Control systems on the Heisenberg group: equivalence and classification

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Abstract. Left-invariant control affine systems on the three-dimensional Heisenberg group are classified under detached feedback equivalence, strongly detached feedback equivalence, and state space equivalence. The corresponding controllable cost-extended systems (associated to left-invariant optimal control problems with quadratic cost) are also classified. As a corollary, a classification of the left-invariant metric point-affine structures is obtained.

1. Introduction

Heisenberg groups, in continuous but also discrete versions, play a significant role in many areas of mathematics, including analysis, geometry and topology, as well as in mathematical physics. Specifically, invariant structures on the Heisenberg groups (or, more generally, Carnot groups) serve as prototypes for various geometries (see, e.g., [12], [15], [17], [18], [19], [24], [25]). In particular, sub-Riemannian structures on the Heisenberg groups have been studied by several authors (see, e.g., [1], [4], [5], [20], [21]). The sub-Riemannian geodesic problem can be regarded as an optimal control problem (with quadratic cost). In this way the Pontryagin Maximum Principle is used to obtain first-order necessary conditions for minimising geodesics of sub-Riemannian structures. In the words of AGRACHEV and GAMKRELIDZE ([2]) "Even in the classical case of Riemannian geometry, the maximum principle approach to finding geodesics leads to a final result much simpler and shorter than the traditional method of using the

Levi–Civita connection." More broadly, geometric control theory offers a natural framework for various (variational) problems in mathematical physics, mechanics, elasticity, and dynamical systems (see, e.g., [3], [11], [16], [22]).

In this paper we investigate invariant control affine systems on the three-dimensional Heisenberg group. Firstly, the full-rank left-invariant control affine systems are classified under three natural equivalence relations (detached feedback equivalence, strongly detached feedback equivalence, and state space equivalence). Secondly, the controllable cost-extended systems (associated to left-invariant optimal control problems with quadratic cost) are classified under cost-equivalence; this classification is based on the classification of the underlying control systems. Third and lastly, the classification of the cost-extended control systems is reinterpreted as a classification of the left-invariant metric point-affine structures. (Metric point-affine structures can be viewed as generalizing sub-Riemannian structures by allowing for point-affine rather than linear distributions.)

2. Preliminaries

2.1. The three-dimensional Heisenberg group. The Heisenberg group H_3 is the only three-dimensional, simply connected, two-step nilpotent Lie group. One can view this group as the collection of 3×3 upper triangular matrices with ones on the main diagonal, i.e.,

$$\mathsf{H}_3 = \left\{ \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

The Lie algebra of H_3 is given by

$$\mathfrak{h}_3 = \left\{ \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} = x_1 E_1 + x_2 E_2 + x_3 E_3 : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

and has nonzero commutator relations $[E_2, E_3] = E_1$. With respect to the ordered basis (E_1, E_2, E_3), the group of automorphisms is given by

$$\mathsf{Aut}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} : \begin{array}{l} v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, \\[4pt] v_2 w_3 - v_3 w_2 \neq 0 \end{array} \right\}.$$

2.2. Invariant control systems. Let G be a (real, finite-dimensional, connected) Lie group with Lie algebra $\mathfrak{g}$. An (ℓ-input) *left-invariant control affine system* Σ on G can be viewed as a family of left-invariant vector fields $\Xi_u = \Xi(\cdot, u)$ on G, affinely parametrized by controls $u \in \mathbb{R}^\ell$. In classical notation, such a system is written as

$$\dot{g} = \Xi(g, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \quad g \in \mathsf{G}, \ u \in \mathbb{R}^\ell.$$

Here $A, B_1, \ldots, B_\ell$ are elements of the Lie algebra $\mathfrak{g}$ with $B_1, \ldots, B_\ell$ linearly independent. The "product" gA denotes the left translation $T_{\mathbf{1}} L_g \cdot A$ of $A \in \mathfrak{g}$ by the tangent map of $L_g : \mathsf{G} \to \mathsf{G}, h \mapsto gh$. (When G is a matrix Lie group, this product is simply matrix multiplication.) An *admissible control* $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ is a piecewise continuous map. The *trajectory* corresponding to an admissible control $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = \Xi(g(t), u(t))$ for almost every $t \in [0, T]$. A system Σ is *controllable* if for any two points $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(T) = g_1$. It is customary to refer to $A = \Xi(\mathbf{1}, 0)$ as the *drift* of Σ. The *trace* Γ of Σ is the affine subspace $A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ of the Lie algebra $\mathfrak{g}$. Σ is called *homogeneous* if $A \in \Gamma^0$ and *inhomogeneous* otherwise. Σ has *full rank* if its trace generates $\mathfrak{g}$, i.e., $\mathrm{Lie}(\Gamma) = \mathfrak{g}$. (Full rank is a necessary condition for controllability.) Σ is completely determined by the specification of its state space G and its parametrization map $\Xi(\mathbf{1}, \cdot)$. When G is fixed, we specify Σ by simply writing

$$\Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell.$$

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\overline{\Sigma} = (\mathsf{G}, \overline{\Xi})$ be left-invariant control affine systems on G with the same input space $\mathbb{R}^\ell$. Then Σ and $\overline{\Sigma}$ are *detached feedback equivalent* (or DF-equivalent for short) if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}$ and an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g \phi \cdot \Xi(g, u) = \overline{\Xi}(\phi(g), \varphi(u))$ for all $g \in \mathsf{G}$, $u \in \mathbb{R}^\ell$. Likewise, Σ and $\overline{\Sigma}$ are said to be

- *strongly detached feedback equivalent* (or SDF-equivalent for short) if $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is a linear map
- *state space equivalent* (or S-equivalent for short) if $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is the identity map.

In each case, the diffeomorphism ϕ establishes a one-to-one correspondence between trajectories of Σ and $\overline{\Sigma}$. We have the following algebraic characterization of these equivalences.

Proposition 1 (cf. [6], [9]). *Let Σ and $\overline{\Sigma}$ be two full-rank control systems on a simply connected Lie group G.*

1. Σ and $\overline{\Sigma}$ are DF-equivalent if and only if there exists a Lie algebra automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma = \overline{\Gamma}$.

2. Σ and $\overline{\Sigma}$ are SDF-equivalent if and only if there exists a Lie algebra automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma = \overline{\Gamma}$ and $\psi \cdot A = \overline{A}$.

