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Continuum-wise expansive homoclinic classes for generic diffeomorphisms

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Abstract. Let $f : M \to M$ be a diffeomorphism on a closed smooth $n(n \geq 2)$-dimensional Riemannian manifold M. For C^1 generic f, if a homoclinic class $H_f(p)$ is continuum-wise expansive then it is hyperbolic. Moreover, we show that if a diffeomorphism $f : M \to M$ exhibiting a homoclinic tangency associated to a hyperbolic periodic point p, there is g C^1 close to f such that g is not continuum-wise expansive.

1. Introduction

Let M be a closed smooth $n(n \geq 2)$-dimensional Riemannian manifold without boundary, and let $f : M \to M$ be a diffeomorphism. Denote $\mathrm{Diff}(M)$ the space of diffeomorphisms of M with the C^1 topology. Let d be the distance on M induced from the Riemannian metric $\|\cdot\|$ on the tangent bundle TM. For any closed f-invariant set $\Lambda \subset M$, we say that Λ is *expansive* for f if there is $e > 0$ such that for any $x, y \in \Lambda$ if $d(f^n(x), f^n(y)) \leq e$ for all $n \in \mathbb{Z}$ then $x = y$. If $\Lambda = M$ then f is expansive. Roughly speaking, a system is expansive if two points stay near for future and past iterates then they must be equal. This notion was introduced by Utz [21]. In a dynamical system, the notion of expansiveness is a very useful tool to investigate of the stability theory. For instance, Mañé [12] proved that a diffeomorphism f belongs to the C^1-interior of the set of all expansive diffeomorphisms if and only if it is quasi-Anosov. We say that f is *quasi-Anosov* if for any $v \in TM \setminus \{0\}$, the set $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded.

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For expansivity, Kato [7] introduced the generalized concept of expansivity which is called *continuum-wise expansive*. A set Λ is *nondegenerate* if the set Λ is not reduced to one point. We say that $\Lambda \subset M$ is a *subcontinuum* if it is a compact connected nondegenerate subset Λ of M. A diffeomorphism f on M is said to be *continuum-wise expansive* if there is a constant $e > 0$ such that for any nondegenerate subcontinuum A there is an integer $n = n(A)$ such that $\operatorname{diam} f^n(A) \geq e$, where $\operatorname{diam} A = \sup\{d(x,y) : x, y \in A\}$ for any subset A of M. Such the constant e is called a *continuum-wise expansive constant* for f. Note that every expansive homeomorphism is continuum-wise expansive, but its converse is not true (see [6]). In fact, we consider that if for a unit $\mathbf{S}^2$, $f : \mathbf{S}^2 \to \mathbf{S}^2$ is a diffeomorphism, then it is well-known that f does not admit an expansive diffeomorphism, but it admits a continuum-wise expansive diffeomorphisms. Note that by Mañé's result, a robustly expansive diffeomorphism is a quasi-Anosov diffeomorphism. For continuum-wise expansiveness, Sakai [18] proved that if a diffeomorphism f belongs to the C^1-interior of the set of all continuum-wise expansive diffeomorphisms then it is quasi-Anosov. Thus we know that a robustly expansive diffeomorphism is a robustly continuum-wise expansive diffeomorphism.

2. Statement of the main results

A point $x \in M$ is *non-wandering point* of f if a neighborhood U of x there is $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. Denote by $\Omega(f)$ the set of all non-wandering points of f. We say that Λ is *hyperbolic* for f if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \quad \text{and} \quad \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$ then f is *Anosov*.

We say that f satisfies *Axiom A* if its periodic points are dense in the set of non-wandering points $\Omega(f)$, and f is hyperbolic on $\Omega(f)$.

We say that a subset $\mathcal{G} \subset \operatorname{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\operatorname{Diff}(M)$; in this case $\mathcal{G}$ is dense in $\operatorname{Diff}(M)$. A property "P" is said to be (C^1) *generic* if "P" holds for all diffeomorphisms which belong to some residual subset of $\operatorname{Diff}(M)$. Arbieto [2] proved that for C^1 generic f, if f is expansive then it satisfies both Axiom A and the no-cycle condition. In [10], Lee proved that for C^1-generic f, if f is continuum-wise expansive then it satisfies both Axiom A and the no-cycle condition.

