Publ. Math. Debrecen
45 / 3-4 (1994), 407–413

Local degrees of simplicial mappings

By YU. A. SHASHKIN (Ekaterinburg)

In this paper we give a difference formula for local degrees of a simplicial mapping between pseudomanifolds (Theorem 1). This formula generalizes combinatorial lemmas of Sperner [8] and Tucker [9], as well as some results of Fan [4,5].

§**1** We begin with recalling the necessary definitions. A finite non more than n-dimensional simplicial complex K^n is said to be an *n-dimensional simplicial pseudomanifold* (or briefly, *n-pseudomanifold*), if: 1) every simplex (of any dimension) of K^n is a face of at least one n-simplex of K^n, 2) every $(n-1)$-simplex of K^n is a face of at most two n-simplices of K^n, 3) given two n-simplices σ^n and τ^n of K^n, there is a finite chain $\{\sigma^n = \sigma_1^n, \sigma_2^n, \dots, \sigma_{m-1}^n, \sigma_m^n = \tau^n\}$ of n-simplices, such that the intersection of any two neighboring simplices of the chain is their common $(n-1)$-face.

An n-pseudomainfold K^n may or may not have a boundary ∂K^n. By definition, the *boundary* ∂K^n of K^n consists of all its $(n-1)$-simplices σ^{n-1} (and their faces) such that σ^{n-1} is a face of exactly one n-simplex of K^n.

A pseudomanifold K^n is said to have a *coherent orientation*, if: 1) all its n-simplices and $(n-1)$-simplices are oriented, 2) whenever an $(n-1)$-simplex $\sigma^{n-1} \in K^n$ is a common face of two n-simplices σ_1^n and σ_2^n, the orientations of σ_1^n and σ_2^n induce opposite orientations on σ^{n-1}, 3) every simplex $\sigma^{n-1} \in \partial K^n$ has the orientation induced by that of the unique simplex σ^n incident to σ^{n-1}. A pseudomanifold is said to be *orientable* if it has a coherent orientation.

Let K^n and M^n be two n-pseudomanifolds. A mapping $f : K^n \to M^n$ is called *simplicial*, if: 1) for every simplex $\sigma^m \in K^n$ with vertices

This work was supported by the Foundation of Fundamental Researches of Russia Grant 93-011-1401.

$v_0, v_1, \dots, v_m$ the points $f(v_0), f(v_1), \dots, f(v_m)$ are vertices (not necessarily distinct) of some simplex of M^n, 2) the mapping f is linear on every simplex $\sigma^m \in K^n$.

Let K^n and M^n be simplicial n-pseudomanifolds, each with a fixed coherent orientation, and $f : K^n \to M^n$ a simplicial mapping. Let us call an n-simplex $\sigma^n \in K^n$ *positive*: $\sigma^n > 0$ (resp., *negative*: $\sigma^n < 0$) if f maps it onto an n-simplex $\tau^n \in M^n$ with preserving (resp., with reversing) of the orientation. For a fixed n-simplex $\tau^n \in M^n$, the difference between the numbers of positive and negative n-simplices of K^n that are mapped onto τ^n, is said to be the *local degree* of the mapping f on τ^n. If these differences are the same for all n-simplices $\tau^n \in M^n$, their common value is called the *degree* $\deg f$ of the simplicial mapping $f : K^n \to M^n$.

If the pseudomanifolds K^n and M^n do not have a coherent (or in general any) orientation, it is natural to speak about degree (or local degree) modulo 2; it assumes the values 0 and 1 and defines the parity of the number of simplices $\sigma^n \in K^n$ that are mapped onto a simples $\tau^n \in M^n$.

All statements of this paper in which the notion of degree is used, are formulated and proved for the *oriented* case: to preserve their validity in the non-oriented case, all equalities should be reduced modulo 2.

