0$. So $\mathrm{Tor}_1(M, E) = 0$ since E is isomorphic to a direct summand of E^{++}. Thus M is copure flat. □

Theorem 2.7. *Let R be a left coherent ring. The following are equivalent for a left R-module M:*

(1) *M^+ is copure flat.*

(2) *For every exact sequence $0 \to M \to N \to L \to 0$ of left R-modules with N absolutely pure (resp. injective), $N \to L$ is an absolutely pure (resp. injective) adjoint precover of L.*

(3) *M is a kernel of an absolutely pure (resp. injective) adjoint precover $f : A \to B$.*

Proof. (1) $\Longrightarrow$ (2) The exact sequence $0 \to M \to N \to L \to 0$ induces the exact sequence $0 \to L^+ \to N^+ \to M^+ \to 0$. For any absolutely pure (resp. injective) left R-module C, we get the induced exact sequence

$$0 = \mathrm{Tor}_1(M^+, C) \to L^+ \otimes C \to N^+ \otimes C.$$

So $N \to L$ is an absolutely pure (resp. injective) adjoint precover of L.

(2) $\Longrightarrow$ (3) is obvious since there exists an exact sequence $0 \to M \to A \to B \to 0$ with A injective.

(3) $\Longrightarrow$ (1) There are an epimorphism $\pi : A \to \mathrm{im}(f)$ and a monomorphism $\lambda : \mathrm{im}(f) \to B$ such that $f = \lambda\pi$. For any injective left R-module E, we obtain the following commutative diagram:

$$\begin{array}{ccc} B^+ \otimes E & \xrightarrow{f^+ \otimes 1_E} & A^+ \otimes E \\ {\scriptstyle \lambda^+ \otimes 1_E}\downarrow & \nearrow {\scriptstyle \pi^+ \otimes 1_E} & \\ (\mathrm{im}(f))^+ \otimes E & & \end{array}$$

Since $B^+ \otimes E \xrightarrow{f^+ \otimes 1_E} A^+ \otimes E$ is monic and $B^+ \otimes E \xrightarrow{\lambda^+ \otimes 1_E} (\mathrm{im}(f))^+ \otimes E$ is epic, $(\mathrm{im}(f))^+ \otimes E \xrightarrow{\pi^+ \otimes 1_E} A^+ \otimes E$ is monic. The exact sequence $0 \to M \to A \xrightarrow{\pi} \mathrm{im}(f) \to 0$ induces the exact sequence $0 \to (\mathrm{im}(f))^+ \xrightarrow{\pi^+} A^+ \to M^+ \to 0$. Note that A^+ is flat by [4, Theorem 1]. So we obtain the induced exact sequence

$$0 = \mathrm{Tor}_1(A^+, E) \to \mathrm{Tor}_1(M^+, E) \to (\mathrm{im}(f))^+ \otimes E \xrightarrow{\pi^+ \otimes 1_E} A^+ \otimes E,$$

whence $\mathrm{Tor}_1(M^+, E) = 0$. Thus M^+ is copure flat. □

Recall that a left R-module M is *copure injective* [10] if $\mathrm{Ext}^1(E, M) = 0$ for any injective left R-module E.

Corollary 2.8. *Let R be a commutative Artinian ring. The following are equivalent for an R-module M:*

(1) *M is copure injective.*

(2) *For every exact sequence $0 \to M \to N \to L \to 0$ with N injective, $N \to L$ is an injective adjoint precover of L.*

(3) *M is a kernel of an injective adjoint precover $f : A \to B$.*

Proof. By [10, Lemma 3.6], M is copure injective if and only if M^+ is copure flat. So it is an immediate consequence of Theorem 2.7. □

3. Relationships between adjoint preenvelopes (adjoint precovers) and preenvelopes (precovers)

Let $\mathcal{C}$ be a class of left R-modules and $\mathcal{D}$ be a class of right R-modules. We write ${}^{\top}\mathcal{C} = \{X \in \text{Mod-}R : \mathrm{Tor}_1(X, C) = 0 \text{ for all } C \in \mathcal{C}\}$ and $\mathcal{D}^{\top} = \{X \in R\text{-Mod} : \mathrm{Tor}_1(D, X) = 0 \text{ for all } D \in \mathcal{D}\}$.

Lemma 3.1. *Let $\mathcal{C}$ be a class of left R-modules and $0 \to M \to N \to L \to 0$ with $N \in \mathcal{C}$ be an exact sequence of left R-modules.*

(1) *If $L \in (\mathcal{C}^+)^{\top}$, then $M \to N$ is a $\mathcal{C}$-adjoint preenvelope of M.*

(2) *If $M^+ \in {}^{\top}\mathcal{C}$, then $N \to L$ is a $\mathcal{C}$-adjoint precover of L.*

Proof. (1) For any $C \in \mathcal{C}$, the exact sequence $0 \to M \to N \to L \to 0$ induces the exact sequence

$$0 = \mathrm{Tor}_1(C^+, L) \to C^+ \otimes M \to C^+ \otimes N.$$

So $M \to N$ is a $\mathcal{C}$-adjoint preenvelope of M.

(2) The exact sequence $0 \to M \to N \to L \to 0$ induces the exact sequence $0 \to L^+ \to N^+ \to M^+ \to 0$. For any $C \in \mathcal{C}$, we get the induced exact sequence

$$0 = \mathrm{Tor}_1(M^+, C) \to L^+ \otimes C \to N^+ \otimes C.$$

So $N \to L$ is a $\mathcal{C}$-adjoint precover of L. □

We call the $\mathcal{C}$-adjoint preenvelope $M \to N$ in Lemma 3.1(1) a *special $\mathcal{C}$-adjoint preenvelope* and the $\mathcal{C}$-adjoint precover $N \to L$ in Lemma 3.1(2) a *special $\mathcal{C}$-adjoint precover*.