3. Σ and $\overline{\Sigma}$ are S-equivalent if and only if there exists a Lie algebra automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Xi(\mathbf{1}, \cdot) = \overline{\Xi}(\mathbf{1}, \cdot)$.

PROOF SKETCH. (2) Suppose Σ and $\overline{\Sigma}$ are SDF-equivalent. By composing ϕ with an appropriate left translation, we may assume $\phi(\mathbf{1}) = \mathbf{1}$. Hence $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \overline{\Xi}(\mathbf{1}, \varphi(u))$ for $u \in \mathbb{R}^{\ell}$. Thus $T_{\mathbf{1}}\phi \cdot \Gamma = \overline{\Gamma}$ and, as φ is linear, $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, 0) = \overline{\Xi}(\mathbf{1}, 0)$. Moreover, as the elements $\Xi(\mathbf{1}, u) \in \mathfrak{g}$, $u \in \mathbb{R}^{\ell}$ generate $\mathfrak{g}$ and the push forward (by ϕ) of the left-invariant vector fields $\Xi_u = \Xi(\cdot, u)$ are left-invariant vector fields, it follows that ϕ is a group isomorphism (see, e.g., [7]). Consequently $T_{\mathbf{1}}\phi$ is the required Lie algebra isomorphism.

On the other hand, suppose there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma = \overline{\Gamma}$ and $\psi \cdot \Xi(\mathbf{1}, 0) = \overline{\Xi}(\mathbf{1}, 0)$. As G is simply connected, there exists a Lie group automorphism ϕ such that $T_{\mathbf{1}}\phi = \psi$. Furthermore, as $\psi \cdot \Gamma = \overline{\Gamma}$, there exists a unique affine isomorphism $\varphi : \mathbb{R}^{\ell} \to \mathbb{R}^{\ell}$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \overline{\Xi}(\mathbf{1}, \varphi(u))$. Also, as $\psi \cdot \Xi(\mathbf{1}, 0) = \overline{\Xi}(\mathbf{1}, 0)$, it follows that φ is linear. By left-invariance (and the fact that ϕ is an automorphism) it then follows that $T_g\phi \cdot \Xi(g, u) = \overline{\Xi}(\phi(g), \varphi(u))$.

Proofs for (1) and (3) are similar. $\qquad\square$

Corollary 1. *If Σ and $\overline{\Sigma}$ are S-equivalent, then they are SDF-equivalent. Likewise, if Σ and $\overline{\Sigma}$ are SDF-equivalent, then they are DF-equivalent.*

Accordingly, an SDF-equivalence classification may be based upon a DF-equivalence classification in the following manner. Let $\{\Sigma_i : i \in I\}$ be a list of DF-equivalence normal forms of control systems on a simply connected Lie group G (i.e., any system on G is DF-equivalent to exactly one Σ_i). If Σ is DF-equivalent to Σ_i, then it is SDF-equivalent to some system $\overline{\Sigma}$ with trace $\overline{\Gamma} = \Gamma_i$. Hence, in order to find SDF-equivalence normal forms, it suffices to classify (for each $i \in I$) the systems with trace Γ_i. Furthermore, two systems Σ and $\overline{\Sigma}$, both with trace Γ_i, are SDF-equivalent if and only if there exist $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma_i = \Gamma_i$ and $\psi \cdot \Xi(\mathbf{1}, 0) = \overline{\Xi}(\mathbf{1}, 0)$. In other words, it suffices to classify the orbits of the elements $A \in \Gamma_i$ under the subgroup of automorphisms preserving Γ_i.

Likewise, an S-equivalence classification may be based upon an SDF-equivalence classification. Let $\{\Sigma_i : i \in I\}$ be a list of SDF-equivalence normal forms. If Σ is SDF-equivalent to Σ_i, then it is S-equivalent to some system $\overline{\Sigma}$ with drift $\overline{\Xi}(\mathbf{1}, 0) = \Xi_i(\mathbf{1}, 0)$ and trace $\overline{\Gamma} = \Gamma_i$. Hence, in order to find S-equivalence normal forms, it suffices to classify (for each $i \in I$) the systems with trace Γ_i

and drift $\Xi_i(\mathbf{1},0)$. Furthermore, two systems $\Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell$ and $\overline{\Sigma} : A + u_1 \overline{B}_1 + \cdots + u_\ell \overline{B}_\ell$, both with trace Γ_i and drift $A = \Xi_i(\mathbf{1},0)$, are *SDF*-equivalent if and only if there exist $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma_i = \Gamma_i$, $\psi \cdot \Xi_i(\mathbf{1},0) = \Xi_i(\mathbf{1},0)$ and $\psi \cdot B_j = \overline{B}_j$, $j = 1, \ldots, \ell$. In other words, it suffices to classify the parametrizations of the subspace Γ_i^0 up to composition with an automorphism ψ preserving both $\Xi_i(\mathbf{1},0)$ and Γ_i.

2.3. Cost-extended control systems. A left-invariant optimal control problem may be specified by (i) a left-invariant control affine system $\Sigma = (\mathsf{G},\Xi)$, (ii) an affine quadratic cost function $\chi : \mathbb{R}^\ell \to \mathbb{R}$, and (*iii*) boundary data: $g(0) = g_0$, $g(T) = g_1$, and fixed terminal time $T > 0$. Formally,

$$\dot{g}(t) = \Xi(g(t), u(t)), \quad g(\cdot) : [0,T] \to \mathsf{G}, \quad u(\cdot) : [0,T] \to \mathbb{R}^\ell \tag{1}$$

$$g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{G} \tag{2}$$

$$\mathcal{J}(u(\cdot)) = \int_0^T \chi(u(t)) \mathrm{dt} \to \min. \tag{3}$$

Here $\chi : \mathbb{R}^\ell \to \mathbb{R}, u \mapsto (u - \mu)^\top Q(u - \mu)$, $\mu \in \mathbb{R}^\ell$ and Q is a positive definite $\ell \times \ell$ matrix. To each optimal control problem (1)–(2)–(3), we associate the *cost-extended control system* (Σ, χ), where Σ is the invariant control system (1) and χ is the cost, as given in (3). Each cost-extended system corresponds to a family of invariant optimal control problems; by specification of the boundary data (g_0, g_1, T), the associated problem is uniquely determined.