§**2.** Let $f : K^n \to M^n$ be a simplicial mapping. Let us consider a finite chain

$$\{\tau_1^n, \tau_1^{n-1}, \tau_2^n, \tau_2^{n-1}, \dots, \tau_{p-1}^{n-1}, \tau_p^n\} \tag{1}$$

of alternately n- and $(n-1)$-simplices of M^n, such that every $(n-1)$-simplex is the common face of the two neighboring n-simplices and these two n-simplices are distinct. We assume in what follows that the simplex τ_i^{n-1} $(i = 1, \dots, p-1)$ in the chain (1) has the orientation induced by that of the simplex τ_{i+1}^n. Let $d(\tau^n)$ denote the local degree of the mapping f on $\tau^n \in M^n$, and let $d(\tau_i^{n-1})$ denote the local degree of the restriction $f \mid \partial K^n$ on τ_i^{n-1} $(i = 1, \dots, p-1)$. Here $f \mid \partial K^n$ means the mapping $f \mid \partial K^n : \partial K^n \to skel_{n-1} M^n$.

Theorem 1. *We have the following equality*

$$d(\tau_p^n) - d(\tau_1^n) = \sum_{i=1}^{p-1} d(\tau_i^{n-1}). \tag{2}$$

Since the left hand side of (2) depends on τ_1^n and τ_p^n only, the right hand side is independent of the choice of the chain (1) joining the simplices τ_1^n and τ_p^n.

PROOF. We first consider the case when the chain (1) is of the form

$$\left\{\tau_1^n, \tau^{n-1}, \tau_2^n\right\}. \tag{1a}$$

Let $\sigma_1^n \in K^n$ be a simplex such that either $f(\sigma_1^n) = \tau_1^n$ or $f(\sigma_1^n) = \tau_2^n$ and its face σ_1^{n-1} is mapped by f onto τ^{n-1}. It follows immediately from the definition of a simplicial mapping that in the pseudomanifold K^n there is either a chain

$$\left\{\sigma_1^n, \sigma_1^{n-1}, \sigma_2^n, \sigma_2^{n-1}, \ldots, \sigma_{m-1}^{n-1}, \sigma_m^n\right\}, \ m \geq 2, \tag{3}$$

such that: a) no simplex σ_i^{n-1} $(i = 1, \ldots, m-1)$ lies on the boundary ∂K^n, b) all intermediate simplices are mapped onto τ^{n-1}:

$$f(\sigma_1^{n-1}) = f(\sigma_2^n) = \cdots = f(\sigma_{m-1}^n) = f(\sigma_{m-1}^{n-1}) = \tau^{n-1},$$

and c) one of the following three conditions holds:

$$f(\sigma_1^n) = \tau_1^n, \quad f(\sigma_m^n) = \tau_2^n, \tag{3.1}$$

$$f(\sigma_1^n) = f(\sigma_m^n) = \tau_1^n, \tag{3.2}$$

$$f(\sigma_1^n) = f(\sigma_m^n) = \tau_2^n, \tag{3.3}$$

or a chain

$$\{\sigma_1^n, \sigma_1^{n-1}, \sigma_2^n, \sigma_2^{n-1}, \ldots, \sigma_m^n, \sigma_m^{n-1}\}, \quad m \geq 1, \tag{4}$$

such that: a) no simplex σ_i^{n-1}, $i < m$, lies on the boundary ∂K^n, b) the simplex σ_m^{n-1} lies on this boundary, c) all simplices except for σ_1^n, are mapped onto τ^{n-1}:

$$f(\sigma_1^{n-1}) = f(\sigma_2^n) = \cdots = f(\sigma_m^n) = f(\sigma_m^{n-1}) = \tau^{n-1},$$

and d) one of the following two conditions holds:

$$f(\sigma_1^n) = \tau_1^n, \tag{4.1}$$

$$f(\sigma_1^n) = \tau_2^n. \tag{4.2}$$

It is easy to see that the following implications are valid:

in the chain (3.1):	$(\sigma_1^n > 0) \implies (\sigma_m^n > 0)$,
	$(\sigma_1^n < 0) \implies (\sigma_m^n < 0)$,
in the chain (3.2) or (3.3):	$(\sigma_1^n > 0) \implies (\sigma_m^n < 0)$,
	$(\sigma_1^n < 0) \implies (\sigma_m^n > 0)$,
in the chain (4.1):	$(\sigma_1^n > 0) \implies (\sigma_m^{n-1} < 0)$,
	$(\sigma_1^n < 0) \implies (\sigma_m^{n-1} > 0)$,
in the chain (4.2):	$(\sigma_1^n > 0) \implies (\sigma_m^{n-1} > 0)$,
	$(\sigma_1^n < 0) \implies (\sigma_m^{n-1} < 0)$

(let us recall that in the chain (1a) the orientation of the simplex τ^{n-1} is induced by that of τ_2^n).