Following [11, Definition 7.1.6], given a class $\mathcal{D}$ of right R-modules, a right R-module M is said to have a *special $\mathcal{D}$-precover* if there is an exact sequence $0 \to K \to D \to M \to 0$ with $D \in \mathcal{D}$ and $K \in \mathcal{D}^{\perp} = \{X \in \text{Mod-}R : \text{Ext}^1(D, X) = 0$ for all $D \in \mathcal{D}\}$. M is said to have a *special $\mathcal{D}$-preenvelope* if there is an exact sequence $0 \to M \to D \to L \to 0$ with $D \in \mathcal{D}$ and $L \in {}^{\perp}\mathcal{D} = \{X \in \text{Mod-}R : \text{Ext}^1(X, D) = 0$ for all $D \in \mathcal{D}\}$.

There exist close connections between (resp. special) adjoint preenvelopes (adjoint precovers) and (resp. special) preenvelopes (precovers) as follows.

Theorem 3.2. *Let $\mathcal{C}$ be a class of left R-modules.*

(1) *If $\mathcal{C}^{++} \subseteq \mathcal{C}$ and $\phi : M \to N$ is a (resp. special) $\mathcal{C}$-preenvelope of M, then $\phi : M \to N$ is a (resp. special) $\mathcal{C}$-adjoint preenvelope of M.*

(2) *If every left R-module in $\mathcal{C}$ is pure-injective and $\phi : M \to N$ is a (resp. special) $\mathcal{C}$-adjoint preenvelope of M, then $\phi : M \to N$ is a (resp. special) $\mathcal{C}$-preenvelope of M.*

(3) *If $\mathcal{C}^{++} \subseteq \mathcal{C}$, then $\phi : M \to N$ with $M \in \mathcal{C}$ is a (resp. special) $\mathcal{C}$-adjoint precover of N if and only if $\phi^{++} : M^{++} \to N^{++}$ is a (resp. special) $\mathcal{C}$-precover of N^{++}.*

Proof. (1) We first assume that $\phi : M \to N$ is a $\mathcal{C}$-preenvelope of M. For any $C \in \mathcal{C}$, the sequence $\text{Hom}(N, C) \to \text{Hom}(M, C) \to 0$ is exact. Since $\mathcal{C}^{++} \subseteq \mathcal{C}$, we get the exact sequence $\text{Hom}(N, C^{++}) \to \text{Hom}(M, C^{++}) \to 0$.

Consider the following commutative diagram:

$$\begin{array}{ccccc} (C^+ \otimes N)^+ & \longrightarrow & (C^+ \otimes M)^+ & & \\ \cong\downarrow & & \downarrow\cong & & \\ \text{Hom}(N, C^{++}) & \longrightarrow & \text{Hom}(M, C^{++}) & \longrightarrow & 0. \end{array}$$

So the sequence $(C^+ \otimes N)^+ \to (C^+ \otimes M)^+ \to 0$ is exact. Thus $0 \to C^+ \otimes M \to C^+ \otimes N$ is exact, i.e., $\phi : M \to N$ is a $\mathcal{C}$-adjoint preenvelope of M.

Now we assume that $\phi : M \to N$ is a special $\mathcal{C}$-preenvelope of M. Then there exists an exact sequence of left R-modules $0 \to M \to N \to L \to 0$ with $N \in \mathcal{C}$ and $L \in {}^{\perp}\mathcal{C}$. For any $C \in \mathcal{C}$, $C^{++} \in \mathcal{C}$. By [3, VI. 5.1] or [18, p. 360], we have

$$\text{Tor}_1(C^+, L)^+ \cong \text{Ext}^1(L, C^{++}) = 0.$$

Thus $\text{Tor}_1(C^+, L) = 0$ and hence $\phi : M \to N$ is a special $\mathcal{C}$-adjoint preenvelope of M.

(2) We first assume that $\phi : M \to N$ is a $\mathcal{C}$-adjoint preenvelope of M. Then for any $C \in \mathcal{C}$, the sequence $0 \to C^+ \otimes M \to C^+ \otimes N$ is exact. So we get the exact sequence $(C^+ \otimes N)^+ \to (C^+ \otimes M)^+ \to 0$. Thus the sequence $\mathrm{Hom}(N, C^{++}) \to \mathrm{Hom}(M, C^{++}) \to 0$ is exact. Since C is pure-injective, there exists $\pi : C^{++} \to C$ such that $\pi\delta_C = 1$. So the sequence $\mathrm{Hom}(M, C^{++}) \to \mathrm{Hom}(M, C) \to 0$ is exact.

Consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}(N, C^{++}) & \longrightarrow & \mathrm{Hom}(M, C^{++}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}(N, C) & \longrightarrow & \mathrm{Hom}(M, C) & & \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}$$

Thus the sequence $\mathrm{Hom}(N, C) \to \mathrm{Hom}(M, C) \to 0$ is exact, whence $\phi : M \to N$ is a $\mathcal{C}$-preenvelope of M.

Now we assume that $\phi : M \to N$ is a special $\mathcal{C}$-adjoint preenvelope of M. Then there exists an exact sequence of left R-modules $0 \to M \to N \to L \to 0$ with $N \in \mathcal{C}$ and $L \in (\mathcal{C}^+)^\top$. For any $C \in \mathcal{C}$, we have

$$\mathrm{Ext}^1(L, C^{++}) \cong \mathrm{Tor}_1(C^+, L)^+ = 0.$$

Since $\delta_C : C \to C^{++}$ is split, $\mathrm{Ext}^1(L, C) = 0$. This means that $\phi : M \to N$ is a special $\mathcal{C}$-preenvelope of M.