Two cost-extended control systems (Σ, χ) and $(\overline{\Sigma}, \overline{\chi})$ on G are *cost-equivalent* (or *C*-equivalent for short) if there exists a Lie group automorphism $\phi : \mathsf{G} \to \mathsf{G}$ and an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g \phi \cdot \Xi(g, u) = \overline{\Xi}(\phi(g), \varphi(u))$ and $r\chi = \overline{\chi} \circ \varphi$ for some $r > 0$. Cost equivalence establishes a one-to-one correspondence between the associated optimal trajectories, as well as the associated extremal curves ([10]). The following proposition is easy to prove.

Proposition 2. *Two cost-extended control systems (Σ, χ) and $(\overline{\Sigma}, \overline{\chi})$ on a simply connected Lie group G are C-equivalent if and only if there exists a Lie algebra isomorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ and an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \overline{\Xi}(\mathbf{1}, \varphi(u))$ and $\overline{\chi} \circ \varphi = r\chi$ for some $r > 0$.*

Corollary 2. *If (Σ, χ) and $(\overline{\Sigma}, \overline{\chi})$ are C-equivalent, then Σ and $\overline{\Sigma}$ are DF-equivalent.*

We say that a cost-extended system (Σ, χ) has *homogeneous cost* if $\chi(0) = 0$. Any cost-extended system is *C*-equivalent to one with homogeneous cost (cf. [10]).

Corollary 3. *If (Σ, χ) and $(\overline{\Sigma}, \overline{\chi})$ both have homogeneous cost and are C-equivalent, then Σ and $\overline{\Sigma}$ are SDF-equivalent.*

The classification of a class of cost-extended systems may be based upon the classification of the associated control systems. Let $\{\Sigma_i : i \in I\}$ be a list of DF-equivalence normal forms of control systems on a simply connected Lie group G. Any cost-extended system must then be equivalent to a cost-extended system (Σ_i, χ) for some $i \in I$ and some cost χ. Note that (Σ_i, χ) and $(\Sigma_j, \overline{\chi})$ cannot be equivalent unless $i = j$. Therefore, the classification problem reduces to finding normal forms of the cost χ for each Σ_i. Accordingly, we characterize cost equivalence between cost-extended systems for which the underlying system is identical.

Let (Σ, χ) and $(\Sigma, \overline{\chi})$ be two cost-extended systems (with identical underlying systems on a simply connected Lie group). Let $\mathcal{T}_\Sigma$ be the group of feedback transformations leaving Σ invariant. More precisely,

$$\mathcal{T}_\Sigma = \{\varphi \in \mathsf{Aff}(\mathbb{R}^\ell) : \exists \psi \in \mathsf{Aut}(\mathfrak{g}), \ \psi \cdot \Gamma = \Gamma, \ \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))\}.$$

(Here $\mathsf{Aff}(\mathbb{R}^\ell)$ is the group of affine isomorphisms of $\mathbb{R}^\ell$.) The following result is easy to prove.

Proposition 3. *(Σ, χ) and $(\Sigma, \overline{\chi})$ are cost equivalent if and only if there exists an element $\varphi \in \mathcal{T}_\Sigma$ such that $\overline{\chi} = r\chi \circ \varphi$ for some $r > 0$.*

3. Classification of control systems

A classification of the full-rank (left-invariant control affine) systems on H_3 with respect to DF-equivalence was obtained in [8] (see also [9]).

Theorem 1. *Any full-rank system on H_3 is DF-equivalent to exactly one of the systems*

$$\Sigma^{(1,1)} : E_2 + uE_3 \qquad\qquad \Sigma^{(2,0)} : u_1 E_2 + u_2 E_3$$

$$\Sigma_1^{(2,1)} : E_1 + u_1 E_2 + u_2 E_3 \qquad\qquad \Sigma_2^{(2,1)} : E_2 + u_1 E_1 + u_2 E_3$$

$$\Sigma^{(3,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3.$$

We make use of this result in order to get classifications under (the stronger equivalence relations) SDF-equivalence and S-equivalence. As described in Section 2.2, we base our SDF-equivalence classification upon the above list of DF-equivalence normal forms. Likewise, we base our S-equivalence classification upon the list of SDF-equivalence normal forms.

Type	DF	SDF	S $(\alpha, \alpha_i \neq 0,\ \gamma_i \in \mathbb{R})$
$(1,1)$	$E_2 + \langle E_3 \rangle$	$\begin{cases} A = E_2 \\ \Gamma^0 = \langle E_3 \rangle \end{cases}$	$\left[\begin{array}{c\|c} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{array}\right]$
$(2,0)$	$\langle E_2, E_3 \rangle$	$\begin{cases} A = 0 \\ \Gamma^0 = \langle E_2, E_3 \rangle \end{cases}$	$\left[\begin{array}{c\|cc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$
		$\begin{cases} A = E_2 \\ \Gamma^0 = \langle E_2, E_3 \rangle \end{cases}$	$\left[\begin{array}{c\|cc} 0 & 0 & 0 \\ 1 & \alpha & 0 \\ 0 & 0 & 1 \end{array}\right]$
			$\left[\begin{array}{c\|cc} 0 & 0 & 0 \\ 1 & 0 & \alpha \\ 0 & 1 & \gamma_1 \end{array}\right]$
$(2,1)$	$E_1 + \langle E_2, E_3 \rangle$	$\begin{cases} A = E_1 \\ \Gamma^0 = \langle E_2, E_3 \rangle \end{cases}$	$\left[\begin{array}{c\|cc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{array}\right]$
		$\begin{cases} A = E_1 + E_2 \\ \Gamma^0 = \langle E_2, E_3 \rangle \end{cases}$	$\left[\begin{array}{c\|cc} 1 & 0 & 0 \\ 1 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{array}\right]$
			$\left[\begin{array}{c\|cc} 1 & 0 & 0 \\ 1 & 0 & \alpha_2 \\ 0 & \alpha_1 & \gamma_1 \end{array}\right]$
	$E_2 + \langle E_1, E_3 \rangle$	$\begin{cases} A = E_2 \\ \Gamma^0 = \langle E_1, E_3 \rangle \end{cases}$	$\left[\begin{array}{c\|cc} 0 & \alpha & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$
			$\left[\begin{array}{c\|cc} 0 & 0 & \alpha \\ 1 & 0 & 0 \\ 0 & 1 & \gamma_1 \end{array}\right]$