According to these implications, we shall say that the chain of type (3.1) defines a path joining $+\tau_1^n$ with $+\tau_2^n$ or $-\tau_1^n$ with $-\tau_2^n$, the chain of type (3.2) or (3.3) defines a path joining $+\tau_1^n$ with $-\tau_1^n$ or $+\tau_2^n$ with $-\tau_2^n$, and so on. Therefore we obtain paths of the following eight types:

$$\begin{array}{ll} \text{from } +\tau_1^n \text{ to } +\tau_2^n, & \text{from } -\tau_1^n \text{ to } -\tau_2^n, \\ \text{from } +\tau_1^n \text{ to } -\tau_1^n, & \text{from } +\tau_2^n \text{ to } -\tau_2^n, \\ \text{from } +\tau_1^n \text{ to } -\tau^{n-1}, & \text{from } -\tau_1^n \text{ to } +\tau^{n-1}, \\ \text{from } +\tau_2^n \text{ to } +\tau^{n-1}, & \text{from } -\tau_2^n \text{ to } -\tau^{n-1}. \end{array}$$

Since the directions of these paths are inessential, we do not distinguish between the path from $+\tau_1^n$ to $+\tau_2^n$ and the path from $+\tau_2^n$ to τ_1^n, and so on. Let α_j $(j = 1, 2, \dots, 8)$ denote the number of paths of the corresponding type in the pseudomanifold K^n. Then we have

$$\begin{aligned} d(\tau_1^n) &= \alpha_1 - \alpha_2 + \alpha_3 - \alpha_3 + \alpha_5 - \alpha_6, \\ d(\tau_2^n) &= \alpha_1 - \alpha_2 + \alpha_4 - \alpha_4 + \alpha_7 - \alpha_8, \\ d(\tau^{n-1}) &= -\alpha_5 + \alpha_6 + \alpha_7 - \alpha_8, \end{aligned}$$

and therefore

$$d(\tau_2^n) - d(\tau_1^n) = d(\tau^{n-1}). \tag{5}$$

This proves the equality (2) in the particular case $p = 2$. We can obtain the general case by summing equalities of type (5) over all simplices τ_i^{n-1} $(i = 1, \dots, p-1)$. □

Corollary 1. *Let $f : K^n \to M^n$ be a simplicial mapping such that $f(\partial K^n) \subseteq \partial M^n$ (in particular, let ∂K^n be empty). Then the local degrees are the same for all simplices $\tau^n \in M^n$. In other words, the mapping f has the degree* $\deg f$.

§**3.** In what follows (except for Corollary 6) the boundaries ∂K^n and ∂M^n of the pseudomanifolds K^n and M^n are supposed to be $(n-1)$-pseudomanifolds with the orientations induced by the orientations of K^n and M^n respectively. Moreover, if we have $f : K^n \to M^n$ and $f(\partial K^n) \subseteq \partial M^n$, then $f \mid \partial K^n$ is considered as a map $\partial K^n \to \partial M^n$, and $\deg(f \mid \partial K^n)$ is understood in this sense.

Theorem 2. *Let the boundaries ∂K^n and ∂M^n be non-empty $(n-1)$-pseudomanifolds. Let $f : K^n \to M^n$ be a simplicial mapping such that $f(\partial K^n) \subseteq \partial M^n$. Then there exists the degree $\deg(f \mid \partial K^n)$ and*

$$\deg f = \deg(f \mid \partial K^n). \tag{6}$$

PROOF. The existence of $\deg(f \mid \partial K^n)$ follows from Corollary 1 and the equality $\partial\partial K^n = 0$. The proof of the equality (6) goes by analogy with the proof of the equality (2), but instead of the chain (1a) we must consider the chain consisting of an arbitrary simplex $\tau^{n-1} \in \partial M^n$ and the unique simplex $\tau^n \in M^n$ incident to τ^{n-1}. $\square$

Since n-simplices degenerate under the simplicial mapping $f : K^n \to \partial K^n$, we deduce from Theorem 2 the following

Corollary 2. *For a simplicial mapping $f : K^n \to \partial K^n$ the equality $\deg(f \mid \partial K^n) = 0$ holds.*

Here we consider $f \mid \partial K^n$ as a mapping $\partial K^n \to \partial K^n$.