(3) For any $C \in \mathcal{C}$, the sequence $0 \to N^+ \otimes C \to M^+ \otimes C$ is exact if and only if the sequence $(M^+ \otimes C)^+ \to (N^+ \otimes C)^+ \to 0$ is exact.

Consider the following commutative diagram:

$$\begin{array}{ccc}
(M^+ \otimes C)^+ & \longrightarrow & (N^+ \otimes C)^+ \\
\cong \downarrow & & \downarrow \cong \\
\mathrm{Hom}(C, M^{++}) & \longrightarrow & \mathrm{Hom}(C, N^{++}).
\end{array}$$

Since $\mathcal{C}^{++} \subseteq \mathcal{C}$, $\phi : M \to N$ with $M \in \mathcal{C}$ is a $\mathcal{C}$-adjoint precover of N if and only if $\phi^{++} : M^{++} \to N^{++}$ is a $\mathcal{C}$-precover of N^{++}.

On the other hand, for $K = \ker(\phi)$ and any $C \in \mathcal{C}$, we have the isomorphism

$$\mathrm{Ext}^1(C, K^{++}) \cong \mathrm{Tor}_1(K^+, C)^+.$$

Then there exists an exact sequence of left R-modules $0 \to K \to M \to N \to 0$ with $K^+ \in {}^\top\mathcal{C}$ if and only if there exists an exact sequence of left R-modules $0 \to K^{++} \to M^{++} \to N^{++} \to 0$ with $K^{++} \in \mathcal{C}^\perp$. So $\phi : M \to N$ with $M \in \mathcal{C}$ is a special $\mathcal{C}$-adjoint precover of N if and only if $\phi^{++} : M^{++} \to N^{++}$ is a special $\mathcal{C}$-precover of N^{++}. □

Remark 3.3. (1) The condition "$\mathcal{C}^{++} \subseteq \mathcal{C}$" in Theorem 3.2(1) is not superfluous. For example, if R is a ring which is not left coherent, then there exists a left R-module M which has not a (special) absolutely pure (injective) adjoint preenvelope by Theorem 4.7 in the next section although M has a (special) absolutely pure (injective) preenvelope by [13, Theorem 4.1.6].

(2) The condition "every left R-module in $\mathcal{C}$ is pure-injective" in Theorem 3.2(2) is not superfluous. For example, if R is a left coherent ring but not left Noetherian, then there exists an absolutely pure left R-module M which is not injective by [16, Theorem 3]. Let $M \to E(M)$ be the injective envelope of M. It is clear that $M \to E(M)$ is a (special) absolutely pure adjoint preenvelope. But $M \to E(M)$ is clearly not a (special) absolutely pure preenvelope.

Specializing Theorem 3.2 to the case $\mathcal{C}$ = the class of all pure-injective left R-modules, we have

Corollary 3.4. *Let $\phi : M \to N$ be a left R-module homomorphism.*

(1) *$\phi : M \to N$ is a (resp. special) pure-injective adjoint preenvelope of M if and only if $\phi : M \to N$ is a (resp. special) pure-injective preenvelope of M.*

(2) *$\phi : M \to N$ with M pure-injective is a (resp. special) pure-injective adjoint precover of N if and only if $\phi^{++} : M^{++} \to N^{++}$ is a (resp. special) pure-injective precover of N^{++}.*

Theorem 3.5. *Let $\mathcal{C}$ be a class of left R-modules and $\phi : M \to N$ be a left R-module homomorphism. Then*

(1) *$\phi : M \to N$ is a (resp. special) $\mathcal{C}$-adjoint preenvelope of M if and only if $\phi^+ : N^+ \to M^+$ is a (resp. special) $\mathcal{C}^+$-precover of M^+.*

(2) *$\phi : M \to N$ is a (resp. special) $\mathcal{C}$-adjoint precover of N if and only if $\phi^+ : N^+ \to M^+$ is a (resp. special) $\mathcal{C}^+$-preenvelope of N^+.*

Proof. (1) For any $C \in \mathcal{C}$, we have the following commutative diagram:

$$\begin{array}{ccc} (C^+ \otimes N)^+ & \longrightarrow & (C^+ \otimes M)^+ \\ \cong \big\downarrow & & \cong \big\downarrow \\ \mathrm{Hom}(C^+, N^+) & \longrightarrow & \mathrm{Hom}(C^+, M^+). \end{array}$$

So the sequence $0 \to C^+ \otimes M \to C^+ \otimes N$ is exact if and only if the sequence $(C^+ \otimes N)^+ \to (C^+ \otimes M)^+ \to 0$ is exact if and only if the sequence

$\mathrm{Hom}(C^+, N^+) \to \mathrm{Hom}(C^+, M^+) \to 0$ is exact. Thus $\phi : M \to N$ is a $\mathcal{C}$-adjoint preenvelope of M if and only if $\phi^+ : N^+ \to M^+$ is a $\mathcal{C}^+$-precover of M^+.

On the other hand, by [3, VI. 5.1] or [18, p. 360], for $L = \mathrm{coker}(\phi)$, we have the standard isomorphism

$$\mathrm{Ext}^1(C^+, L^+) \cong \mathrm{Tor}_1(C^+, L)^+.$$

So an exact sequence of left R-modules $0 \to M \to N \to L \to 0$ satisfies $N \in \mathcal{C}$ and $L \in (\mathcal{C}^+)^\top$ if and only if the induced exact sequence $0 \to L^+ \to N^+ \to M^+ \to 0$ satisfies $N^+ \in \mathcal{C}^+$ and $L^+ \in (\mathcal{C}^+)^\perp$. Thus $\phi : M \to N$ is a special $\mathcal{C}$-adjoint preenvelope of M if and only if $\phi^+ : N^+ \to M^+$ is a special $\mathcal{C}^+$-precover of M^+.