Table 1. Classification of full-rank control systems

Type	DF	SDF	S $(\alpha, \gamma_i \in \mathbb{R})$
$(3,0)$	$\langle E_1, E_2, E_3 \rangle$	$\begin{cases} A = 0 \\ \Gamma^0 = \langle E_1, E_2, E_3 \rangle \end{cases}$	$\left[\begin{array}{c\|ccc} 0 & \alpha & 0 & 0 \\ 0 & \gamma_1 & 1 & 0 \\ 0 & \gamma_2 & 0 & 1 \end{array} \right]$
			$\left[\begin{array}{c\|ccc} 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \gamma_1 \end{array} \right]$
			$\left[\begin{array}{c\|ccc} 0 & 0 & \alpha & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$
		$\begin{cases} A = E_1 \\ \Gamma^0 = \langle E_1, E_2, E_3 \rangle \end{cases}$	$\left[\begin{array}{c\|ccc} 1 & \gamma_1 & \gamma_4 & 0 \\ 0 & \gamma_2 & \gamma_5 & 0 \\ 0 & \gamma_3 & 0 & 1 \end{array} \right]$ $\gamma_1 \gamma_5 - \gamma_2 \gamma_4 \neq 0$
			$\left[\begin{array}{c\|ccc} 1 & 0 & \gamma_1 & \gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_2 & \gamma_4 \end{array} \right]$ $\gamma_2 \gamma_3 - \gamma_1 \gamma_4 \neq 0$
			$\left[\begin{array}{c\|ccc} 1 & \gamma_1 & 0 & \gamma_4 \\ 0 & \gamma_2 & 0 & \gamma_5 \\ 0 & \gamma_3 & 1 & \gamma_6 \end{array} \right]$ $\gamma_2 \gamma_4 - \gamma_1 \gamma_5 \neq 0$
		$\begin{cases} A = E_2 \\ \Gamma^0 = \langle E_1, E_2, E_3 \rangle \end{cases}$	$\left[\begin{array}{c\|ccc} 0 & \gamma_1 & \gamma_4 & 0 \\ 1 & \gamma_2 & \gamma_5 & 0 \\ 0 & \gamma_3 & 0 & 1 \end{array} \right]$ $\gamma_1 \gamma_5 - \gamma_2 \gamma_4 \neq 0$
			$\left[\begin{array}{c\|ccc} 0 & 0 & \gamma_1 & \gamma_3 \\ 1 & 0 & \gamma_2 & \gamma_4 \\ 0 & 1 & 0 & 0 \end{array} \right]$ $\gamma_1 \gamma_4 - \gamma_2 \gamma_3 \neq 0$
			$\left[\begin{array}{c\|ccc} 0 & \gamma_1 & 0 & \gamma_4 \\ 1 & \gamma_2 & 0 & \gamma_5 \\ 0 & \gamma_3 & 1 & \gamma_6 \end{array} \right]$ $\gamma_2 \gamma_4 - \gamma_1 \gamma_5 \neq 0$

Table 2. Classification of full-rank control systems (cont.)

The full classification is exhibited in Tables 1 and 2. For the DF-equivalence normal forms, we tabulate only the trace Γ of the system (as systems with the same trace are DF-equivalent). Likewise for the SDF-equivalence normal forms, we tabulate only the drift $A = \Xi(\mathbf{1}, 0)$ and direction subspace Γ^0. The tables are organised so that the SDF-equivalence normal forms have trace corresponding to that given in the column for DF-equivalence normal forms. On the other hand, for the S-equivalence normal forms, a system specified by $\Sigma : \sum_{i=1}^{3} a_i E_i + u_1 \sum_{i=1}^{3} b_i E_i + u_2 \sum_{i=1}^{3} c_i E_i + u_3 \sum_{i=1}^{3} d_i E_i$ is represented as a matrix

$$\left[\begin{array}{c|ccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array} \right].$$

This representation is also useful in computations as the evaluation $\psi \cdot \Xi(\mathbf{1}, u)$, where $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$, becomes matrix multiplication. The tables are organised so that the S-equivalence normal forms have drift and trace (or rather direction subspace) corresponding to that given in the column for SDF-equivalence normal forms.

As a typical case, we give a full treatment of the classifications up to SDF- and S-equivalence only for those systems which are DF-equivalent to $\Sigma_1^{(2,1)}$.

Theorem 2. *Let Σ be a system on H_3.*

1. *If Σ is DF-equivalent to $\Sigma_1^{(2,1)} : E_1 + u_1 E_2 + u_2 E_3$, then Σ is SDF-equivalent to either $\Sigma_1^{(2,1)}$ or $\Sigma_{12}^{(2,1)} : E_1 + E_2 + u_1 E_2 + u_2 E_3$.*

2. *If Σ is SDF-equivalent $\Sigma_1^{(2,1)}$, then Σ is S-equivalent to exactly one of the systems*

$$\Sigma_\alpha : E_1 + u_1 E_2 + u_2 \alpha E_3.$$

If Σ is SDF-equivalent to $\Sigma_{12}^{(2,1)}$, then Σ is S-equivalent to exactly one of the systems

$$\Sigma'_{\boldsymbol{\alpha}} : E_1 + E_2 + u_1 \alpha_1 E_2 + u_2 \alpha_2 E_3$$

$$\Sigma''_{\boldsymbol{\alpha}, \gamma} : E_1 + E_2 + u_1 \alpha_1 E_3 + u_2 (\alpha_2 E_2 + \gamma E_3).$$

Here $\alpha, \alpha_1, \alpha_2 \neq 0, \gamma \in \mathbb{R}$ parametrize families of distinct class representatives.