If f satisfies Axiom A then the non-wandering set $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_m$, where Λ_i are compact, disjoint, invariant sets, and each Λ_i contains dense periodic orbits. The sets $\Lambda_1, \dots, \Lambda_m$ are called the *basic sets*. It is well known that if p is a hyperbolic periodic point of f with period k then the sets $W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\}$ and $W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$ are C^1-injectively immersed submanifolds of M. A point $x \in W^s(p) \pitchfork W^u(p)$ is called a *homoclinic point* of f associated to p. The closure of the homoclinic points of f associated to p is called the *homoclinic class* of f associated to p, and it is denoted by $H_f(p)$. It is well-known that the basic sets is a homoclinic class $H_f(p)$. This set like occurs for instance in Smale's horseshoe. Actually, in [14] we can see various examples. For homoclinic classes and expansivity, there are many results published in [6], [8], [9], [10], [11], [14], [15], [19], [20], [23]. Among that we introduce two results. First, Das, Lee and Lee [6] proved that if the homoclinic class $H_f(p)$ is C^1-persistently expansive and the chain condition then it is hyperbolic. Here a homoclinic class $H_f(p)$ satisfy the chain condition if for any g C^1-close to f, the homoclinic class $H_g(p_g)$ is the chain component, say, $C_g(p_g)$. Finally, Yang and Gan [23] showed that for C^1 generic f, expansive homoclinic classes are hyperbolic. From the results, we have the following which is a main result of the paper. It is a general result of [23].

Theorem A. *For C^1 generic f, if a homoclinic class $H_f(p)$ is continuum-wise expansive then it is hyperbolic.*

Let p be a hyperbolic periodic point. We say that f is a *homoclinic tangency* if there is a hyperbolic periodic point p whose invariant manifolds $W^s(p)$ and $W^u(p)$ have a non-transverse intersection. Denote by $\mathcal{HT}$ the set of all homoclinic tangency diffeomorphisms. We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. The set Λ is *partially hyperbolic* if the tangent bundle $T_\Lambda M$ has a dominated splitting $E^s \oplus E^c \oplus E^u$ and there exist $C > 0$, and $0 < \lambda < 1$ such that E^s is contracting, E^u is expanding, and for any vector in E^c is less expand than vector in E^u and less contracted than vectors in E^s. In [5], the authors proved that for C^1 generic f, if $f \in \mathrm{Diff}(M) \setminus \overline{\mathcal{HT}}$ then f is partially hyperbolic, where $\overline{A}$ is the closure of A.

Let M be a compact smooth 2-dimensional manifold, and let $f : M \to M$ be a diffeomorphism. Pacifico and Vieites [17] proved that if f having a homoclinic tangency associated to a hyperbolic periodic point p, then there is a g C^1-close to f such that g is not measure expansive. Here, measure expansive was introduce by [13]. By Artigue and Carrasco-Olivera [3, Lemma 2.3], we know that continuum-wise expansive is a more general notion than measure expansive. Thus the following is a general result of [17, Theorem B].

Theorem B. *Let M be a compact smooth $n(\geq 2)$-dimensional manifold, and let $f : M \to M$ be a diffeomorphism. If f has a homoclinic tangency associated to a hyperbolic periodic point p, then there is a g C^1-close to f such that g is not continuum-wise expansive.*

3. Proof of Theorem A

Let p and q be hyperbolic periodic points. We write $p \sim q$ if $W^s(p) \pitchfork W^u(q) \neq \emptyset$ and $W^u(p) \pitchfork W^s(q) \neq \emptyset$. We say that p and q are *homoclinic related* if $p \sim q$.