(2) For any $C \in \mathcal{C}$, we have the following commutative diagram:

$$\begin{array}{ccc} (M^+ \otimes C)^+ & \longrightarrow & (N^+ \otimes C)^+ \\ \cong\downarrow & & \cong\downarrow \\ \mathrm{Hom}(M^+, C^+) & \longrightarrow & \mathrm{Hom}(N^+, C^+). \end{array}$$

Hence the sequence $0 \to N^+ \otimes C \to M^+ \otimes C$ is exact if and only if the sequence $(M^+ \otimes C)^+ \to (N^+ \otimes C)^+ \to 0$ is exact if and only if the sequence $\mathrm{Hom}(M^+, C^+) \to \mathrm{Hom}(N^+, C^+) \to 0$ is exact. So $\phi : M \to N$ is a $\mathcal{C}$-adjoint precover of N if and only if $\phi^+ : N^+ \to M^+$ is a $\mathcal{C}^+$-preenvelope of N^+.

On the other hand, for $K = \ker(\phi)$, we have the isomorphism

$$\mathrm{Ext}^1(K^+, C^+) \cong \mathrm{Tor}_1(K^+, C)^+.$$

So an exact sequence of left R-modules $0 \to K \to M \to N \to 0$ satisfies $M \in \mathcal{C}$ and $K^+ \in {}^\top\mathcal{C}$ if and only if the induced exact sequence $0 \to N^+ \to M^+ \to K^+ \to 0$ satisfies $M^+ \in \mathcal{C}^+$ and $K^+ \in {}^\perp(\mathcal{C}^+)$. Thus $\phi : M \to N$ is a special $\mathcal{C}$-adjoint precover of N if and only if $\phi^+ : N^+ \to M^+$ is a special $\mathcal{C}^+$-preenvelope of N^+. □

Recall that a pair $(\mathcal{A}, \mathcal{B})$ of classes of R-modules is a Tor-*pair* [13] if $\mathcal{A} = {}^\top\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\top$.

Proposition 3.6. *Let $(\mathcal{A}, \mathcal{B})$ be a* Tor*-pair.*

(1) *If $(\mathcal{B}^\perp)^{++} \subseteq \mathcal{B}^\perp$, then every left R-module has a special $\mathcal{B}^\perp$-adjoint preenvelope.*

(2) *If every left R-module has a special $\mathcal{B}$-adjoint preenvelope, then every left R-module has a special $(\mathcal{B}^+)^\top$-adjoint precover.*

PROOF. (1) By [13, Lemma 2.2.3 and Theorem 3.2.15], every left R-module has a special $\mathcal{B}^{\perp}$-preenvelope. By Theorem 3.2(1), every left R-module has a special $\mathcal{B}^{\perp}$-adjoint preenvelope.

(2) For any left R-module M, there is an exact sequence $0 \to K \to P \to M \to 0$ with P projective. By hypothesis, K has a special $\mathcal{B}$-adjoint preenvelope, i.e., there is an exact sequence $0 \to K \to G \to L \to 0$ with $G \in \mathcal{B}$ and $L \in (\mathcal{B}^+)^{\top}$. So we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & G & \longrightarrow & H & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

For any $B \in \mathcal{B}$, the exact sequence $0 \to P \to H \to L \to 0$ gives rise to the exactness of the sequence

$$0 = \mathrm{Tor}_1(B^+, P) \to \mathrm{Tor}_1(B^+, H) \to \mathrm{Tor}_1(B^+, L) = 0.$$

So $H \in (\mathcal{B}^+)^{\top}$. Since $G^+ \in \mathcal{B}^+$ and $\mathcal{B}^+ \subseteq {}^{\top}((\mathcal{B}^+)^{\top})$, $G^+ \in {}^{\top}((\mathcal{B}^+)^{\top})$. Thus $H \to M$ is a special $(\mathcal{B}^+)^{\top}$-adjoint precover of M. □

4. Applications

In this section, we will characterize some important rings such as left coherent rings, left Noetherian rings, left Artinian rings and left semihereditary rings in terms of adjoint preenvelopes and adjoint precovers.

We first exhibit the dual property between adjoint preenvelopes and adjoint precovers. Recall that a left R-module M is $\mathbb{Q}/\mathbb{Z}$-*reflexive* [1] if $\delta_M : M \to M^{++}$ is an isomorphism.

Theorem 4.1. *Let $\mathcal{C}$ be a class of left R-modules, $\mathcal{D}$ be a class of right R-modules such that $\mathcal{C}^+ \subseteq \mathcal{D}$ and $\mathcal{D}^+ \subseteq \mathcal{C}$ and $\phi : M \to N$ be a left R-module homomorphism.*

(1) *$\phi : M \to N$ with $M \in \mathcal{C}$ is a $\mathcal{C}$-adjoint precover of N if and only if $\phi^+ : N^+ \to M^+$ is a $\mathcal{D}$-adjoint preenvelope of N^+.*

(2) *If $\phi^+ : N^+ \to M^+$ is a $\mathcal{D}$-adjoint precover of M^+ with $N \in \mathcal{C}$, then $\phi : M \to N$ is a $\mathcal{C}$-adjoint preenvelope of M. The converse holds if M is $\mathbb{Q}/\mathbb{Z}$-reflexive.*

Proof. (1)"$\Longrightarrow$" For any $C \in \mathcal{C}$, $0 \to N^+ \otimes C \xrightarrow{\phi^+ \otimes 1_C} M^+ \otimes C$ is exact. Since $\mathcal{D}^+ \subseteq \mathcal{C}$, we have $0 \to N^+ \otimes D^+ \xrightarrow{\phi^+ \otimes 1_{D^+}} M^+ \otimes D^+$ is exact for any $D \in \mathcal{D}$. Note that $M^+ \in \mathcal{D}$. Thus $\phi^+ : N^+ \to M^+$ is a $\mathcal{D}$-adjoint preenvelope of N^+.