PROOF. (1) Suppose Σ is DF-equivalent to $\Sigma_1^{(2,1)}$. We may assume that the trace Γ of Σ coincides with the trace of $\Sigma_1^{(2,1)}$, i.e., $\Gamma = E_1 + \langle E_2, E_3 \rangle$. The

subgroup of automorphisms preserving Γ is given by

$$
\mathsf{Aut}_\Gamma(\mathfrak{h}_3) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} : v_2, v_3, w_2, w_3 \in \mathbb{R}, v_2 w_3 - v_3 w_2 = 1 \right\}.
$$

If $\overline{\Sigma} : \overline{A} + u_1 E_2 + u_2 E_3$ has trace Γ, then $\Sigma : A + u_1 E_2 + u_2 E_3$ is equivalent to $\overline{\Sigma}$ if and only if there exists $\psi \in \mathsf{Aut}_\Gamma(\mathfrak{h}_3)$, such that $\psi \cdot A = \overline{A}$. As the drift A of Σ is an element of Γ, we have that $A = E_1 + a_1 E_2 + a_2 E_3$ for some $a_1, a_2 \in \mathbb{R}$. If $a_1 = a_2 = 0$, then $\Sigma = \Sigma_1^{(2,1)}$. On the other hand, if $a_1^2 + a_2^2 \neq 0$, then

$$
\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{a_1}{a_1^2 + a_2^2} & \dfrac{a_2}{a_1^2 + a_2^2} \\ 0 & -a_2 & a_1 \end{bmatrix} \in \mathsf{Aut}_\Gamma(\mathfrak{h}_3)
$$

and $\psi \cdot A = E_1 + E_2$; hence Σ is SDF-equivalent to $\Sigma_{12}^{(2,1)}$. Since $\psi \cdot E_1 = E_1$ for any $\psi \in \mathsf{Aut}_\Gamma(\mathfrak{h}_3)$, it follows that $\Sigma_1^{(2,1)}$ is not SDF-equivalent to $\Sigma_{12}^{(2,1)}$.

(2) Suppose Σ is SDF-equivalent to $\Sigma_1^{(2,1)}$. Then we may assume that the trace Γ and the drift A of Σ coincide with the trace and the drift of $\Sigma_1^{(2,1)}$, respectively, i.e., $\Gamma = E_1 + \langle E_2, E_3 \rangle$ and $A = E_1$.

The subgroup of automorphisms preserving both A and Γ is given by

$$
\mathsf{Aut}_{A,\Gamma}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} : v_2, v_3, w_2, w_3 \in \mathbb{R}, \; v_2 w_3 - w_2 v_3 = 1 \right\}.
$$

If $\overline{\Sigma} : E_1 + u_1 \overline{B} + u_2 \overline{C}$ has trace Γ and drift E_1, then $\Sigma : E_1 + u_1 B + u_2 C$ is S-equivalent to $\overline{\Sigma}$ if and only if there exists $\psi \in \mathsf{Aut}_{A,\Gamma}(\mathfrak{h}_3)$ such that $\psi \cdot B = \overline{B}$ and $\psi \cdot C = \overline{C}$. As $\langle B, C \rangle = \langle E_2, E_3 \rangle$, Σ has matrix form

$$
\Sigma : \quad \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{array} \right]
$$

where $b_2 c_3 - c_2 b_3 \neq 0$. We have

$$
\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{c_3}{b_2 c_3 - b_3 c_2} & \dfrac{-c_2}{b_2 c_3 - b_3 c_2} \\ 0 & -b_3 & b_2 \end{bmatrix} \in \mathsf{Aut}_{A,\Gamma}(\mathfrak{h}_3)
$$

and

$$\psi \begin{bmatrix} 1 & 0 & 0 \\ \hline 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & b_2c_3 - b_3c_2 \end{bmatrix}.$$

Thus Σ is S-equivalent to Σ_α, $\alpha = b_2c_3 - b_3c_2 \neq 0$. We claim that Σ_α is S-equivalent to $\Sigma_{\overline{\alpha}}$ only if $\alpha = \overline{\alpha}$. Indeed, if

$$\psi \begin{bmatrix} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & \overline{\alpha} \end{bmatrix}$$

for some $\psi \in \mathsf{Aut}_{A,\Gamma}(\mathfrak{h}_3)$, then $\alpha = \overline{\alpha}$ (this is most easily seen by taking determinants both sides).

Suppose Σ is SDF-equivalent to $\Sigma_{12}^{(2,1)}$. We may assume that the trace Γ of Σ is $E_1 + \langle E_2, E_3 \rangle$ and the drift A of Σ is $E_1 + E_2$. The subgroup of automorphisms preserving both A and Γ is given by

$$\mathsf{Aut}_{A,\Gamma}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & w_2 \\ 0 & 0 & 1 \end{bmatrix} : w_2 \in \mathbb{R} \right\}.$$

Σ has matrix form

$$\Sigma : \quad \begin{bmatrix} 1 & 0 & 0 \\ \hline 1 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{bmatrix}$$

where $b_2c_3 - c_2b_3 \neq 0$. If $b_3 = 0$, then

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\dfrac{c_2}{c_3} \\ 0 & 0 & 1 \end{bmatrix} \in \mathsf{Aut}_{A,\Gamma}(\mathfrak{h}_3), \qquad \psi \begin{bmatrix} 1 & 0 & 0 \\ \hline 1 & b_2 & c_2 \\ 0 & 0 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \hline 1 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

and so Σ is S-equivalent to $\Sigma'_{\boldsymbol{\alpha}}$, $\alpha_1 = b_2$, $\alpha_2 = c_3$. Likewise if $b_3 \neq 0$, then

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\dfrac{b_2}{b_3} \\ 0 & 0 & 1 \end{bmatrix} \in \mathsf{Aut}_{A,\Gamma}(\mathfrak{h}_3), \quad \psi \begin{bmatrix} 1 & 0 & 0 \\ \hline 1 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \hline 1 & 0 & -\dfrac{b_2c_3 - c_2b_3}{b_3} \\ 0 & b_3 & c_3 \end{bmatrix}$$

and so Σ is S-equivalent to $\Sigma''_{\boldsymbol{\alpha},\gamma}$, $\alpha_1 = b_3, \alpha_2 = c_2 - \frac{b_2 c_3}{b_3}$, $\gamma = c_3$. It is not difficult to show that $\Sigma'_{\boldsymbol{\alpha}}$ is S-equivalent to $\Sigma'_{\overline{\boldsymbol{\alpha}}}$ only if $\boldsymbol{\alpha} = \overline{\boldsymbol{\alpha}}$; similarly $\Sigma''_{\boldsymbol{\alpha},\gamma}$ is S-equivalent to $\Sigma''_{\overline{\boldsymbol{\alpha}},\overline{\gamma}}$ only if $\boldsymbol{\alpha} = \overline{\boldsymbol{\alpha}}$ and $\gamma = \overline{\gamma}$. Since