By Oseledet's theorem, any f-invariant probability μ, almost every point admits a splitting of tangent space

$$T_xM = E_x^1 \oplus \cdots \oplus E_x^k, \quad k = k(x)$$

and real numbers $\chi_1(x, v) \leq \chi_2(x, v) \leq \cdots \chi_k(x, v)$ such that

$$\chi(x, v) = \lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v_i\|,$$

for every non-zero $v_i \in E_x^i$. These objects are uniquely defined and they vary measurably with the point x. If μ is ergodic then the Lyapunov exponents $\chi_i(x, v)$ are constant on orbits. Thus they are constant μ-almost every where if μ is ergodic.

For a homoclinic class $H_f(p)$, Wang [22] proved the following.

Theorem 3.1. *For C^1 generic f, a homoclinic class $H_f(p)$ either is hyperbolic, or contains periodic orbits with arbitrarily long periods that are homoclinically related to p and have a Lyapunov exponent arbitrarily close to* 0.

Let p be a periodic point of f. For any $\delta \in (0, 1)$, we say that p has a *δ-weak eigenvalue* if $D_pf^{\pi(p)}$ has an eigenvalue λ such that $(1-\delta)^{\pi(p)} < |\lambda| < (1+\delta)^{\pi(p)}$.

Remark 3.2. A periodic point p of f is hyperbolic if and only if all the Lyapunov exponents of p are nonzero. Thus a periodic point q has a Lyapunov exponent arbitrarily close to 0 means that q has a δ-weak eigenvalue.

The following notion was introduced by Yang and Gan [23]. For any $\gamma > 0$, a C^1 curve ζ is called *γ-simply periodic curve* of f if (i) ζ is diffeomorphic to $[0, 1]$ and its two endpoints are hyperbolic periodic points of f, (ii) ζ is periodic with period $\pi(\zeta)$ and the length of ζ, that is, $L(f^i(\zeta)) < \gamma$ for any $i \in \{1, 2, \cdots, \pi(\zeta)\}$, where $L(\zeta)$ denotes the length of ζ, and (iii) ζ is normally hyperbolic. For the γ-simply periodic curve and δ-weak eigenvalue, we have the following which was proved by Yang and Gan [23, Lemma 2.1].

Lemma 3.3. *For C^1 generic f, and any hyperbolic periodic point p of f, we have the following:*

(a) *for any $\gamma > 0$, if for any C^1-neighborhood $\mathcal{U}(f)$ of f some $g \in \mathcal{U}(f)$ has a γ-simply periodic curve ς such that two endpoints of ς are homoclinically related with p_g then f has an 2γ-simply periodic curve ζ such that the two endpoints of ζ are homoclinically related to p.*

(b) *for any $\delta > 0$, if for any C^1-neighborhood $\mathcal{U}(f)$ of f, some $g \in \mathcal{U}(f)$ has a periodic $q \sim p_g$ with δ-weak eigenvalue, then f has a periodic point $q_f \sim p$ with 2δ-weak eigenvalue and every eigenvalue of q_f is real.*

Lemma 3.4. *For C^1 generic $f \in \mathrm{Diff}(M)$, if a homoclinic class $H_f(p)$ is continuum-wise expansive then a periodic point q contained in $H_f(p)$ with homoclinically related to p has no a δ-weak eigenvalue.*

Proof. Suppose, by contradiction, that there is a periodic point $q \in H_f(p)$ with homoclinically related to p such that q has a δ-weak eigenvalue. For any $\gamma > 0$, there is g C^1 close to f such that g has a $\gamma/2$-simply periodic curve ς such that two endpoints of ς are homoclinically related with p_g, where p_g is the continuation of p (see [19, Theorem 2]). By Lemma 3.3, f has an γ-simply periodic curve ζ such that the two endpoints of ζ are homoclinically related to p. By [4], $H_f(p) = C_f(p)$, and so, we know $\zeta \subset H_f(p)$. Take $e \geq \gamma$. Then we have

$$\mathrm{diam}(f^{\pi(\zeta)i}(\zeta)) \leq e, \quad \text{for all } i \in \mathbb{Z}. \tag{1}$$