"$\Longleftarrow$" Since $\mathcal{C}^+ \subseteq \mathcal{D}$, we have $0 \to N^+ \otimes C^{++} \xrightarrow{\phi^+ \otimes 1_{C^{++}}} M^+ \otimes C^{++}$ is exact for any $C \in \mathcal{C}$. Note that $N^+ \otimes C \xrightarrow{1_{N^+} \otimes \delta_C} N^+ \otimes C^{++}$ is monic and so we get the following commutative diagram:

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & N^+ \otimes C & \xrightarrow{\phi^+ \otimes 1_C} & M^+ \otimes C \\ & & \downarrow{\scriptstyle 1_{N^+} \otimes \delta_C} & & \downarrow{\scriptstyle 1_{M^+} \otimes \delta_C} \\ 0 & \longrightarrow & N^+ \otimes C^{++} & \xrightarrow{\phi^+ \otimes 1_{C^{++}}} & M^+ \otimes C^{++}. \end{array}$$

Consequently $0 \to N^+ \otimes C \xrightarrow{\phi^+ \otimes 1_C} M^+ \otimes C$ is exact. Thus $\phi : M \to N$ is a $\mathcal{C}$-adjoint precover of N.

(2) If $\phi^+ : N^+ \to M^+$ is a $\mathcal{D}$-adjoint precover of M^+ with $N \in \mathcal{C}$, then $0 \to C^+ \otimes M^{++} \xrightarrow{1_{C^+} \otimes \phi^{++}} C^+ \otimes N^{++}$ is exact for any $C \in \mathcal{C}$ since $\mathcal{C}^+ \subseteq \mathcal{D}$. Also $C^+ \otimes M \xrightarrow{1_{C^+} \otimes \delta_M} C^+ \otimes M^{++}$ is monic. So we obtain the following commutative diagram:

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & C^+ \otimes M & \xrightarrow{1_{C^+} \otimes \phi} & C^+ \otimes N \\ & & \downarrow{\scriptstyle 1_{C^+} \otimes \delta_M} & & \downarrow{\scriptstyle 1_{C^+} \otimes \delta_N} \\ 0 & \longrightarrow & C^+ \otimes M^{++} & \xrightarrow{1_{C^+} \otimes \phi^{++}} & C^+ \otimes N^{++}. \end{array}$$

It means that $0 \to C^+ \otimes M \xrightarrow{1_{C^+} \otimes \phi} C^+ \otimes N$ is exact. Thus $\phi : M \to N$ is a $\mathcal{C}$-adjoint preenvelope of M.

Conversely, if $\phi : M \to N$ is a $\mathcal{C}$-adjoint preenvelope of M and M is $\mathbb{Q}/\mathbb{Z}$-reflexive, then $0 \to D^{++} \otimes M \xrightarrow{1_{D^{++}} \otimes \phi} D^{++} \otimes N$ is exact for any $D \in \mathcal{D}$ since $\mathcal{D}^{+} \subseteq \mathcal{C}$.

Consider the following commutative diagram:

$$\begin{array}{ccccc}
D^{++} \otimes M & \xleftarrow{\delta_D \otimes 1_M} & D \otimes M & \xrightarrow{\cong} & D \otimes M^{++} \\
\downarrow{\scriptstyle 1_{D^{++}} \otimes \phi} & & \downarrow{\scriptstyle 1_D \otimes \phi} & & \downarrow{\scriptstyle 1_D \otimes \phi^{++}} \\
D^{++} \otimes N & \xleftarrow{\delta_D \otimes 1_N} & D \otimes N & \xrightarrow{1_D \otimes \delta_N} & D \otimes N^{++}.
\end{array}$$

Since $0 \to D \otimes M \xrightarrow{\delta_D \otimes 1_M} D^{++} \otimes M$ is exact, $0 \to D \otimes M \xrightarrow{1_D \otimes \phi} D \otimes N$ is exact. Because $0 \to D \otimes N \xrightarrow{1_D \otimes \delta_N} D \otimes N^{++}$ is exact, $0 \to D \otimes M^{++} \xrightarrow{1_D \otimes \phi^{++}} D \otimes N^{++}$ is exact. Hence $\phi^{+} : N^{+} \to M^{+}$ is a $\mathcal{D}$-adjoint precover of M^{+}. □

As applications, we list some corollaries of the Theorem 4.1 above.

Corollary 4.2. *Let $\phi : M \to N$ be a left R-module homomorphism.*

(1) *$\phi : M \to N$ with M pure-injective (resp. cotorsion) is a pure-injective (resp. cotorsion) adjoint precover of N if and only if $\phi^{+} : N^{+} \to M^{+}$ is a pure-injective (resp. cotorsion) adjoint preenvelope of N^{+}.*

(2) *If $\phi^{+} : N^{+} \to M^{+}$ with N pure-injective (resp. cotorsion) is a pure-injective (resp. cotorsion) adjoint precover of M^{+}, then $\phi : M \to N$ is a pure-injective (resp. cotorsion) adjoint preenvelope of M.*

Proof. It is obvious by letting $\mathcal{C}$ = the class of all pure-injective (resp. cotorsion) left R-modules and $\mathcal{D}$ = the class of all pure-injective (resp. cotorsion) right R-modules in Theorem 4.1. □