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & w_2 \\ 0 & 0 & 1 \end{bmatrix}
\left[\begin{array}{c|cc} 1 & 0 & 0 \\ 1 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{array} \right]
= \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 1 & \alpha_1 & w_2\alpha_2 \\ 0 & 0 & \alpha_2 \end{array} \right]
$$

$$
\neq \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 1 & w_2\overline{\alpha}_1 & \overline{\alpha}_2 + w_2\overline{\gamma} \\ 0 & \overline{\alpha}_1 & \overline{\gamma} \end{array} \right].
$$

for any $\overline{\alpha}_1 \neq 0$, it follows that $\Sigma'_{\boldsymbol{\alpha}}$ is not S-equivalent to $\Sigma''_{\overline{\boldsymbol{\alpha}},\overline{\gamma}}$. $\square$

4. Classification of cost-extended control systems

We now proceed to classify the cost-extended systems on H_3. We shall consider only those systems which are controllable. A system on the Heisenberg group is controllable if and only if the direction space Γ^0 generates $\mathfrak{h}_3$, i.e., $\mathsf{Lie}(\Gamma^0) = \mathfrak{h}_3$ ([23]). Accordingly, the classification problem reduces to that of classifying the cost-extended systems associated with the DF-equivalence normal forms $\Sigma^{(2,0)}$, $\Sigma_1^{(2,1)}$, and $\Sigma^{(3,0)}$ (see Section 2.3). However, we prefer C-equivalent normal forms with underlying system not necessarily being $\Sigma^{(2,0)}$, $\Sigma_1^{(2,1)}$ or $\Sigma^{(3,0)}$ (see Remark 1).

Theorem 3. *Any controllable cost-extended system on* H_3 *is* C-*equivalent to exactly one of the following cost-extended systems*

$$
(\Sigma^{(2,0)}, \chi^{(2)}) : \begin{cases} \Xi^{(2,0)}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3 \\ \chi^{(2)}(u) = u_1^2 + u_2^2 \end{cases}
$$

$$
(\Sigma_2^{(2,0)}, \chi^{(2)}) : \begin{cases} \Xi_2^{(2,0)}(\mathbf{1}, u) = E_2 + u_1 E_2 + u_2 E_3 \\ \chi^{(2)}(u) = u_1^2 + u_2^2 \end{cases}
$$

$$
(\Sigma_{1,\alpha}^{(2,1)}, \chi^{(2)}) : \begin{cases} \Xi_{1,\alpha}^{(2,1)}(\mathbf{1}, u) = E_1 + \alpha E_2 + u_1 E_2 + u_2 E_3 \\ \chi^{(2)}(u) = u_1^2 + u_2^2 \end{cases}
$$

$$
(\Sigma_{\boldsymbol{\alpha}}^{(3,0)}, \chi^{(3)}) : \begin{cases} \Xi_{\boldsymbol{\alpha}}^{(3,0)}(\mathbf{1}, u) = \alpha_1 E_1 + \alpha_2 E_2 + u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \chi^{(3)}(u) = u_1^2 + u_2^2 + u_3^2. \end{cases}
$$

Here $\alpha, \alpha_1, \alpha_2 \geq 0$ *parametrize families of distinct class representatives.*

PROOF. Let (Σ, χ) be a controllable cost-extended system on H_3. Σ is DF-equivalent to exactly one of $\Sigma^{(2,0)}$, $\Sigma_1^{(2,1)}$, and $\Sigma^{(3,0)}$.

Suppose Σ is DF-equivalent to $\Sigma^{(2,0)}$. Then (Σ, χ) is C-equivalent to $(\Sigma^{(2,0)}, \chi_0)$ for some cost $\chi_0 : u \mapsto (u - \mu)^\top Q(u - \mu)$ where $Q = \left[\begin{smallmatrix} a_1 & b \\ b & a_2 \end{smallmatrix}\right]$ is positive definite and $\mu \in \mathbb{R}^2$. The group of feedback transformations leaving $\Sigma^{(2,0)}$ invariant is $\mathcal{T}_{\Sigma^{(2,0)}} = \mathsf{GL}(2, \mathbb{R})$. We have

$$\varphi_1 = \begin{bmatrix} \dfrac{1}{\sqrt{a_1 - \frac{b^2}{a_2}}} & 0 \\[4mm] \dfrac{-b}{a_2\sqrt{a_1 - \frac{b^2}{a_2}}} & \dfrac{1}{\sqrt{a_2}} \end{bmatrix} \in \mathcal{T}_{\Sigma^{(2,0)}}$$

and $\chi_1 = (\chi_0 \circ \varphi_1)(u) = (u - \mu')^\top (u - \mu')$ for some $\mu' \in \mathbb{R}^2$. If $\mu' = \mathbf{0}$, then (Σ, χ) is C-equivalent to $(\Sigma^{(2,0)}, \chi^{(2)})$ (by Proposition 3). Suppose $\mu' \neq \mathbf{0}$. There exists $\alpha > 0$ and $\theta \in \mathbb{R}$ such that $\mu_1' = \alpha \cos\theta$ and $\mu_2' = \alpha \sin\theta$. Hence, $\varphi_2 = \left[\begin{smallmatrix} \alpha\cos\theta & -\alpha\sin\theta \\ \alpha\sin\theta & \alpha\cos\theta \end{smallmatrix}\right] \in \mathcal{T}_{\Sigma^{(2,0)}}$ and

$$\chi_2(u) = \frac{1}{\alpha^2}(\chi_1 \circ \varphi_2)(u) = \left(u - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^\top \left(u - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = (u_1 - 1)^2 + u_2^2.$$

Therefore (Σ, χ) is C-equivalent to $(\Sigma^{(2,0)}, \chi_2)$ (by Proposition 3). Lastly, it is easy to show that $\phi = \mathrm{id}_{\mathsf{H}_3}$ and $\varphi : \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u_1 - 1 \\ u_2 \end{smallmatrix}\right]$ defines a C-equivalence between $(\Sigma^{(2,0)}, \chi_2)$ and $(\Sigma_2^{(2,0)}, \chi^{(2)})$. As $\Sigma^{(2,0)}$ and $\Sigma_2^{(2,0)}$ are not SDF-equivalent (see Table 3), it follows by Corollary 3 that $(\Sigma^{(2,0)}, \chi_2)$ and $(\Sigma_2^{(2,0)}, \chi^{(2)})$ are not C-equivalent.