Note that f is continuum-wise expansive if and only if f^n is continuum-wise expansive $n \in \mathbb{Z}\backslash\{0\}$ (see [7, Proposition 2.6]). Since ζ is γ-simply periodic curve, we know

$$f^{\pi(\zeta)}(\zeta) = f^{\pi(\zeta)i}(\zeta) = \zeta,$$

for all $i \in \mathbb{Z}$. Then (1) should be continuum-wise expansive which is a contradiction since ζ is not one point set. $\square$

Proof of Theorem A. By Theorem 3.1, we will prove that for C^1 generic f, if a homoclinic class $H_f(p)$ is measure expansive then there is $\delta > 0$ such that for any $q \sim p$, q has no δ-weak eigenvalue. Suppose, by contradiction, that for any $\delta > 0$, $H_f(p)$ contains a periodic point $q \sim p$ with δ-weak eigenvalue. Then by Lemma 3.4, this is a contradiction. Thus for C^1 generic f, if a homoclinic class $H_f(p)$ is measure expansive then there is no periodic point $q \in H_f(p)$ with homoclinically related to p such that q has a δ-weak eigenvalue, and so, a measure expansive homoclinic class $H_f(p)$ is hyperbolic. □

Let Λ be a closed f-invariant set. We say that Λ is *transitive* if there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the omega-limit set of x. If $\Lambda = M$ then f is transitive. Note that the homoclinic class $H_f(p)$ is a closed, invariant and transitive. We say that Λ is *locally maximal* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$.

Corollary 3.5. *For C^1 generic f, if a transitive diffeomorphism f is continuum-wise expansive then it is Anosov.*

Proof. By [1, Theorem 4.10], for C^1 generic f, a locally maximal homoclinic class $H_f(p)$ is a transitive set. Thus C^1 generic f, if f is transitive then it is a homoclinic class $H_f(p)$. By Theorem A, f is Anosov. □

4. Proof of Theorem B

In this section, we prove that for a diffeomorphism $f : M \to M$ with a homoclinic tangency associated to a hyperbolic point p, if there is a g C^1-close to f such that g exhibits a homoclinic tangency then it is not continuum-wise expansive. To show that we need the following lemma which was proved by Pacifico and Vieitez [16] and also founded in [17, Lemma 4.2].

Lemma 4.1 ([16, Proposition 2.6]). *Let $f : M \to M$ be a diffeomorphism with a homoclinic tangency associated to a hyperbolic periodic point p. Then there is a g C^1-close to f such that g has a small arc $\mathcal{J}$ contained in $W^s(p_g, g) \cap W^u(p_g, g)$, where p_g is the continuation of p.*

Proof of Theorem B. Let f having a homoclinic tangency associated to a hyperbolic periodic point p, and let $\mathcal{U}(f)$ be a C^1-neighborhood of f. Suppose that for any $g \in \mathcal{U}(f)$, g is continuum-wise expansive. Since f has a homoclinic tangency associated to a hyperbolic periodic point p, by Lemma 4.1, there is $h \in \mathcal{U}(f)$ which has a small arc $\mathcal{J}$ contained in $W^s(p_h, h) \cap W^u(p_h, h)$. Clearly, it is not an one point set. Put $\mathrm{diam}(\mathcal{J}) = \alpha$. Let $e = \alpha/4$ be a continuum-wise expansive constant. Since $\mathcal{J} \subset W^s(p_h, h) \cap W^u(p_h, h)$, there is $N > 0$ such that (i) $\mathrm{diam}\, h^i(\mathcal{J}) \leq e/2$ for $-N \leq i \leq N$, and (ii) $h^i(\mathcal{J}) \subset W^s_{e/2}(p_h, h) \cap W^u_{e/2}(p_h, h)$, for $|i| > N$. This means that $\mathrm{diam}\, h^i(\mathcal{J}) \leq e$ for all $i \in \mathbb{Z}$. Since $\mathcal{J}$ is not an one point set, this is a contradiction. □

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