It is known that R is a left Noetherian ring if and only if a left R-module M being injective is equivalent to M^{+} being flat (see [4, Theorem 2]). So we have

Corollary 4.3. *The following are equivalent for a ring R:*

(1) *R is a left Noetherian ring.*

(2) *A left R-module homomorphism $\phi : M \to N$ is an injective adjoint precover of N if and only if $\phi^{+} : N^{+} \to M^{+}$ is a flat adjoint preenvelope of N^{+}.*

(3) *A left R-module homomorphism $\phi : M \to N$ is an injective adjoint preenvelope of M if and only if $\phi^{+} : N^{+} \to M^{+}$ is a flat adjoint precover of M^{+}.*

Proof. (1) $\Longrightarrow$ (2) Letting $\mathcal{C}$ = the class of all injective left R-modules and $\mathcal{D}$ = the class of all flat right R-modules. Then $\mathcal{C}^+ \subseteq \mathcal{D}$ and $\mathcal{D}^+ \subseteq \mathcal{C}$ by [4, Theorem 2]. So (2) follows from Theorem 4.1(1).

(2) $\Longrightarrow$ (1) M is an injective left R-module if and only if $M \xrightarrow{1_M} M$ is an injective adjoint precover if and only if $M^+ \xrightarrow{1_{M^+}} M^+$ is a flat adjoint preenvelope if and only if M^+ is a flat right R-module. Hence R is a left Noetherian ring.

(1) $\Longrightarrow$ (3) " $\Longrightarrow$ " If $\phi : M \to N$ is an injective adjoint preenvelope of M, then ϕ is monic by Proposition 2.3(1). So $\phi^{++} : M^{++} \to N^{++}$ is monic. Since N^+ is flat, $\phi^+ : N^+ \to M^+$ is a flat adjoint precover of M^+.

" $\Longleftarrow$ " It follows from Theorem 4.1(2) by letting $\mathcal{C}$ = the class of all injective left R-modules and $\mathcal{D}$ = the class of all flat right R-modules.

(3) $\Longrightarrow$ (1) M is an injective left R-module if and only if $M \xrightarrow{1_M} M$ is an injective adjoint preenvelope if and only if $M^+ \xrightarrow{1_{M^+}} M^+$ is a flat adjoint precover if and only if M^+ is a flat right R-module. Hence R is a left Noetherian ring. □

Corollary 4.4. *The following are equivalent for a ring R:*

(1) *R is a left Artinian ring.*

(2) *A left R-module homomorphism $\phi : M \to N$ is an injective adjoint precover of N if and only if $\phi^+ : N^+ \to M^+$ is a projective adjoint preenvelope of N^+.*

(3) *A left R-module homomorphism $\phi : M \to N$ is an injective adjoint preenvelope of M if and only if $\phi^+ : N^+ \to M^+$ is a projective adjoint precover of M^+.*

Proof. By [4, Theorem 4], R is left Artinian if and only if a left R-module M being injective is equivalent to M^+ being projective. So the proof is similar to that of Corollary 4.3 by letting $\mathcal{C}$ = the class of all injective left R-modules and $\mathcal{D}$ = the class of all projective right R-modules in Theorem 4.1. □

Corollary 4.5. *The following are equivalent for a ring R:*

(1) *R is a left coherent right perfect ring.*

(2) *A left R-module homomorphism $\phi : M \to N$ is an absolutely pure adjoint precover of N if and only if $\phi^+ : N^+ \to M^+$ is a projective adjoint preenvelope of N^+.*

(3) *A left R-module homomorphism $\phi : M \to N$ is an absolutely pure adjoint preenvelope of M if and only if $\phi^+ : N^+ \to M^+$ is a projective adjoint precover of M^+.*

Proof. By [4, Theorem 3], R is left coherent right perfect if and only if a left R-module M being absolutely pure is equivalent to M^+ being projective.

So the proof is similar to that of Corollary 4.3 by letting $\mathcal{C}$ = the class of all absolutely pure left R-modules and $\mathcal{D}$ = the class of all projective right R-modules in Theorem 4.1. □

Lemma 4.6. *Let R be a ring.*

(1) *A finitely presented left R-module M has a flat adjoint preenvelope $M \to N$ if and only if M has a projective adjoint preenvelope $M \to P$ with P finitely generated.*

(2) *A pure-injective left R-module N has an absolutely pure adjoint precover $M \to N$ if and only if N has an injective adjoint precover $E \to N$.*

Proof. (1)" $\Longrightarrow$ " Let $\phi : M \to N$ be a flat adjoint preenvelope. Then there exists a finitely generated projective left R-module P, $\alpha : M \to P$ and $\beta : P \to N$ such that $\phi = \beta\alpha$. For any projective left R-module Q, we get the following commutative diagram:

$$\begin{array}{ccc} Q^+ \otimes M & \xrightarrow{1_{Q^+}\otimes\phi} & Q^+ \otimes N \\ {\scriptstyle 1_{Q^+}\otimes\alpha}\big\downarrow & \nearrow {\scriptstyle 1_{Q^+}\otimes\beta} & \\ Q^+ \otimes P & & \end{array}$$

Thus $1_{Q^+} \otimes \alpha : Q^+ \otimes M \to Q^+ \otimes P$ is monic since $1_{Q^+} \otimes \phi : Q^+ \otimes M \to Q^+ \otimes N$ is monic. Hence $\alpha : M \to P$ is a projective adjoint preenvelope of M.