Suppose Σ is DF-equivalent to $\Sigma_1^{(2,1)}$. A similar argument shows that (Σ, χ) is C-equivalent to $(\Sigma_{1,\alpha}^{(2,1)}, \chi^{(2)})$ for some $\alpha \geq 0$; moreover, $(\Sigma_{1,\alpha}^{(2,1)}, \chi^{(2)})$ is C-equivalent to $(\Sigma_{1,\overline{\alpha}}^{(2,1)}, \chi^{(2)})$ only if $\alpha = \overline{\alpha}$ (cf. [10]).

Suppose Σ is DF-equivalent to $\Sigma^{(3,0)}$. Then (Σ, χ) is C-equivalent to $(\Sigma^{(3,0)}, \chi_0)$ for some cost $\chi_0 : u \mapsto (u - \mu)^\top Q(u - \mu)$, where

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}$$

is positive definite and $\mu \in \mathbb{R}^3$. It is easy to show that the group of feedback transformations leaving $\Sigma^{(3,0)}$ invariant is

$$\mathcal{T}_{\Sigma^{(3,0)}} = \left\{ \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} : \begin{array}{l} v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, \\ v_2 w_3 - v_3 w_2 \neq 0 \end{array} \right\} \cong \mathsf{Aut}(\mathfrak{h}_3).$$

We have

$$\varphi_1 = \begin{bmatrix} a_1 & \dfrac{-a_3 b_1 + b_2 b_3}{a_1 a_3 - b_2^2} & -b_2 \\ 0 & 1 & 0 \\ 0 & \dfrac{b_1 b_2 - a_1 b_3}{a_1 a_3 - b_2^2} & a_1 \end{bmatrix} \in \mathcal{T}_{\Sigma^{(3,0)}}$$

and

$$\chi_1(u) = \frac{1}{a_1^3}(\chi_0 \circ \varphi_1)(u) = (u - \mu')^\top \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_2' & 0 \\ 0 & 0 & a_3' \end{bmatrix} (u - \mu')$$

for some $a_2', a_3' > 0, \mu' \in \mathbb{R}^3$. Now

$$\varphi_2 = \begin{bmatrix} \sqrt{a_2' a_3'} & 0 & 0 \\ 0 & \sqrt{a_3'} & 0 \\ 0 & 0 & \sqrt{a_2'} \end{bmatrix} \in \mathcal{T}_{\Sigma^{(3,0)}}$$

and $\chi_2(u) = \frac{1}{\sqrt{a_2' a_3'}}(\chi_1 \circ \varphi_2)(u) = (u - \mu'')^\top (u - \mu'')$ for some $\mu'' \in \mathbb{R}^3$. We may assume $\mu_1'' \geq 0$; if $\mu_1'' < 0$, then the feedback transformation

$$\bar{\varphi} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in \mathcal{T}_{\Sigma^{(3,0)}}$$

serves to change its sign. Let $\alpha_1 = \mu_1''$. There exists $\alpha_2 \geq 0$ and $\theta \in \mathbb{R}$ such that $\mu_2'' = \alpha_2 \cos\theta$ and $\mu_3'' = \alpha_2 \sin\theta$. Hence

$$\varphi_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \in \mathcal{T}_{\Sigma^{(3,0)}}$$

and

$$\chi_{3,\boldsymbol{\alpha}}(u) = (\chi_2 \circ \varphi_3)(u) = (u_1 - \alpha_1)^2 + (u_2 - \alpha_2)^2 + u_3^2.$$

Thus, by Proposition 3, (Σ, χ) is C-equivalent to $(\Sigma^{(3,0)}, \chi_{3,\boldsymbol{\alpha}})$. We claim that $(\Sigma^{(3,0)}, \chi_{3,\boldsymbol{\alpha}})$, $\alpha_1, \alpha_2 \geq 0$ is C-equivalent to $(\Sigma^{(3,0)}, \chi_{3,\bar{\boldsymbol{\alpha}}})$, $\bar{\alpha}_1, \bar{\alpha}_2 \geq 0$ only if $\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}}$. Indeed, suppose $(\Sigma^{(3,0)}, \chi_{3,\boldsymbol{\alpha}})$ is C-equivalent to $(\Sigma^{(3,0)}, \chi_{3,\bar{\boldsymbol{\alpha}}})$. Then $\chi_{3,\bar{\boldsymbol{\alpha}}} = r\chi_{3,\boldsymbol{\alpha}} \circ \varphi$ for some $r > 0$ and $\varphi \in \mathcal{T}_{\Sigma^{(3,0)}}$. A straightforward but tedious computation shows that this implies $\alpha_1 = \bar{\alpha}_1$ and $\alpha_2 = \bar{\alpha}_2$. Finally, $\phi = \mathrm{id}_{\mathsf{H}_3}$ and $\varphi : \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mapsto \begin{bmatrix} u_1 - \alpha_1 \\ u_2 - \alpha_2 \\ u_3 \end{bmatrix}$ defines a C-equivalence between $(\Sigma^{(3,0)}, \chi_{3,\boldsymbol{\alpha}})$ and $(\Sigma_{\boldsymbol{\alpha}}^{(3,0)}, \chi^{(3)})$. $\qquad \square$

Remark 1. The choice of normal forms is motivated by making the cost and parametrization of Γ^0 as simple as possible (parameters are moved to the drift whenever possible). However, other choices for normal forms are feasible (e.g., making the drift and the cost as simple as possible, or taking the system as the DF-equivalence normal form and leaving all additional parameters in the cost). For instance, for $\alpha_1, \alpha_2 > 0$, the cost extended system

$$(\Sigma_{\boldsymbol{\alpha}}^{(3,0)}, \chi^{(3)}) : \begin{cases} \Xi_{\boldsymbol{\alpha}}^{(3,0)}(\mathbf{1}, u) = \alpha_1 E_1 + \alpha_2 E_2 + u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \chi^{(3)}(u) = u_1^2 + u_2^2 + u_3^2 \end{cases}$$

is C-equivalent to each of the following cost-extended systems

$$\begin{cases} \Xi(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \chi(u) = (u_1 - \alpha_1)^2 + (u_2 - \alpha_2)^2 + u_3^2 \end{cases}$$

$$\begin{cases} \Xi(\mathbf{1}, u) = E_1 + E_2 + \dfrac{1}{\alpha_2} u_1 E_1 + \dfrac{\alpha_1}{\alpha_2^2} u_2 E_2 + u_3 E_3 \\ \chi(u) = u_1^2 + u_2^2 + u_3^2 \end{cases}$$

$$\begin{cases} \Xi(\mathbf{1}, u) = E_1 + E_2 + u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \chi(u) = \alpha_2^2 u_1^2 + \dfrac{\alpha_2^4}{\alpha_1^2} u_2^2 + u_3^2. \end{cases}$$