Let $K^n = \sigma^n$ be a simplex with some triangulation. Since the degree of the identical mapping $\mathrm{id} : \partial\sigma^n \to \partial\sigma^n$ is equal to 1, we obtain from Corollary 2 the following

Corollary 3. *There is no simplicial retraction of a simplex onto its boundary.*

Let $B^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}$ be the unit ball of the space $\mathbb{R}^{n+1}$ with some (in particular, polyhedral) norm and $S^n = \{x \in \mathbb{R}^{n+1} : \|x\|{=}1\}$ the boundary of this ball. A continuous mapping $f : S^n \to S^n$ is said to be odd if $f(-x) = -f(x)$ for every $x \in S^n$. The classic theorem of BORSUK ([2], Satz 1) states that the degree of every odd mapping is odd. The simplicial version of this theorem and Corollary 2 imply

Corollary 4. *There is no simplicial mapping $f : B^{n+1} \to S^n$ such that its restriction $f \mid S^n$ were odd.*

This is just the combinatorical lemma of TUCKER [9] expressed in terms of mappings.

§**4.** Let us show that Theorem 2 contains the combinatorial lemma of SPERNER in its non-oriented [8] and orienred [3] variants. This lemma is usually formulated in terms of labels corresponding to vertices of a triangulation of a simplex. We formulate it in terms of simplicial mappings, this being equivalent to the original form. Let σ^n be an n-simplex. We shall denote it by σ_1^n or σ_2^n according to whether or not it has a non-trivial triangulation.

Sperner's lemma. *Let $f : \sigma_1^n \to \sigma_2^n$ be a simplicial mapping such that every $(n-1)$-face of σ^n is mapped into itself. Then there is an n-face of triangulation σ_1^n that is mapped onto σ_2^n. More precisely,* $\deg f = 1$.

It is easy to see that under the assumptions of the lemma, each $(n-1)$-face of σ^n, as well as each face of any lower dimension, is mapped into itself. Now the proof goes by multiple application of Theorem 2 with the transition from faces of lower to faces of higher dimension.

We can deduce from Theorem 2 also the following analogue of Sperner's lemma.

Corollary 5 (Fan [6], Lemma 2)**.** *Let $f : \sigma_1^n \to \sigma_2^n$ be a simplicial mapping such that $f(\partial\sigma_1^n) \subseteq \partial\sigma_2^n$. Then, for some integer k, $0 \le k \le n$, there is a k-face σ^k of the simplex σ_2^n and a k-face τ^k of its triangulation σ_1^n, such that $\tau^k \subseteq \sigma^k$ and $f(\tau^k) = \sigma^k$.*

Proof. Let these faces be absent for every k, $0 \le k \le n-1$. Let $f \mid \partial\sigma_1^n$ be the restriction of the mapping f to $\partial\sigma_1^n$ that is homeomorphic to the $(n-1)$-dimensional sphere. This restriction maps the boundary $\partial\sigma^n$ into itself without fixed points. Therefore we have $\deg(f \mid \partial\sigma^n) = (-1)^n$ ([7], p. 136). Then Theorem 2 implies $\deg f = (-1)^n \neq 0$, and hence there is an n-face of triangulation σ_1^n that is mapped onto σ_2^n. □

The next Corollary generalizes Lemma 1 from [1] which is in turn a generalization of Sperner's lemma. Here $f \mid K_i^{n-1}$ means a mapping $K_i^{n-1} \to \partial M^n$ and $\deg(f \mid K_i^{n-1})$ should be understood in this sense.