" $\Longleftarrow$ " Let $f : M \to P$ be a projective adjoint preenvelope of M. For any flat left R-module F, there is an epimorphism $g : B \to F$ with B projective. So we get the split monomorphism $g^+ : F^+ \to B^+$. Consider the following commutative diagram:

$$\begin{array}{ccc} F^+ \otimes M & \xrightarrow{1_{F^+}\otimes f} & F^+ \otimes P \\ {\scriptstyle g^+\otimes 1_M}\big\downarrow & & \big\downarrow{\scriptstyle g^+\otimes 1_P} \\ B^+ \otimes M & \xrightarrow{1_{B^+}\otimes f} & B^+ \otimes P. \end{array}$$

Since $1_{B^+} \otimes f : B^+ \otimes M \to B^+ \otimes P$ and $g^+ \otimes 1_M : F^+ \otimes M \to B^+ \otimes M$ are monic, $1_{F^+} \otimes f : F^+ \otimes M \to F^+ \otimes P$ is monic. Hence $f : M \to P$ is a flat adjoint preenvelope of M.

(2)" $\Longrightarrow$ " Let $\varphi : M \to N$ be an absolutely pure adjoint precover of N. Then there exist an injective left R-module E, $\psi : M \to E$ and $\theta : E \to N$ such that $\varphi = \theta\psi$. For any injective left R-module H, we get the following commutative diagram:

$$\begin{array}{ccc} & & N^+\otimes H \\ & \swarrow^{\varphi^+\otimes 1_H} & \downarrow{\scriptstyle \theta^+\otimes 1_H} \\ M^+\otimes H & \xleftarrow{\psi^+\otimes 1_H} & E^+\otimes H. \end{array}$$

Thus $\theta^+\otimes 1 : N^+\otimes H \to E^+\otimes H$ is monic since $\varphi^+\otimes 1 : N^+\otimes H \to M^+\otimes H$ is monic. Hence $\theta : E \to N$ is an injective adjoint precover of N.

" $\Longleftarrow$ " Let $\gamma : C \to N$ be an injective adjoint precover of N. For any absolutely pure left R-module D, there is a monomorphism $\eta : D \to G$ with G injective. Consider the following commutative diagram:

$$\begin{array}{ccc} N^+\otimes D & \xrightarrow{\gamma^+\otimes 1_D} & C^+\otimes D \\ {\scriptstyle 1_{N^+}\otimes\eta}\downarrow & & \downarrow{\scriptstyle 1_{C^+}\otimes\eta} \\ N^+\otimes G & \xrightarrow{\gamma^+\otimes 1_G} & C^+\otimes G. \end{array}$$

Note that $1_{N^+}\otimes\eta : N^+\otimes D \to N^+\otimes G$ and $\gamma^+\otimes 1_G : N^+\otimes G \to C^+\otimes G$ are monic. So $\gamma^+\otimes 1_D : N^+\otimes D \to C^+\otimes D$ is monic. Hence $\gamma : C \to N$ is an absolutely pure adjoint precover of N. □

It is known that every left R-module has an absolutely pure precover if R is a left coherent ring [17]. But the converse is still open. The following theorem shows that R is a left coherent ring is equivalent to the condition that every left R-module has an absolutely pure adjoint preenvelope.

Theorem 4.7. *The following are equivalent for a ring R:*

(1) *R is a left coherent ring.*

(2) *Every left R-module has an absolutely pure adjoint preenvelope.*

(3) *Every left R-module has an injective adjoint preenvelope.*

(4) *Every right R-module has a flat adjoint preenvelope.*

(5) *A left R-module homomorphism $\phi : M \to N$ is an absolutely pure adjoint precover of N if and only if $\phi^+ : N^+ \to M^+$ is a flat adjoint preenvelope of N^+.*

(6) *A left R-module homomorphism $\phi : M \to N$ is an absolutely pure adjoint preenvelope of M if and only if $\phi^+ : N^+ \to M^+$ is a flat adjoint precover of M^+.*

Proof. (1) $\Longrightarrow$ (2) For any absolutely pure left R-module E, E^{++} is absolutely pure by [4, Theorem 1]. Since every left R-module M has an absolutely pure preenvelope $M \to N$ by [11, Proposition 6.2.4], $M \to N$ is an absolutely pure adjoint preenvelope of M by Theorem 3.2(1).

(2) $\Longrightarrow$ (1) We will show that M^+ is flat for any absolutely pure left R-module M. Let $0 \to A \to B \to C \to 0$ be any exact sequence of left R-modules. Then A has a monic absolutely pure adjoint preenvelope $A \to N$ by (2) and Proposition 2.3(1). So we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & H & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

which yields the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & M^+ \otimes A & \longrightarrow & M^+ \otimes B & \longrightarrow & M^+ \otimes C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M^+ \otimes N & \longrightarrow & M^+ \otimes H & \longrightarrow & M^+ \otimes C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & M^+ \otimes L & = & M^+ \otimes L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Hence $0 \to M^+ \otimes A \to M^+ \otimes B \to M^+ \otimes C \to 0$ is exact. So M^+ is flat. Thus R is a left coherent ring by [4, Theorem 1].

(1) $\Longrightarrow$ (3) For any injective left R-module E, E^{++} is injective by [4, Theorem 1]. Since every left R-module M has an injective preenvelope $M \to N$, $M \to N$ is an injective adjoint preenvelope of M by Theorem 3.2(1).

(3) $\Longrightarrow$ (1) Let N be an absolutely pure left R-module. Then there exists a pure exact sequence $0 \to N \to E \to L \to 0$ with E injective. Thus we get the split exact sequence $0 \to L^+ \to E^+ \to N^+ \to 0$, By (3) and the proof of (2) $\Longrightarrow$ (1), E^+ is flat, and so is N^+. Thus R is a left coherent ring by [4, Theorem 1].