5. Metric point-affine distributions

One can associate to any cost-extended system $(\Sigma = (\mathsf{G}, \Xi), \chi)$ with homogeneous cost a quadruple $(\mathsf{G}, X, \mathcal{D}, \mathbf{g})$. Here $X = \Xi_0$ is a distinguished left-invariant vector field; $\mathcal{D}$ is a left-invariant distribution specified by $\mathcal{D}(g) = \mathrm{im}(\Xi(g, \cdot) - X(g)) \subseteq T_g \mathsf{G}$; $\mathbf{g}$ is a left-invariant Riemannian metric on $\mathcal{D}$ specified by $\mathbf{g}(\Xi_u - X, \Xi_u - X) = \chi(u)$. We shall refer to such a quadruple as a *left-invariant metric point-affine structure* (or LiMA structure for short). The case when $X = 0$ corresponds to a left-invariant sub-Riemannian structure. Metric point-affine structures (on low-dimensional manifolds) have been investigated in [14] (see also [13]).

Suppose $(\mathsf{G}, X, \mathcal{D}, \mathbf{g})$ and $(\overline{\mathsf{G}}, \overline{X}, \overline{\mathcal{D}}, \overline{\mathbf{g}})$ are two LiMA structures. A diffeomorphism $\phi : \mathsf{G} \to \overline{\mathsf{G}}$ is said to be a *point-affine isometry* if $\phi_* X = \overline{X}$, $\phi_* \mathcal{D} = \overline{\mathcal{D}}$, and $\mathbf{g} = \phi^* \overline{\mathbf{g}}$. It is not difficult to show that if two cost-extended systems are C-equivalent, then their associated LiMA structures are point-affine isometric up to rescaling (i.e., $\mathbf{g} = r\phi^* \overline{\mathbf{g}}$ for some $r > 0$). It turns out that for controllable

cost-extended systems on H_3, the converse also holds. We briefly substantiate this claim below.

Suppose $\phi : \mathsf{H}_3 \to \mathsf{H}_3$ is a point-affine isometry between two LiMA structures $(\mathsf{H}_3, X, \mathcal{D}, \mathbf{g})$ and $(\mathsf{H}_3, \overline{X}, \overline{\mathcal{D}}, \overline{\mathbf{g}})$ associated to some controllable cost-extended systems. (Then $\mathcal{D}$ and $\overline{\mathcal{D}}$ are bracket generating.) We may assume that $\phi(\mathbf{1}) = \mathbf{1}$ (by composition with an appropriate left translation). In the two-input homogeneous case, it then follows that ϕ defines a sub-Riemannian isometry between $(\mathsf{H}_3, \mathcal{D}, \mathbf{g})$ and $(\mathsf{H}_3, \overline{\mathcal{D}}, \overline{\mathbf{g}})$ and hence ϕ is a Lie group automorphism (as it is an isometry between Carnot groups, see [17] and also [5]). For the two-input inhomogeneous case, let (Y_1, Y_2) be an orthonormal frame of left-invariant vector fields for $(\mathcal{D}, \mathbf{g})$. Let $\widehat{\mathbf{g}}$ denote the Riemannian structure on H_3 with orthonormal frame (X, Y_1, Y_2); similarly, let $\widehat{\overline{\mathbf{g}}}$ denote the Riemannian structure on H_3 with orthonormal frame $(\overline{X}, \overline{Y}_1, \overline{Y}_2)$, where $(\overline{Y}_1, \overline{Y}_2)$ is an orthonormal frame for $(\overline{\mathcal{D}}, \overline{\mathbf{g}})$. Then ϕ is an isometry between left-invariant Riemannian structures $\widehat{\mathbf{g}}$ and $\widehat{\overline{\mathbf{g}}}$ on a nilpotent Lie group and so ϕ is a Lie group automorphism ([25], see also [5]). Likewise, in the three-input case, we have that $\mathcal{D} = T\mathsf{H}_3$ and so ϕ defines a Riemannian isometry between $\mathbf{g}$ and $\overline{\mathbf{g}}$; hence ϕ is a Lie group automorphism. It is a simple matter to show that if there exists a point-affine isometry (between $(\mathsf{H}_3, X, \mathcal{D}, \mathbf{g})$ and $(\mathsf{H}_3, \overline{X}, \overline{\mathcal{D}}, \overline{\mathbf{g}})$) which is also a Lie group automorphism, then the associated cost-extended systems are C-equivalent.

Consequently, by Theorem 3, we obtain the following classification of LiMA structures.

Corollary 4. *Any LiMA structure on* H_3, *with bracket generating distribution, is point-affine isometric up to rescaling to exactly one of the following structures*

$$(\mathsf{H}_3, 0, \mathcal{D}^{(1)}, \mathbf{g}^{(1)}), \quad (\mathsf{H}_3, E_2, \mathcal{D}^{(1)}, \mathbf{g}^{(1)}), \quad (\mathsf{H}_3, E_1 + \alpha E_2, \mathcal{D}^{(1)}, \mathbf{g}^{(1)}), \alpha \geq 0,$$

$$(\mathsf{H}_3, \alpha_1 E_1 + \alpha_2 E_2, \mathcal{D}^{(0)}, \mathbf{g}^{(0)}), \alpha_1, \alpha_2 \geq 0.$$

Here $(\mathcal{D}^{(1)}, \mathbf{g}^{(1)})$ *is the structure admitting orthonormal frame* (E_2, E_3); $(\mathcal{D}^{(0)}, \mathbf{g}^{(0)})$ *is the structure admitting orthonormal frame* (E_1, E_2, E_3).

Corollary 5 (cf. [1], [5]). *Any left-invariant Riemannian structure on* H_3 *is isometric, up to rescaling, to* $\mathbf{g}^{(0)}$. *Likewise, any left-invariant sub-Riemannian structure on* H_3 *is isometric, up to rescaling, to* $(\mathcal{D}^{(1)}, \mathbf{g}^{(1)})$.

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