Corollary 6. *Let the boundary ∂M^n be an $(n-1)$-pseudomanifold, let the boundary ∂K^n consists of finitely many of non-empty pairwise disjoint $(n-1)$-pseudomanifolds:*

$$\partial K^n = K_1^{n-1} \cup \cdots \cup K_p^{n-1}, \quad p \ge 1,$$

and suppose that for a simplicial mapping $f : K^n \to M^n$ we have $f(\partial K^n) \subseteq \partial M^n$. Then there exist $\deg(f \mid K_1^{n-1}), \ldots, \deg(f \mid K_p^{n-1})$ and

$$\deg f = \deg(f \mid K_1^{n-1}) + \cdots + \deg(f \mid K_p^{n-1}).$$

The proof goes by analogy with the proof of Theorem 1. Now one must consider the chains of type (4.1) only and remark that the set of these chains is partitioned into non-intersecting classes corresponding to connected components of the boundary ∂K^n.

§**5.** Our Theorem 1 generalizes the particular case (when $m = n$) of Fan's Theorem 1 [5], which concerns the oriented variant of the problem (concerning the non-oriented variant see Theorem 1 in [4]). We shall verify this only in the simplest case when $m = n = 2$. In this case, Fan considers a simplicial mapping of a pseudomanifold K^2 into the unit sphere S^2 of the space $\mathbb{R}^3$ equipped with the octahedral triangulation. Our chain (1) is here of the form

$$c\{-e_1, e_2, -e_3\},\ c\{e_2, -e_3\},\ c\{e_1, e_2, -e_3\},\ c\{e_1, -e_3\},$$
$$c\{e_1, -e_2, -e_3\},\ c\{e_1, -e_2\},\ c\{e_1, -e_2, e_3\},$$

where $\{e_1, e_2, e_3\}$ denotes the system of unit coordinate vectors in the space $\mathbb{R}^3$ and c is the operator of the convex hull. Fan defines the orientation of the sphere S^2 by means of the following convention: a 2-face (or a 1-face) of the sphere is oriented in the positive sense if the labels corresponding to its vertices increase in absolute value. This orientation agrees with the coherent one on the faces $c\{-e_1, e_2, -e_3\}$ and $c\{e_1, -e_2, -e_3\}$, but they are opposite on the faces $c\{e_1, e_2, -e_3\}$ and $c\{e_1, -e_2, e_3\}$. This implies the appearance of the factor $(-1)^n$ in formula (5) of Theorem 1 [5]. Now it is clear that our Theorem 1 generalizes the particular case of Fan's theorem. To obtain the proof of his general case (when $m > n$), one must take finitely many of our chains (1) and then sum the equalities (2) corresponding to these chains.

The author would like to thank the referee for helpful remarks.

References

[1] F. Bagemihl, An extension of Sperner's lemma, with applications to closed-set coverings and fixed points, *Fund. Math.* **40** (1953), 3–12.

[2] K. Borsuk, Drei Sätze über die n-dimensionale Sphäre, *Fund. Math.* **20** (1933), 177–190.

[3] A. B. Brown and S. S. Cairns, Strengthening of Sperner's lemma applied to homology theory, *Proc. Nat. Acad. Sci. USA* **47** (1961), 113–114.

[4] Ky Fan, Combinatorial properties of certain simplicial and cubical vertex maps, *Arch. Math.* **11** No. 5 (1960), 368–377.

[5] Ky Fan, Simplicial maps from an orientable n-pseudomanifold into S^m with the octahedral triangulation, *J. Comb. Theory* **2** No. 4 (1967), 588–602.

[6] Ky Fan, Some properties of convex sets related to fixed point theorems, *Math. Ann.* **266** No. 4 (1984), 519–537.

[7] Л. С. Понтрягин , Основы комбинаторной топологии, Наука, Москва, 1976.

[8] E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, *Abh. Math. Semin. Univ. Hamburg* **6** (1928), 265–272.

[9] A. W. Tucker, Some topological properties of disc and sphere, Proc. 1st Canadian Math. Congress, Montreal, 1945. Toronto, *Univ. of Toronto Press*, 1946, pp. 285–309.

Yu. A. Shashkin
Institute of Mathematics and Mechanics
ul. S. Kovalevskoi 16, GSP–384
620219, Ekaterinburg, Russia

(Received August 1, 1993; revised June 20, 1994)