(1) $\Longrightarrow$ (4) Every right R-module A has a flat preenvelope $A \to B$ by [9, Proposition 5.1]. For any flat right R-module F, F^{++} is flat. So $A \to B$ is a flat adjoint preenvelope of A by Theorem 3.2(1).

(4) $\Longrightarrow$ (1) Since every finitely presented right R-module A has a flat adjoint preenvelope $A \to B$, A has a projective adjoint preenvelope $A \to C$ with C finitely generated by Lemma 4.6(1). For any projective right R-module P, $A \otimes P^+ \to C \otimes P^+$ is monic. So we get the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & A \otimes P^+ & \longrightarrow & C \otimes P^+ \\ & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ & & \mathrm{Hom}(A, P)^+ & \longrightarrow & \mathrm{Hom}(C, P)^+. \end{array}$$

By [6, Lemma 2], τ_1 and τ_2 are isomorphisms. Hence $\mathrm{Hom}(A, P)^+ \to \mathrm{Hom}(C, P)^+$ is monic and so $\mathrm{Hom}(C, P) \to \mathrm{Hom}(A, P)$ is epic. Therefore $A \to C$ is a projective preenvelope. Thus R is a left coherent ring by [7, Corollary 3.11].

(1) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) The proof is similar to that of Corollary 4.3 by [4, Theorem 1] and letting $\mathcal{C}$ = the class of all absolutely pure left R-modules and $\mathcal{D}$ = the class of all flat right R-modules in Theorem 4.1. □

As a consequence of Theorem 4.7, we have

Corollary 4.8. *Let R be a ring. Then*

(1) *Every right R-module has a monic flat adjoint preenvelope if and only if R is a left coherent ring and ${}_RR$ is absolutely pure.*

(2) *Every right R-module has an epic flat adjoint preenvelope if and only if R is a left semihereditary ring.*

PROOF. (1)"$\Longrightarrow$" R is a left coherent ring by Theorem 4.7. Since the injective right R-module $({}_RR)^+$ has a monic flat adjoint preenvelope $({}_RR)^+ \to F$, $({}_RR)^+$ is flat. Thus ${}_RR$ is an absolutely pure left R-module.

"$\Longleftarrow$" By Theorem 4.7, any right R-module M has a flat adjoint preenvelope $\phi : M \to F$. Since $({}_RR)^+$ is flat, ϕ is monic by Proposition 2.3(1).

(2)" $\Longrightarrow$ " R is a left coherent ring by Theorem 4.7.

Suppose that A is a submodule of a flat right R-module B and $i : A \to B$ is the inclusion. Let $\varphi : A \to F$ be an epic flat adjoint preenvelope. Then $A \otimes B^+ \to F \otimes B^+$ is monic. So $\mathrm{Hom}(B^+, F^+) \to \mathrm{Hom}(B^+, A^+)$ is epic. Thus there exists $\alpha : B^+ \to F^+$ such that $i^+ = \varphi^+\alpha$. Since i^+ is epic, φ^+ is epic. Hence φ is monic. Thus $A \cong F$ is flat. So R is a left semihereditary ring by [14, Theorem 4.67].

" $\Longleftarrow$ " By Theorem 4.7, every right R-module has a flat adjoint preenvelope $\phi : M \to F$. Note that $\mathrm{im}(f)$ is flat by [14, Theorem 4.67]. It is easy to check that $M \to \mathrm{im}(f)$ is an epic flat adjoint preenvelope of M. □

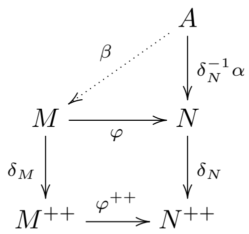
We end this paper by giving the existence of absolutely pure adjoint precovers.

Proposition 4.9. *Let R be a ring.*

(1) *If R is left coherent, then every $\mathbb{Q}/\mathbb{Z}$-reflexive left R-module has an absolutely pure (injective) adjoint precover.*

(2) *If every left R-module has a monic absolutely pure adjoint precover, then R is a left semihereditary ring.*

Proof. (1) Let N be any $\mathbb{Q}/\mathbb{Z}$-reflexive left R-module. Then N has an absolutely pure precover $\varphi : M \to N$ by [17, Theorem 4.9]. For any absolutely pure left R-module A and any homomorphism $\alpha : A \to N^{++}$, there exists $\beta : A \to M$ making



into a commutative diagram. Thus we have

$$\varphi^{++}(\delta_M\beta) = \delta_N\varphi\beta = \delta_N\delta_N^{-1}\alpha = \alpha.$$

So $\varphi^{++} : M^{++} \to N^{++}$ is an absolutely pure precover of N^{++}. Thus $\varphi : M \to N$ is an absolutely pure adjoint precover of N by Theorem 3.2(3).

By Lemma 4.6(2), N has an injective adjoint precover.

(2) Let $\pi : M \to N$ be an epimorphism with M an absolutely pure left R-module. By hypothesis, N has a monic absolutely pure adjoint precover $\lambda : A \to N$.

Then $0 \to N^+ \otimes M \to A^+ \otimes M$ is monic. So $\mathrm{Hom}(A^+, M^+) \to \mathrm{Hom}(N^+, M^+)$ is epic. Thus there exists $\gamma : A^+ \to M^+$ such that $\pi^+ = \gamma\lambda^+$. Since π^+ is monic, λ^+ monic and so λ is epic. Hence $N \cong A$ is absolutely pure. It follows that R is a left semihereditary ring by [16, Theorem 2]. □

Acknowledgements. This research was supported by NSFC (No. 11171149, 11371187). The author would like to thank the referees for the valuable comments and suggestions in shaping the paper into its present form.

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(Received February 26, 2015; revised September 30, 2015)